

## Elliptic Curves. Solutions to Sheet 1.

**1. (a).** The only points on  $Y^2 = X^3 + 2X$  over  $\mathbb{F}_5$  are:  $\mathbf{o}, (0,0)$ . The group table is the same as that of  $C_2$  (“cyclic 2” group; i.e. the integers modulo 2 under addition), with 0, 1 replaced by  $\mathbf{o}, (0,0)$ , respectively.

**(b).** The only points on  $Y^2 = X^3 + 1$  over  $\mathbb{F}_5$  are:  $\mathbf{o}, (0,1), (0,4), (4,0), (2,2), (2,3)$ . The point  $(2,2)$  has order 6, and the group table is the same as that of  $C_6$  (“cyclic 6” group; i.e. the integers modulo 6 under addition), with 0, 1, 2, 3, 4, 5 replaced by  $\mathbf{o}, (2,2), (0,4), (4,0), (0,1), (2,3)$ , respectively. Note that it’s also correct to say that the group is  $C_2 \times C_3$ , since  $C_2 \times C_3$  is isomorphic to  $C_6$ .

**2.** We have:  $2YY' = 3X^2 + 4$ , and so the slope at the point  $(2,4)$  is:  $(3 \cdot 2^2 + 4)/(2 \cdot 4) = 2$ . The line tangent to the curve at  $(2,4)$  is:  $Y = 2X$ . The  $x$ -coordinate of the third point of intersection is therefore:  $2^2 - 2 - 2 = 0$ , with  $y$ -coordinate  $y = 2 \cdot 0 = 0$ . Hence,  $(2,4) + (2,4) + (0,0) = \mathbf{o}$ , and so  $(2,4) + (2,4) = (0,-0) = (0,0)$ . Now, note that we always have  $-(x,y) = (x,-y)$  and so  $-(0,0) = (0,0)$ , giving:  $2(0,0) = \mathbf{o}$ . Hence,  $4(2,4) = 2(0,0) = \mathbf{o}$ . Also, check that  $3(2,4) = 4(2,4) - (2,4) = -(2,4) = (2,-4)$ . Summarising: the first 4 multiples of  $(2,4)$  are:  $1(2,4) = (2,4)$ ,  $2(2,4) = (0,0)$ ,  $3(2,4) = (2,-4)$ ,  $4(2,4) = \mathbf{o}$ , and so 4 is the smallest positive multiple of  $(2,4)$  equal to  $\mathbf{o}$ .

**3.**

**(a).** Case 1.  $m$  is odd. The birational transformation  $(X,Y) \mapsto (X/f_m, Y/f_m^{(m+1)/2})$  [with inverse transformation:  $(X,Y) \mapsto (Xf_m, Yf_m^{(m+1)/2})$ ] maps the given curve to:  $f_m^{(m+1)}Y^2 = f_m^{(m+1)}X^m + \dots + f_0$ , which is the same as:  $Y^2 = X^m + c_{m-1}X^{m-1} + \dots + c_0$ , where  $c_i = f_i/f_m^{m+1-i} \in \mathbb{Q}$  [N.B. our birational transformation is legitimate, since  $(m+1)/2$  is an integer]. So, our new right hand side is monic and is still defined over  $\mathbb{Q}$ . Now, write each  $c_i = n_i/d_i$ , where each  $n_i, d_i \in \mathbb{Z}$ , with each  $n_i, d_i$  coprime. Let  $d$  be the least common multiple of  $d_0, \dots, d_{m-1}$ ; that is to say,  $d$  is the smallest integer divisible by all of the denominators of the coefficients  $c_i$ . Now, the birational transformation  $(X,Y) \mapsto (Xd^2, Yd^m)$  [with inverse transformation:  $(X,Y) \mapsto (X/d^2, Y/d^m)$ ] maps to the curve:  $Y^2/d^{2m} = X^m/d^{2m} + c_{m-1}X^{m-1}/d^{2m-2} + \dots + c_0$ , which is the same as:  $Y^2 = X^m + g_{m-1}X^{m-1} + \dots + g_0$ , where  $g_i = c_id^{2(m-i)}$ . But now, each  $d_i|d$  and so each  $c_id \in \mathbb{Z}$ , giving that each  $g_i \in \mathbb{Z}$ , as required.

Case 2.  $f_m$  is a square. Let  $w$  be such that  $f_m = w^2$ . The birational transformation  $(X,Y) \mapsto (X, Y/w)$  [with inverse transformation:  $(X,Y) \mapsto (X, Yw)$ ] maps the given curve to:  $f_mY^2 = f_mX^m + \dots + f_0$ , which is the same as:  $Y^2 = X^m + c_{m-1}X^{m-1} + \dots + c_0$ , where  $c_i = f_i/f_m \in \mathbb{Q}$ . We are again in the situation where our right hand side is monic and is still defined over  $\mathbb{Q}$ . So now apply the same second birational transformation as in Case 1.

**(b).** The birational transformation  $(X,Y) \mapsto (5X, 5^2Y)$  [with inverse  $(X,Y) \mapsto ((1/5)X, (1/5^2)Y)$ ] maps the given curve to the curve:  $Y^2 = X^3 + 75X^2 + 625$ . The birational transformation  $(X,Y) \mapsto (X + 25, Y)$  [with inverse  $(X,Y) \mapsto (X - 25, Y)$ ] maps this to the curve:  $Y^2 = X^3 - 1875X + 31875$ , as required.

4.

(a). Since  $p \equiv 2 \pmod{3}$ , we have  $\gcd(3, p-1) = 1$ , and so there exist  $\lambda, \mu \in \mathbb{Z}$  such that  $3\lambda + (p-1)\mu = 1$ . For any  $x, y \in \mathbb{F}_p^*$ , we have  $x^{p-1} = y^{p-1} = 1$  [by Fermat's Little Theorem], and so:

$$\begin{aligned} x^3 = y^3 &\Rightarrow (x^3)^\lambda \cdot 1^\mu = (y^3)^\lambda \cdot 1^\mu \Rightarrow (x^3)^\lambda \cdot (x^{p-1})^\mu = (y^3)^\lambda \cdot (y^{p-1})^\mu \\ &\Rightarrow x^{3\lambda+(p-1)\mu} = y^{3\lambda+(p-1)\mu} \Rightarrow x = y. \end{aligned}$$

Thus the map  $x \mapsto x^3$  is injective on  $\mathbb{F}_p^*$  and so is a bijection. Since  $0^3 = 0$ , the map is also a bijection on  $\mathbb{F}_p$ . Now, rearrange the equation of the curve as:  $X^3 = Y^2 - A$ . We see that, for each  $Y = 0, \dots, p-1$  in  $\mathbb{F}_p$ , there is precisely one  $X \in \mathbb{F}_p$  such that  $X^3 = Y^2 - A$ , giving rise to a total of precisely  $p$  affine points, which becomes  $p+1$  points when we include the point at infinity.

