Linear Algebra 6: The Primary Decomposition Theorem
Friday 11 November 2005
Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- The Primary Decomposition Theorem, Mark 1
- The Primary Decomposition Theorem, Mark 2
- The Primary Decomposition Theorem, Mark 3
- An application: diagonalisability

Note: Throughout this lecture $F$ is a field, $V$ is a finite-dimensional vector space over $F$, and $T: V \rightarrow V$ is a linear transformation.
The Primary Decomposition Theorem, Mark 1

Theorem: Suppose that \( f(T) = 0 \), where \( f \in F[x] \). Suppose also that \( f(x) = g(x)h(x) \), where \( g, h \in F[x] \) and \( g, h \) are co-prime. Then there are \( T \)-invariant subspaces \( U, W \) of \( V \) such that \( V = U \oplus W \) and \( g(T|_U) = 0 \), \( h(T|_W) = 0 \).

Proof.

Challenge. Let \( P \) be the projection of \( V \) onto \( U \) along \( W \). Express \( P \) as \( p(T) \) for some \( p \in F[x] \). (Worth a Marsbar.)
The Primary Decomposition Theorem, Mark 2

Theorem. If \( m_T(x) = g(x)h(x) \) where \( g, h \in F[x] \) are monic and co-prime, then \( g \) is the minimal polynomial of \( T|_U \) and \( h \) is the minimal polynomial of \( T|_W \).

Proof.

Example. If \( m_T(x) = x^2 - x \) then (as we already know) there exist \( U, W \leq V \) such that \( V = U \oplus W \), \( T|_U = I_U \) and \( T|_W = 0_W \).
The Primary Decomposition Theorem, Mark 3

The Primary Decomposition Theorem. Suppose that

\[ m_T(x) = f_1(x)^{m_1} f_2(x)^{m_2} \cdots f_k(x)^{m_k}, \]

where \( f_1, f_2, \ldots, f_k \) are distinct monic irreducible polynomials over \( F \). Then

\[ V = V_1 \oplus V_2 \oplus \cdots \oplus V_k, \]

where \( V_1, V_2, \ldots, V_k \) are \( T \)-invariant subspaces and the minimal polynomial of \( T|_{V_i} \) is \( f_i^{m_i} \) for \( 1 \leq i \leq k \).

Proof.
An application: diagonalisability

Definition: The linear transformation $T$ is said to be diagonalisable if there is a basis of $V$ consisting of eigenvectors of $T$.

Note: Matrix $A \in M_{n \times n}(F)$ is said to be diagonalisable if there exists an invertible $n \times n$ matrix $P$ over $F$ such that $P^{-1}AP$ is diagonal. And $T$ is diagonalisable if and only if there is a basis of $V$ with respect to which its matrix is diagonal.

Theorem. Our transformation $T$ is diagonalisable if and only if $m_T(x)$ may be factorised as a product of distinct linear factors in $F[x]$.

Proof.