

Elliptic Curves. Sheet 2.

1. Let K be a field with non-Archimedean valuation $|\cdot|$.
 - (a). For any $x, y \in K$ show that, if $|x| \neq |y|$ then $|x \pm y| = \max(|x|, |y|)$.
 - (b). If $x_1, \dots, x_n \in K$ and if there exists ℓ such that $|x_\ell| > |x_i|$ for all $i \neq \ell$, then show that $|x_1 + \dots + x_n| = |x_\ell|$.
 - (c). Suppose that $s_n \rightarrow s$ in $K, |\cdot|$. Show that $|s_n| \rightarrow |s|$ in $\mathbb{R}, |\cdot|_\infty$. When $s \neq 0$, show that there exists N such that, for all $n > N$, $|s_n| = |s|$.
- 2(a). Find: $|3/50|_5, |3/50|_3, |3/50|_7, d_5(2/3, 1/5), d_7(2/3, 1/5), d_{11}(2/3, 1/5)$.
- (b). Describe $|3/7|_p$ for all p . What is the product $\prod |3/7|_i$, taken over $i = p$, for all primes p , and $i = \infty$? Given any $x \in \mathbb{Q}$ ($x \neq 0$), what is $\prod |x|_i$?
3. Which of the following are convergent in $\mathbb{Q}_5$?
 - (a). $1/5^n$. (b). n . (c). $n!$ (d). $3 + 10^n$. (e). $\sum_0^\infty 10^n$. (f). $\sum_0^\infty 7^n$.
4. For each p, m, r , either find an $x \in \mathbb{Z}$ such that $|x^2 - r|_p \leq p^{-m}$ or show that no such x exists.
 - (a). $p = 5, r = -1, m = 4$. (b). $p = 3, r = 7/8, m = 7$. (c). $p = 5, r = 5/4, m = 4$.
5. Find the 7-adic expansion of each of: 200 and $3/14$. Determine the member of $\mathbb{Q}$ expressed by the 5-adic expansion $2, \overline{34}$.
6. Let $x \in \mathbb{Q}$. Show that $x \in \mathbb{Z} \iff (x \in \mathbb{Z}_p \text{ for all } p)$.
7. Decide whether there exists $x \in \mathbb{Q}_p$ such that $x^2 = -28$ for each of: $p = 2, 3, 5, 7, 11$.
8. Show that $(X^2 - 2)(X^2 - 17)(X^2 - 34)$ has a root in $\mathbb{R}$ and in every $\mathbb{Q}_p$, but not in $\mathbb{Q}$.
9. Is 4 a cube in $\mathbb{Q}_3$? Is 28 a cube in $\mathbb{Q}_3$? Is 13 a cube in $\mathbb{Q}_7$?

The following question is compulsory for students taking the MSc in MFoCS (Mathematics and the Foundations of Computer Science). For everyone else, it is optional.

10. Show that $|n!|_p = p^{-M}$ where $M = \sum_{i=1}^\infty \left\lfloor \frac{n}{p^i} \right\rfloor$ (where $[x]$ denotes the greatest integer $\leq x$). Let K be a field containing $\mathbb{Q}_p$, let $|\cdot|$ be a non-Archimedean valuation on K which extends $|\cdot|_p$, and assume that K is complete with respect to this valuation. For any $x \in K$, show that $\exp_p(x) = \sum_{n=0}^\infty \frac{x^n}{n!}$ converges if and only if $|x| < p^{-\frac{1}{p-1}}$. When $K = \mathbb{Q}_p$ ($p \neq 2$), show that $\exp_p(x)$ converges if and only if $|x|_p < 1$. When $K = \mathbb{Q}_2$, show that $\exp_2(x)$ converges if and only if $|x|_2 < \frac{1}{2}$.