(b). Since  $p \equiv 3 \pmod{4}$ , we have that  $\left(\frac{-1}{p}\right) = -1$  [i.e.  $-1$  is not congruent to a square mod  $p$ ]. First note that  $X = 0$  gives rise to precisely one point  $(0, 0)$ . Now, pair each  $X \in \mathbb{F}_p^*$  with its negative  $-X \pmod{p}$ . If  $X(X^2 + B) = 0$  then the pair  $X, -X$  gives rise to precisely two points:  $(X, 0), (-X, 0)$ . If  $X(X^2 + B) \neq 0$  then one of  $X(X^2 + B)$  and  $-X(X^2 + B)$  will be a nonzero quadratic residue mod  $p$  and the other will be a quadratic nonresidue mod  $p$  [using the fact that  $-1$  is a quadratic nonresidue mod  $p$  and the standard property of Legendre symbols that:  $\left(\frac{mn}{p}\right) = \left(\frac{m}{p}\right)\left(\frac{n}{p}\right)$ ]. Whichever one is a nonresidue will contribute no points; which one is a residue will contribute two points:  $(X, Y), (X, -Y)$ . In all cases, each pair  $X, -X$  will contribute two points. This gives rise to a total of precisely  $p$  affine points, which becomes  $p+1$  points when we include the point at infinity.

5.

(a). We first solve for  $g_1, g_0$  and  $h_1, h_0$  such that  $Q(X) = (X^2 + g_1X + g_0)^2 + (h_1X + h_0)$ . Equating coefficients of  $X^3$  forces  $g_1 = f_3/2$ . Successively equating coefficients of  $X^2, X^1, X^0$  now solves for  $g_0, h_1, h_0$ , giving:

$$g_1 = f_3/2, \quad g_0 = (4f_2 - f_3^2)/8, \quad h_1 = f_1 - f_3(4f_2 - f_3^2)/8, \quad h_0 = f_0 - (4f_2 - f_3^2)^2/64.$$

Now define  $G(X) = X^2 + g_1X + g_0$  and  $H(X) = h_1X + h_0$ , where  $g_1, g_0, h_1, h_0$  are as in the above equation, and we see that indeed  $Q(X) = G(X)^2 + H(X)$ . Our curve can be written as:  $(Y + G(X))(Y - G(X)) = H(X)$ . The birational transformation  $(X, Y) \mapsto (Y + G(X), X(Y + G(X)))$  [that is:  $(X, Y) \mapsto (T, S)$ , given in the hint] has inverse  $(X, Y) \mapsto (Y/X, X - G(Y/X))$  and transforms the given curve to:  $X(X - 2G(Y/X)) = H(Y/X)$ . After multiplying both sides by  $X$  and expanding, this can be rewritten as:

$$2Y^2 + 2g_1XY + h_1Y = X^3 - 2g_0X^2 - h_0X.$$

[You could write this as:  $(Y + G(X))(Y - G(X)) = H(X) \leftrightarrow T(T - 2G(S/T)) = H(S/T)$  [where  $T = Y + G(X)$ ,  $S = X(Y + G(X))$ ,  $X = S/T$ ,  $Y = T - G(S/T)$ ], which in turn is:  $2S^2 + 2g_1TS + h_1S = T^3 - 2g_0T^2 - h_0T$ , if you prefer]. One can now easily proceed to remove the term  $2g_1XY$  by  $(X, Y) \mapsto (X, Y + g_1X/2)$  and the term  $h_1Y$  by

$(X, Y) \mapsto (X, Y + h_1/4)$ , giving the curve:  $Y^2 = (1/2)X^3 + \dots$ . Finally, remove the  $X^2$  term and then make the right hand side monic & defined over  $\mathbb{Z}$ , exactly as in Question 1 (a).

(b). First apply to  $\mathcal{C}_1$  the birational transformation  $(X, Y) \mapsto (1/X, Y/X^2)$  [self-inverse] which maps  $\mathcal{C}_1$  to:  $Y^2 = 9X^4 + 2$ . The map  $(X, Y) \mapsto (X, Y/3)$  [with inverse  $(X, Y) \mapsto (X, 3Y)$ ] maps this to  $Y^2 = X^4 + 2/9$ . In the notation of part (a), we can take  $G(X) = X^2$  and  $H(X) = 2/9$ , and so  $g_1 = g_0 = h_1 = 0$  and  $h_0 = 2/9$ . The curve is therefore birationally equivalent to:  $2Y^2 = X^3 - (2/9)X$ ; that is,  $Y^2 = (1/2)X^3 - (1/9)X$ . We finally apply  $(X, Y) \mapsto (18X, 108Y)$  [with inverse  $(X, Y) \mapsto (X/18, Y/108)$ ], which maps this curve to:  $(Y/108)^2 = (1/2)(X/18)^3 - (1/9)(X/18)$ , which is the same as:  $Y^2 = X^3 - 72X$ , as required.

For  $\mathcal{C}_2$ , the important thing to recognise is that it has the  $\mathbb{Q}$ -rational point  $(1, 3)$ , which allows us to make the constant term a square after a linear change in  $X$ . Namely,  $(X, Y) \mapsto (X - 1, Y)$  [with inverse  $(X, Y) \mapsto (X + 1, Y)$ ] maps  $\mathcal{C}_2$  to the curve:  $Y^2 = 2(X + 1)^4 + 7$ ; that is,  $Y^2 = 2X^4 + 8X^3 + 12X^2 + 8X + 9$ . Now,  $(X, Y) \mapsto (1/X, Y/X^2)$  [self-inverse] maps this to  $Y^2 = 9X^4 + 8X^3 + 12X^2 + 8X + 2$ . Since the coefficient of  $X^4$  is now a square, we can clearly proceed as for  $\mathcal{C}_2$  to get the curve in the required form.

