

Elliptic Curves. Sheet 3.

1. Show that the curve $2Y^2 = X^4 - 17$ has points in $\mathbb{R}$ and every $\mathbb{Q}_p$, but not in $\mathbb{Q}$.
[Hint: For showing that there are points in every $\mathbb{Q}_p$, it is helpful to use Theorem 1.15 (note also that the curve is birationally equivalent to $V^2 = 2X^4 - 34$, where $V = 2Y$). For showing there are no points in $\mathbb{Q}$, first show that, if there were points in $\mathbb{Q}$, then there would exist $r, s, t \in \mathbb{Z}$ with $\gcd(r, t) = 1$ such that $2s^2 = t^4 - 17r^4$, and then show that any prime dividing s is a quadratic residue modulo 17].
2. Let $p \equiv 2 \pmod{3}$. For any $a \in \mathbb{Z}$ such that $p \nmid a$, show that there exists $x \in \mathbb{Z}_p$ with $x^3 = a$.
3. Let K be any field with a non-Archimedean valuation $|\cdot|$, and let $R = \{x \in K : |x| \leq 1\}$. Let $f(X) \in R[x]$ have discriminant D , and let $a_0 \in R$ satisfy $|f(a_0)| < |D|^2$. Show that $f(X)$ has a root $a \in R$.
4. Prove that, if $d \in \mathbb{Z}_p$ is non-square, then

$$|a + b\sqrt{d}|_p = |a^2 - b^2d|_p^{1/2}, \text{ for any } a, b \in \mathbb{Q}_p,$$
defines a non-Archimedean valuation on $\mathbb{Q}_p(\sqrt{d})$ which extends the usual $|\cdot|_p$ on $\mathbb{Q}_p$.
[Hint: First show that, for any $\alpha \in \mathbb{Q}_p(\sqrt{d})$, $|\alpha|_p \leq 1 \Rightarrow |\alpha + 1|_p \leq 1$].
5. Let $\mathcal{E} : Y^2 = X^3 + 17$, defined over $\mathbb{Q}$, and $\tilde{\mathcal{E}} : Y^2 = X^3 + 2$, defined over $\mathbb{F}_5$. What does $(-64/25, 59/125) \in \mathcal{E}(\mathbb{Q})$ map to under the reduction map modulo 5?
6. Let $\mathcal{E} : Y^2 = X^3 + p$, defined over $\mathbb{Q}_p$, and $\tilde{\mathcal{E}} : Y^2 = X^3$, defined over $\mathbb{F}_p$, where $p \neq 2$. Show that $(0, 0)$ on $\tilde{\mathcal{E}}$ does not lift to a point in $\mathcal{E}(\mathbb{Q}_p)$.
7. Give examples of elliptic curves defined over $\mathbb{Z}_p$ ($p \neq 2$) such that $\tilde{\mathcal{E}}$, defined over $\mathbb{F}_p$, has:
 - (a). A cusp which lifts to a point in $\mathcal{E}(\mathbb{Q}_p)$.
 - (b). A cusp which does not lift to a point in $\mathcal{E}(\mathbb{Q}_p)$.
 - (c). A node which lifts to a point in $\mathcal{E}(\mathbb{Q}_p)$.
 - (d). A node which does not lift to a point in $\mathcal{E}(\mathbb{Q}_p)$.
8. A *non-commutative formal group* over a ring R is a power series $F(X, Y) \in R[[X, Y]]$ which satisfies:
$$F(X, Y) = X + Y + \text{terms of degree } \geq 2,$$

$$F(X, F(Y, Z)) = F(F(X, Y), Z) \text{ [associativity],}$$
but not $F(X, Y) = F(Y, X)$ [commutativity]. Let $R = \mathbb{F}_p[t]/I$, where $I = t^2\mathbb{F}_p[t]$. Find a non-commutative formal group over R .

The following question is compulsory for students taking the MSc in MFoCS (Mathematics and the Foundations of Computer Science). For everyone else, it is optional.

9. Let $\mathcal{E} : Y^2 = X^3 + AX$, where $A \in \mathbb{Z}$ and $A \neq 0$. Let $F(X, Y)$ be the formal group associated to $\mathcal{E}$ [as in lectures] and let $F(X, Y) = \sum F_n(X, Y)$, where each $F_n(X, Y)$ is homogeneous of degree n . Show that $F_n(X, Y) = 0$ unless $n \equiv 1 \pmod{4}$. What is the similar result when the elliptic curve is of the form $\mathcal{E} : Y^2 = X^3 + B$?