Rings & Arithmetic 3: Ideals and quotient rings

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Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Ideals, examples
- Quotient rings
- Homomorphisms
- Kernel and image
- The First Isomorphism Theorem
- A worked exercise
Ideals

Definition: A subset $A$ of a ring $R$ (commutative, with 1) is said to be an ideal if

$(1)$ $0 \in A$ and $a, b \in A \Rightarrow a + b, -a \in A$ (so $A$ is an additive subgroup);

$(2)$ $(a \in A, x \in R) \Rightarrow xa \in A$.

Note: If $A$ is an ideal and $1 \in A$ then $A = R$. Thus a proper ideal is never a subring.
Examples of ideals

Examples: \( \{0\} \), \( R \) are always ideals.

Examples: \( n\mathbb{Z} \) is an ideal in \( \mathbb{Z} \).

Examples: Generally, if \( R \) is any ring (commutative, with 1) and \( a \in R \) then \( aR \) is an ideal.

Note: Such ideals \( aR \) (or \( Ra \)) are known as principal ideals. Notations \( (a) \) and \( \langle a \rangle_R \) are also used by some mathematicians.
Quotient rings

Definition: Let \( A \) be an ideal in the ring \( R \). The quotient ring \( R/A \) is defined as follows:

\[
\text{Set} := \{ x + A \mid x \in R \} \quad \text{[additive cosets]}
\]

\[
0 := A
\]

\[
1 := 1 + A
\]

\[
(x + A) + (y + A) := (x + y) + A
\]

\[
(x + A)(y + A) := (xy) + A.
\]

Check that this is a ring. The issues are:

- are + and \( \times \) well-defined?
- do the ring axioms hold?

Important example: \( \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \)
Homomorphisms

**Definition:** Let $R, S$ be rings (commutative, with 1). A function $\varphi : R \to S$ is said to be a homomorphism if

1. $\varphi(0) = 0, \quad \varphi(1) = 1$
2. $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in R$
3. $\varphi(ab) = \varphi(a) \varphi(b)$ for all $a, b \in R$

**Example:** The identity map $R \to R$ is a homomorphism.

**Example:** If $R$ is a ring, $A$ an ideal then the map $x \mapsto x + A$ is a homomorphism $R \to R/A$. It is known as the natural projection or natural epimorphism.

**Example:** In particular, the map $\mathbb{Z} \to \mathbb{Z}_n$ where $x \mapsto \overline{x}$ (and $\overline{x}$ is the residue class of $x$ modulo $n$) is a surjective homomorphism.
Notes on homomorphisms

Note: If $\varphi : R \to S$ and $\psi : S \to T$ are ring homomorphisms then also $\psi \circ \varphi : R \to T$ is a homomorphism.

Note: If $\varphi : R \to S$ is a ring homomorphism then $\varphi U(R) \leq U(S)$.

Definition: An isomorphism is an invertible homomorphism. We write $R \cong S$ to mean that there exists an isomorphism $R \to S$ (and then we say that $R$, $S$ are isomorphic).

Note: A ring homomorphism $\varphi : R \to S$ is an isomorphism if and only if it is one-one and onto (injective and surjective).
Image and kernel

Definition: Let $\varphi : R \rightarrow S$ be a ring homomorphism. We define the image and kernel of $\varphi$ by

\[
\text{Im} \varphi := \{ y \in S \mid \exists x \in R : \varphi(x) = y \}
\]
\[
\text{Ker} \varphi := \{ x \in R \mid \varphi(x) = 0 \}.
\]

Important Observation: If $\varphi : R \rightarrow S$ is a ring homomorphism then $\text{Im} \varphi$ is a subring of $S$ and $\text{Ker} \varphi$ is an ideal in $R$.

Proof.
The First Isomorphism Theorem

First Isomorphism Theorem for rings: If \( \varphi : R \to S \) is a ring homomorphism then \( \text{Im}\varphi \cong R/\text{Ker}\varphi \).

Proof.
A worked example

Part of Schools 1987, I, 5. Let $D$ be the ring of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with the operations of pointwise addition and multiplication. Show that

$$I = \{ f \in D : f(0) = f'(0) = 0 \}$$

is an ideal in $D$.

Let $\mathbb{R}[x]$ denote the ring of polynomials in the indeterminate $x$ with real coefficients, and $(x^2)$ the ideal generated by the polynomial $x^2$. Show that there is a homomorphism from $\mathbb{R}[x]$ onto $D/I$ and deduce that $D/I \cong \mathbb{R}[x]/(x^2)$. 