Although the question does not require it, here in fact (for those who are curious) is a series of birational transformations which transform  $Y^2 = 9X^4 + 8X^3 + 12X^2 + 8X + 2$  to the form  $Y^2 = X^3 + AX + B$ . This gives you an idea of how large the numbers can become (although the coefficients of the final answer aren't too bad). In the notation of part (c), we have:  $g_1 = 4/9, g_0 = 46/81, h_1 = 280/729, h_0 = -658/6561$ , and so our curve is birationally equivalent to:  $Y^2 + (4/9)XY + (140/729)Y = (1/2)X^3 - (46/81)X^2 + 329/6561$ . One now removes the  $XY$  and  $Y$  terms by the map  $(X, Y) \mapsto (X, Y + (2/9)X + 70/729)$  [with inverse:  $(X, Y) \mapsto (X, Y - (2/9)X - 70/729)$ ], which birationally transforms this curve to the curve:  $Y^2 = (1/2)X^3 - (14/27)X^2 + (203/2187)X + 4900/531441$ . Now apply:  $(X, Y) \mapsto (X - 28/81, Y)$  [with inverse  $(X, Y) \mapsto (X + 28/81, Y)$ ], which maps this curve to the curve:  $Y^2 = (1/2)X^3 - (7/81)X$ . Finally, the map:  $(X, Y) \mapsto (18X, 108Y)$  [with inverse  $(X, Y) \mapsto (X/18, Y/108)$ ] maps this to the curve:  $Y^2 = X^3 - 56X$ , which is in the required form. In summary:  $\mathcal{C}_2 : Y^2 = 2X^4 + 7$  is birationally equivalent to  $Y^2 = X^3 - 56X$ .

(c). First, the map  $(X, Y) \mapsto (X, Y/i)$  [with inverse:  $(X, Y) \mapsto (X, iY)$ ] maps the given curve to the curve:  $Y^2 = X^4 + 1$ . In the notation of part (a), we can take  $G(X) = X^2$  and  $H(X) = 1$ , so that our curve can be birationally transformed over  $\mathbb{Q}$  to  $2Y^2 = X^3 - X$ ; that is:  $Y^2 = (1/2)X^3 - (1/2)X$ . Finally, applying  $(X, Y) \mapsto (2X, 4Y)$  [with inverse:  $(X, Y) \mapsto (X/2, Y/4)$ ] birationally transforms this curve to:  $Y^2 = X^3 - 4X$ , as required.

It is not possible to birationally transform over  $\mathbb{R}$  the curve  $Y^2 = -X^4 - 1$  to the form  $Y^2 = X^3 + AX + B$ , since any such birational transformation would give a map between the  $\mathbb{R}$ -rational points of the two curves; but the second curve has infinitely many  $\mathbb{R}$ -rational points (take all  $X$  sufficiently large, e.g. all  $X > 2\max(1, |A|, |B|)$ ), whereas the first curve has no  $\mathbb{R}$ -rational points. [Note that this also implies that there is no such birational transformation over  $\mathbb{Q}$ , either.] The curves  $Y^2 = -X^4 - 1$  and  $Y^2 = X^3 - 4X$  are called *twists* of each other, since they are both defined over  $\mathbb{Q}$ , are birationally equivalent over  $\mathbb{C}$ , but are not birationally equivalent over  $\mathbb{Q}$ .

Over  $\mathbb{Q}(\sqrt{-2})$ , the curve  $Y^2 = -X^4 - 1$  has the  $\mathbb{Q}(\sqrt{-2})$ -rational point  $(1, \sqrt{-2})$ . Applying the transformation  $(X, Y) \mapsto (X - 1, Y)$  [with inverse  $(X, Y) \mapsto (X + 1, Y)$ ] we get:  $Y^2 = -X^4 - 4X^3 - 6X^2 - 4X - 2$ . Now, applying  $(X, Y) \mapsto (1/X, Y/X^2)$  [self-inverse], we get:  $Y^2 = -2X^4 - 4X^3 - 6X^2 - 4X - 1$ . Now, we apply  $(X, Y) \mapsto (X, Y/\sqrt{-2})$  [with inverse  $(X, Y) \mapsto (X, Y\sqrt{-2})$ ] to get:  $Y^2 = X^4 + 2X^3 + 3X^2 + 2X + 1/2$ . From part (a), we can now say that this is birationally equivalent (over  $\mathbb{Q}$  and so over  $\mathbb{Q}(\sqrt{-2})$ ) to a curve of the form  $Y^2 = X^3 + AX + B$ , with  $A, B \in \mathbb{Z}$ .

## Elliptic Curves. Solutions to Sheet 2.

1.

(a).  $(2, 0) + (-(2, 0)) = \mathbf{o}$ , where  $-(2, 0)$  is the inverse of  $(2, 0)$ ; but we know that  $-(2, 0) = (2, -0) = (2, 0)$  and so  $(2, 0) + (2, 0) = \mathbf{o}$ ; that is,  $(2, 0)$  is of order 2, as required.

(b). In general, a point  $(x, y)$  on  $Y^2 = X^3 + f_2X^2 + f_1X + f_0$  has inverse:  $(x, -y)$  and so  $(x, y)$  is of order 2 exactly when  $(x, y) + (x, y) = \mathbf{o} \iff (x, y) = (x, -y)$ . But  $y = -y$  precisely when  $y = 0$ . So, we see that all points of order 2 are of the form  $(x, 0)$  where  $x$  is a root of the cubic  $X^3 + f_2X^2 + f_1X + f_0$ . There are at most 3 such points over any field, since a cubic has at most 3 roots in any field. In the complex numbers, of course, there will always be exactly 3 such points. Applying this to the curve  $Y^2 = X(X^2 - 3)$ , we see that  $X(X^2 - 3)$  has one  $\mathbb{Q}$ -rational root  $X = 0$ , giving  $(0, 0)$  as the only  $\mathbb{Q}$ -rational point of order 2; the other two roots give the points  $(\sqrt{3}, 0)$  and  $(-\sqrt{3}, 0)$ , which are the remaining two  $\mathbb{C}$ -rational points of order 2. The curve  $Y^2 = X^3 - 7$  has no  $\mathbb{Q}$ -rational point of order 2 (since  $X^3 - 7$  has no  $\mathbb{Q}$ -rational roots); over  $\mathbb{C}$ , there are the 3 points of order 2 given by:  $(\omega_1, 0), (\omega_2, 0), (\omega_3, 0)$ , where  $\omega_k = 7^{1/3}e^{2\pi ik/3}$ , for  $k = 0, 1, 2$ . Finally, the curve  $Y^2 = X(X - 1)(X - 7)$  has the points of order 2:  $(0, 0), (1, 0), (7, 0)$  all of which are both  $\mathbb{C}$ -rational and  $\mathbb{Q}$ -rational. Group structures of the 2-torsion groups over  $\mathbb{Q}$  for the three curves are clearly:  $C_2, C_1$  and  $C_2 \times C_2$ , respectively (since all members except  $\mathbf{o}$  have order 2).

2. Using  $2YY' = 3X^2$ , we see that the slope of the curve at  $(0, 2)$  is  $3 \cdot 0^2 / (2 \cdot 2) = 0$ , and so the tangent to the curve at  $(0, 2)$  is the line:  $Y = 2$ . The  $x$ -coordinate of the third point of intersection is then:  $0^2 - 0 - 0 = 0$ , with corresponding  $y$ -coordinate 2. In summary: The line  $Y = 2$  intersects the curve at  $(0, 2)$  with multiplicity 3. Hence:  $(0, 2) + (0, 2) + (0, 2) = \mathbf{o}$ , and so:  $3(0, 2) = \mathbf{o}$ . But,  $1(0, 2) = (0, 2)$  and  $2(0, 2) = (0, -2)$ , and so 3 is the smallest  $k \geq 1$  such that  $k(0, 2) = \mathbf{o}$ ; that is,  $(0, 2)$  has order 3.

3.

(a). As usual,  $-(\alpha, 0) = (\alpha, 0)$ , and so  $(\alpha, 0) + (\alpha, 0) = \mathbf{o}$ ; that is,  $(\alpha, 0)$  is a point of order 2, as required. We now compute  $(x', y') = (x, y) + (\alpha, 0)$ . First, the line through  $(x, y)$  and  $(\alpha, 0)$  is:  $Y = (y/(x - \alpha))(X - \alpha)$ . Substituting this into the curve gives:  $X^3 - (m^2 + \alpha - a)X^2 + \dots = 0$ , where  $m = y/(x - \alpha)$ . The sum of the roots of this cubic is:  $m^2 + \alpha - a$ , and so the  $x$ -coordinate of the third point of intersection is given by:  $x' = m^2 + \alpha - a - x - \alpha = m^2 - x - a = y^2/(x - \alpha)^2 - x - a = (\alpha x + a\alpha + b)/(x - \alpha)$ . So, we have:

$$T : x \mapsto \frac{\alpha x + a\alpha + b}{x - \alpha}.$$

We now check  $T^2(x) = T(T(x))$ , which is:

$$T : x \mapsto \frac{\alpha(\alpha x + a\alpha + b)/(x - \alpha) + a\alpha + b}{(\alpha x + a\alpha + b)/(x - \alpha) - \alpha}.$$

On multiplying numerator and denominator by  $x - \alpha$  and simplifying, we get:  $(\alpha^2 + a\alpha + b)x/(\alpha^2 + a\alpha + b)x$ , which is  $x$ , as required [Note that  $\alpha^2 + a\alpha + b \neq 0$ , since the cubic  $(X - \alpha)(X^2 + aX + b)$  has distinct roots].

(b). For  $Y^2 = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)$ , the formula for  $T_1$  is the same as for  $T$  in part (a), but with  $\alpha, a, b$  replaced by  $\alpha_1, -\alpha_2 - \alpha_3, \alpha_2\alpha_3$ , respectively. That is:

$$T_1 : x \mapsto \frac{\alpha_1 x - \alpha_1 \alpha_2 - \alpha_1 \alpha_3 + \alpha_2 \alpha_3}{x - \alpha_1}.$$

The formula for  $T_2$  is the same as for  $T_1$ , but with  $\alpha_1, \alpha_2, \alpha_3$  replaced by  $\alpha_2, \alpha_3, \alpha_1$ , respectively. The formula for  $T_3$  is the same as for  $T_2$ , but (again) with  $\alpha_1, \alpha_2, \alpha_3$  replaced by  $\alpha_2, \alpha_3, \alpha_1$ , respectively. These are then:

$$T_2 : x \mapsto \frac{\alpha_2 x - \alpha_2 \alpha_3 - \alpha_2 \alpha_1 + \alpha_3 \alpha_1}{x - \alpha_2}.$$

$$T_3 : x \mapsto \frac{\alpha_3 x - \alpha_3 \alpha_1 - \alpha_3 \alpha_2 + \alpha_1 \alpha_2}{x - \alpha_3}.$$

On computing  $T_1 T_2(x) = T_1(T_2(x))$ , and simplifying (by removing a common factor of  $\alpha_1 - \alpha_2$  from the numerator and denominator), we get the formula for  $T_3$ ; that is:  $T_1 T_2 = T_3$ . The formula for  $T_3$  is invariant under  $\alpha_1 \leftrightarrow \alpha_2$  and so  $T_2 T_1 = T_3$ , also. Hence  $T_1 T_2 = T_3 = T_2 T_1$ , and so  $T_1$  and  $T_2$  commute. By symmetry,  $T_1$  and  $T_3$  commute, as do  $T_2$  and  $T_3$ . Furthermore,  $T_1 T_2 T_3 = T_3^2$  (since  $T_1 T_2 = T_3$ ), and we know from part (a) that  $T_3^2 : x \mapsto x$ ; hence  $T_1 T_2 T_3 : x \mapsto x$ , as required.

Finally, the fixed points of  $T_1$  are the solutions to  $T_1(x) = x$ , that is:

$$\alpha_1 x - \alpha_1 \alpha_2 - \alpha_1 \alpha_3 + \alpha_2 \alpha_3 = (x - \alpha_1)x,$$

which has solutions:

$$x_1 = \alpha_1 + \sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}, \quad x_2 = \alpha_1 - \sqrt{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)}.$$

Computing  $T_2(x_1)$  and  $T_2(x_2)$  gives  $x_2, x_1$ , respectively.

4. (a). Let  $L$  be the tangent line to the curve at  $P$ , and let  $R$  be the third point of intersection; that is, the line and the curve meet at  $P, P, R$ . Then  $P + P + R = \mathbf{o}$ . Then,  $3P = \mathbf{o} \iff R = P \iff L$  intersects  $\mathcal{E}$  only at  $P$  (3 times).

(b). The Hessian matrix is:

$$\begin{pmatrix} -6X_0 & 0 & -2AX_2 \\ 0 & 2X_2 & 2X_1 \\ -2AX_2 & 2X_1 & -2AX_0 - 6BX_2 \end{pmatrix}.$$

The determinant is:

$$\begin{aligned} & -6X_0(2X_2(-2AX_0 - 6BX_2) - (2X_1)(2X_1)) - 2AX_2(0 - (2X_2)(-2AX_2)) \\ & = 8(3AX_0^2X_2 + 9BX_0X_2^2 + 3X_0X_1^2 - A^2X_2^3). \end{aligned}$$

The only projective point  $(X_0, X_1, X_2)$  on the curve with  $X_2 = 0$  is  $(0, 1, 0) = \mathbf{o}$ , for which the statement is true, since  $3\mathbf{o} = \mathbf{o}$  and  $(X_0, X_1, X_2) = (0, 1, 0)$  makes the Hessian determinant 0. When  $X_2 \neq 0$ , we can write everything in affine form, with  $x = X_0/X_2$  and  $y = X_1/X_2$ , when we see that the Hessian determinant (after dividing through by  $8X_2^3$ ) is 0 exactly when:

$$3Ax^2 + 9Bx + 3xy^2 - A^2 = 0. \quad (1)$$

Also, the point  $P = (x, y) \neq \mathbf{o}$ , written in affine form, has order 3 exactly when  $2(x, y) = (x, -y)$  which happens if and only if the  $x$ -coordinate of  $2(x, y)$  is  $x$ . But, as usual, the  $x$ -coordinate of  $2(x, y)$  is  $m^2 - 2x$ , where  $m = (3x^2 + A)/(2y)$  (note that  $y \neq 0$  here, since  $y = 0$  would make  $P$  be of order 2). So,  $P$  being of order 3 implies  $((3x^2 + A)/(2y))^2 - 2x = x$ , that is:

$$-9x^4 - 6Ax^2 + 12xy^2 - A^2 = 0. \quad (2)$$

Finally, we use the fact the  $(x, y)$  is a point on the curve, so that  $y^2 = x^3 + Ax + B$ , and so we can replace  $y^2$  by  $x^3 + Ax + B$  in equations (1),(2). This makes both equations become the same equation:  $3x^4 + 6Ax^2 + 12Bx - A^2 = 0$ , as required.

**(c).** By part (b), the  $x$ -coordinate of any point of order 3 must be a root of the quartic  $3x^4 + 6Ax^2 + 12Bx - A^2$ , which has at most 4 roots  $x_1, \dots, x_4$ . Each  $x_i$  gives rise to at most two points  $(x_i, y_i), (x_i, -y_i)$  on the curve, giving at most 8 points of order 3. Together with  $\mathbf{o}$ , this gives at most 9 points that are 3-torsion.

## Elliptic Curves. Solutions to Sheet 3.

**1. (a).** Let  $x, y \in K$  be such that  $|x| \neq |y|$ , without loss of generality, say that  $|x| < |y|$ . Since  $K$  is non-Archimedean, we know that  $|x \pm y| \leq \max(|x|, |y|) = |y|$ , and so it is sufficient to show that  $|x \pm y| \not\leq |y|$ . Imagine  $|x \pm y| < |y|$ ; then  $|y| = |(x \pm y) - x| \leq \max(|x \pm y|, |x|) < |y|$ , a contradiction, as required. [Note that an immediate consequence is the implication:  $|u \pm v| < |u| \Rightarrow |u| = |v|$ ].

**(b).** Let  $x_1, \dots, x_n \in K$  be such that  $|x_\ell| > |x_i|$  for all  $i \neq \ell$ . Let  $x = x_1 + \dots + x_{\ell-1} + x_{\ell+1} + \dots + x_n$  and let  $y = x_\ell$ . Then  $|x| \leq \max(|x_i| : 1 \leq i \leq n, i \neq \ell) < |y|$  and so:  $|x_1 + \dots + x_n| = |x + y| = |y| = |x_\ell|$ , by part (a).

**(c).** Since  $K, |\cdot|$  satisfies the triangle inequality  $|x+y| \leq |x|+|y|$ , we can apply the standard trick from first year Analysis:  $|x| \leq |(x-y) + y| \leq |x-y| + |y|$  to give:  $|x| - |y| \leq |x-y|$ ; similarly  $|y| - |x| \leq |x-y|$ , and so:  $||x| - |y|| \leq |x-y|$ . If  $s_n \rightarrow s$  in  $K, |\cdot|$  then  $|s_n - s| \rightarrow 0$  in  $\mathbb{R}$ , and so  $||s_n| - |s|| \leq |s_n - s| \rightarrow 0$ , giving  $|s_n| \rightarrow |s|$  in  $\mathbb{R}, |\cdot|_\infty$ , as required.

Suppose  $s \neq 0$ ; then  $|s| > 0$  and taking  $\epsilon = |s|$ , there exists  $N$  such that  $|s_n - s| < |s|$  for all  $n > N$ , so that  $|s_n| = |s|$  for all  $n > N$  (since if  $|s_n| \neq |s|$  then (a) would give  $|s_n - s| = \max(|s_n|, |s|) \geq |s|$ , a contradiction).

**2. (a).**  $3/50 = 5^{-2} \cdot 3/2$  (where 5 has no common factor with either the numerator and denominator of  $3/2$ ) and so  $|3/50|_5 = 5^2$ . Similarly,  $3/50 = 3^1 \cdot 1/50$  and so  $|3/50|_3 = 3^{-1}$ . Similarly,  $3/50 = 7^0 \cdot 3/50$ , and so  $|3/50|_7 = 7^0 = 1$ .

$d_5(2/3, 1/5) = |2/3 - 1/5|_5 = |7/15|_5 = 5$ ,  $d_7(2/3, 1/5) = |7/15|_7 = 7^{-1}$ ,  $d_{11}(2/3, 1/5) = |7/15|_{11} = 1$ .

**(b).**  $|3/7|_3 = 3^{-1}$ ,  $|3/7|_7 = 7$ . For all other primes  $p$ ,  $\gcd(p, 3) = \gcd(p, 7) = 1$  and so  $|3/7|_p = 1$ . Note also that  $|3/7|_\infty = 3/7$ . The given product  $\prod |3/7|_i$  has all factors equal to 1 apart from the factors:  $|3/7|_3 = 3^{-1}$ ,  $|3/7|_7 = 7$  and  $|3/7|_3 = 3/7$ , whose product is 1. Hence the given product  $\prod |3/7|_i$  is 1. For any  $x \in \mathbb{Q}$ , write  $x = \pm n/d$ , where  $n$  and  $d$  are integers with  $\gcd(n, d) = 1$ , and write  $n, d$  in terms of their primes factorisations:  $n = p_1^{s_1} \dots p_k^{s_k}$  and  $d = q_1^{t_1} \dots q_\ell^{t_\ell}$ , where  $p_1, \dots, p_k, q_1, \dots, q_\ell$  are distinct primes. For any prime  $p$ , if  $p = p_i$  for some  $i$ , then  $|x|_p = p_i^{-s_i}$ . If  $p = q_j$  for some  $j$ , then  $|x|_p = q_j^{t_j}$ . If  $p = \infty$ , then  $|x|_p = n/d = (p_1^{s_1} \dots p_k^{s_k}) / (q_1^{t_1} \dots q_\ell^{t_\ell})$ . For all other primes  $p$ , we have  $|x|_p = 1$ . It follows that the product  $\prod |x|_i$  is equal to  $(p_1^{-s_1} \dots p_k^{-s_k}) \cdot q_1^{t_1} \dots q_\ell^{t_\ell} \cdot (p_1^{s_1} \dots p_k^{s_k}) / (q_1^{t_1} \dots q_\ell^{t_\ell})$ , which is again equal to 1.

**3. (a).**  $|1/5^n|_5 = 5^n \rightarrow \infty$ ; this means that  $|1/5^n|_5$  does not converge, and so  $1/5^n$  does not converge in  $\mathbb{Q}_5$  (since, if  $a_n \rightarrow \ell$  then  $|a_n|_p \rightarrow |\ell|_p$ ).

**(b).** *Method 1.*  $|n|_5 \leq 5^{-1}$  for  $n = 5, 10, 15, \dots$  and  $|n|_5 = 1$  otherwise; this means that  $|n|_5$  does not converge (since there is a subsequence  $|n|_5$  for  $5 \nmid n$  converging to 1 and a subsequence  $|n|_5$  for  $5 \mid n$  and  $5^2 \nmid n$  converging to  $1/5$ ), and so  $n$  does not converge in  $\mathbb{Q}_5$  (since, if  $a_n \rightarrow \ell$  then  $|a_n|_p \rightarrow |\ell|_p$ ).

**(b).** *Method 2.*  $n = \sum 1$ , which does not converge, since  $|1|_5 = 1$  which does not converge to 0 (using the result from lectures which says that  $\sum c_i$  converges iff  $|c_i|_p \rightarrow 0$ ).

(c).  $n! = 5^r \cdot k$ , where  $r > n/5 - 1$ , since every fifth factor of  $1 \cdot 2 \cdot \dots \cdot n$  (i.e. the factors 5, 10, 15, ...) contributes at least one new factor of 5. Hence  $|n! - 0|_5 = |n!|_5 < 5^{-(n/5-1)} \rightarrow 0$ , and so  $n!$  converges to 0 in  $\mathbb{Q}_5$ .

(d).  $|(3 + 10^n) - 3|_5 = |10^n|_5 = |5^n \cdot 2^n|_5 = 5^{-n} \rightarrow 0$ , and so  $3 + 10^n$  converges to 3 in  $\mathbb{Q}_5$ .

(e).  $|10^n|_5 = |5^n \cdot 2^n|_5 = 5^{-n} \rightarrow 0$  in  $\mathbb{Q}_5$  and so  $\sum 10^n$  converges in  $\mathbb{Q}_5$  (using the result from lectures which says that  $\sum c_i$  converges iff  $|c_i|_p \rightarrow 0$ ).

(f).  $|7^n|_5 = 1$  which does not converge to 0, and so  $\sum 7^n$  does not converge in  $\mathbb{Q}_5$  (using the same result from lectures as used in part (e)).

4. (a). We are looking for  $x$  such that  $|x^2 + 1|_5 \leq 5^{-4}$ . Take  $x_0 = a_0 = 2$  as the initial approximation for which  $|a_0^2 + 1|_5 = |5|_5 = 5^{-1}$ . Take  $x_1 = a_0 + 5a_1 = 2 + 5a_1$ . Then we want  $(2 + 5a_1)^2 \equiv -1 \pmod{5^2}$ , and so  $20a_1 \equiv -5 \pmod{5^2}$ , which is the same as  $4a_1 \equiv -1 \pmod{5}$ , and so  $a_1 = 1$ . This gives a better approximation:  $x_1 = 2 + 1 \cdot 5 = 7$ , which satisfies  $|x_1^2 + 1|_5 = 5^{-2}$ . Similarly, we then find  $a_2 = 2$  and so  $x_2 = 7 + 2 \cdot 5^2 = 57$ . Then, finally, we similarly find  $a_3 = 1$  and so  $x_3 = 57 + 1 \cdot 5^3 = 182$ , satisfying  $|x^2 + 1|_5 = 5^{-4}$ , as required.

(b). If there were an integer  $x$  such that  $|x^2 - 7/8|_3 \leq 3^{-7}$ , then we would have  $x^2 \equiv 7/8 \pmod{3^7}$  and so  $x^2 \equiv 7/8 \pmod{3}$ . But  $7/8 \equiv 2 \pmod{3}$ , which is not a quadratic residue (since  $0^2 \equiv 0, 1^2 \equiv 1, 2^2 \equiv 1 \pmod{3}$ ); this means that no such  $x$  can exist.

(c). If there were an integer  $x$  such that  $|x^2 - 5/4|_5 \leq 5^{-4}$  then consider the possibilities for  $|x^2|_5$ . If  $|x^2|_5 > |5/4|_5$  then (using the result that  $|a + b|_p = \max(|a|_p, |b|_p)$  when  $|a|_p \neq |b|_p$ ) we have  $|x^2 - 5/4|_5 = \max(|x^2|_5, |5/4|_5) = |x^2|_5 > |5/4|_5 = 5^{-1} > 5^{-4}$ , contradicting  $|x^2 - 5/4|_5 \leq 5^{-4}$ . If  $|x^2|_5 < |5/4|_5$  then (using the same result) we have  $|x^2 - 5/4|_5 = \max(|x^2|_5, |5/4|_5) = |5/4|_5 = 5^{-1} > 5^{-4}$ , again contradicting  $|x^2 - 5/4|_5 \leq 5^{-4}$ . It must then be that  $|x^2|_5 = |5/4|_5 = 5^{-1}$ , but this is again impossible since  $|x^2|_5 = 5^{-r}$  where  $r$  is an even integer. Hence no such  $x$  exists.

5.  $200 \pmod{7}$  is 4, so write:  $200 = 4 + 7 \cdot 28 = (\text{similarly}) 4 + 7 \cdot (0 + 7 \cdot 4) = 4 + 0 \cdot 7^1 + 4 \cdot 7^2 = 4, 04$ .

For  $3/14$ , it is easier first to write:  $3/14 = 7^{-1} \cdot 3/2$ . First find the 7-adic expansion of  $3/2$ , with the idea that the 7-adic expansion of  $3/14$  will then just be the 7-adic expansion of  $3/2$  shifted one place to the left). Now,  $|3/2|_7 = 1$  and so  $3/2 = a_0 + a_1 7 + a_2 7^2 + \dots$ . Using  $2(a_0 + 7a_1 + 7^2 a_2 + \dots) = 3$ , we first find  $2a_0 \equiv 3 \pmod{7}$  and so  $a_0 = 5$ . Continuing as usual, we find that  $a_1 = 3, a_2 = 3, a_3 = 3, \dots$ . At this point, we suspect that  $3/2 = 5, \bar{3}$ . We can prove this rigorously as follows. Let  $\alpha = 5, \bar{3}$ . Then  $\alpha - 5 = 0, \bar{3}$ , and so  $7^{-1}(\alpha - 5) - 3 = 3, \bar{3} - 3 = 0, \bar{3} = \alpha - 5$ . Hence,  $\alpha - 5 - 21 = 7\alpha - 35$  and so  $\alpha = 3/2$ , proving that  $3/2 = 5, \bar{3}$  in  $\mathbb{Q}_7$ . Finally,  $3/14 = 7^{-1} \cdot 3/2 = 53, \bar{3}$ .

Let  $\beta = 2, \overline{34} \in \mathbb{Q}_5$ . Then  $\beta - 2 = 0, \overline{34}$  and so  $5^{-2}(\beta - 2) = 34, \overline{34}$ , giving:  $5^{-2}(\beta - 2) - 3 \cdot 5^{-1} - 4 \cdot 5^0 = 0, \overline{34} = \beta - 2$ , and so:  $\beta = -67/24$ .

6. Let  $x \in \mathbb{Q}$ . Write  $x = \pm n/d$ , where  $n$  and  $d$  are integers with  $\gcd(n, d) = 1$ , and write  $n, d$  in terms of their primes factorisations:  $n = p_1^{s_1} \cdot \dots \cdot p_k^{s_k}$  and  $d = q_1^{t_1} \cdot \dots \cdot q_\ell^{t_\ell}$ , where  $p_1, \dots, p_k, q_1, \dots, q_\ell$  are distinct primes. If  $x \in \mathbb{Z}$  then  $\ell = 0$  (i.e. there are no occurrences of  $q_j$ ) and we see that when  $p = p_i$  for some  $i$ ,  $|x|_p = p_i^{-s_i} \leq 1$ , and otherwise  $|x|_p = 1$ ; hence  $|x|_p \leq 1$  for all  $p$ ; that is to say:  $x \in \mathbb{Z}_p$  for all  $p$ .

Conversely, assume that  $x \in \mathbb{Z}_p$  for all  $p$ . Then there could not be any occurrence of  $q_j$  [since then at  $p = q_j$  we would have  $|x|_p = q_j^{t_j} > 1$ , contradicting  $x \in \mathbb{Z}_p$ . Hence,  $x$  has no denominator and is in  $\mathbb{Z}$ .

**7.** The number of positive multiples of an integer  $k > 0$  which are  $\leq n$  is clearly  $\left[\frac{n}{k}\right]$ . To count the power of  $p$  dividing  $n!$ , since  $p$  is prime, it is enough to count the powers of  $p$  dividing  $1, 2, 3, \dots, n$  and add these powers up. Now, the number of multiples of  $p$  among  $1, 2, 3, \dots, n$  is  $\left[\frac{n}{p}\right]$ . Each multiple of  $p^2$  among  $1, 2, 3, \dots, n$  gives an additional power of  $p$  dividing into  $n!$ , giving  $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right]$  so far. Continuing in this way we get that the total power of  $p$  is as in the given formula. This gives that  $|n!|_p = p^{-M}$ , where  $M = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \dots \leq \frac{n}{p} + \frac{n}{p^2} + \dots = \frac{n}{p-1}$ . Hence  $|n!| \geq p^{-\frac{n}{p-1}}$ , and so  $|1/n!| \leq p^{\frac{n}{p-1}}$ . Suppose that  $|x| < p^{-\frac{1}{p-1}}$  and let  $\tau = |x|/p^{-\frac{1}{p-1}} < 1$ . Then  $\left|\frac{x^n}{n!}\right| = \tau^n p^{-\frac{n}{p-1}} \left|\frac{1}{n!}\right| \leq \tau^n p^{-\frac{n}{p-1}} p^{\frac{n}{p-1}} = \tau^n \rightarrow 0$ . Hence,  $\exp_p(x)$  converges (using the result from lectures which says that  $\sum c_i$  converges iff  $|c_i|_p \rightarrow 0$ ).

On the other hand, if  $|x| \geq p^{-\frac{1}{p-1}}$ , note that, for any  $n = p^\ell$ , the above  $\frac{n}{p}, \frac{n}{p^2}, \dots, \frac{n}{p^\ell} = p^{\ell-1}, p^{\ell-2}, \dots, 1 \in \mathbb{Z}$  and so  $M = p^{\ell-1} + p^{\ell-2} + \dots + 1 = \frac{p^\ell - 1}{p-1}$ ; this means that the subsequence  $\left|\frac{x^{p^\ell}}{(p^\ell)!}\right| \geq (p^{-\frac{1}{p-1}})^{p^\ell} p^{\frac{p^\ell - 1}{p-1}} = p^{\frac{-1}{p-1}}$ , and so  $\left|\frac{x^n}{n!}\right| \not\rightarrow 0$ . Hence  $\exp_p(x)$  does not converge when  $|x| \geq p^{-\frac{1}{p-1}}$ .

When  $K = \mathbb{Q}_p$  ( $p \neq 2$ ), any  $|x|_p < 1$  must satisfy  $|x|_p \leq p^{-1}$  [since  $|x|_p = p^r$  for some  $r \in \mathbb{Z}$ ]  $< p^{-\frac{1}{p-1}}$ , since  $p \geq 3$ . Any  $|x|_p \geq 1$  satisfies  $|x| \geq p^{-\frac{1}{p-1}}$ . When  $K = \mathbb{Q}_2$ , any  $|x|_2 < 1/2$  must satisfy  $|x|_2 \leq 2^{-2}$  [since  $|x|_p = p^r$  for some  $r \in \mathbb{Z}$ ]  $< p^{-\frac{1}{p-1}} = 2^{-1}$ . Any  $|x|_p \geq 1/2$  satisfies  $|x| \geq p^{-\frac{1}{p-1}} = 2^{-1}$ .

## Elliptic Curves. Solutions to Sheet 4.

**1.** First note that  $-28 = 2^2 \cdot (-7)$ . In  $\mathbb{Q}_2$ , we know that any integer congruent to 1 mod 8 is a square, and so certainly  $-7$  is a square; hence so is  $-28 = 2^2 \cdot (-7)$ .

In  $\mathbb{Q}_3$ , we observe that  $|-28|_3 = 1$  and  $-28 \equiv 2 \pmod{3}$  which is not a quadratic residue mod 3; hence  $-28$  is not a square in  $\mathbb{Q}_3$ . In  $\mathbb{Q}_5$ , we observe that  $|-28|_5 = 1$  and  $-28 \equiv 2 \pmod{5}$  which is not a quadratic residue mod 5; hence  $-28$  is not a square in  $\mathbb{Q}_5$ . In  $\mathbb{Q}_7$ , if there were an  $x$  such that  $x^2 = -28$ , then  $|x^2|_7 = |-28|_7 = 7^{-1}$ , contradicting the fact that  $|x^2|_7 = 7^r$  where  $r$  is an even integer; hence  $-28$  is not a square in  $\mathbb{Q}_7$ . In  $\mathbb{Q}_{11}$ , we observe that  $|-28|_{11} = 1$  and  $-28 \equiv 5 \pmod{11}$  which is a quadratic residue mod 11 since  $5 \equiv 4^2 \pmod{11}$ ; hence  $-28$  is a square in  $\mathbb{Q}_{11}$ . [The above repeatedly used the result that, when  $p \neq 2$  and when  $|a|_p = 1$  then  $a$  is a square in  $\mathbb{Q}_p$  iff it is a square mod  $p$ .]

**2.** In  $\mathbb{R}$ , we have for example the root  $\sqrt{2}$ . In  $\mathbb{Q}_2$ , note that  $17 \equiv 1 \pmod{8}$ , and so 17 is a square in  $\mathbb{Q}_2$ . In  $\mathbb{Q}_{17}$ , note that  $|2|_{17} = 1$  and 2 is a quadratic residue mod 17, so that 2 is a square in  $\mathbb{Q}_{17}$ .

Finally, let  $p \neq 2, 17$ . Then each of  $(\frac{2}{p}), (\frac{17}{p}), (\frac{34}{p})$  is 1 or  $-1$ . They cannot all be  $-1$  since  $(\frac{34}{p}) = (\frac{2}{p})(\frac{17}{p})$  [and  $-1 \neq (-1)(-1)$ ], so at least one of them must be 1. Wlog, say that  $(\frac{2}{p}) = 1$ . Then  $|2|_p = 1$  and 2 is a quadratic residue mod  $p$ , so that 2 must be a square in  $\mathbb{Q}_p$ .

**3.** Suppose there were an  $x \in \mathbb{Q}_3$  such that  $x^3 = 4$  in  $\mathbb{Q}_3$ . Then  $|x|_3^3 = |4|_3 = 1$  and so  $|x|_3 = 1$ , which means that  $x$  can be written  $x = a_0 + a_1a_2 \dots$ . Reducing  $x$  modulo 9 would then give the integer  $a_0 + 3a_1$  whose cube is 4 modulo 9. But  $0, \dots, 8$  square to give: 0, 1, 8, 0, 1, 8, 0, 1, 8, respectively, so that  $x^3 = 4$  is an impossible congruence mod 9. Hence 4 is not a cube in  $\mathbb{Q}_3$ .

Let  $f(x) = x^3 - 28$  and let  $x_0 = 1$ . Then  $|f(x_0)|_3 = |-27|_3 = 3^{-3}$ , whereas  $|f'(x_0)|_3 = |3|_3 = 3^{-1}$ . Hence  $|f(x_0)|_3 < |f'(x_0)|_3^2$ , and so by Hensel's Lemma there must be a root of  $f(x)$  in  $\mathbb{Q}_3$ ; that is 28 must be a cube in  $\mathbb{Q}_3$ .

Let  $f(x) = x^3 - 13$  and let  $x_0 = -1$ . Then  $|f(x_0)|_7 = |-14|_7 = 7^{-1}$ , whereas  $|f'(x_0)|_7 = |3|_7 = 1$ . Hence  $|f(x_0)|_7 < |f'(x_0)|_7^2$ , and so by Hensel's Lemma again there must be a root of  $f(x)$  in  $\mathbb{Q}_7$ ; that is 13 must be a cube in  $\mathbb{Q}_7$ .

**4.** Let  $\mathcal{C}$  be the curve  $2Y^2 = X^4 - 17$ . Over  $\mathbb{R}$ , the curve has the point  $(4, \sqrt{239/2})$ . In  $\mathbb{Q}_2$ , let  $x = 11$ . Then  $11^4 - 17 = 2^5 \cdot 457$ ; but 457 is an integer congruent to 1 mod 8, and so, from lectures, there exists  $\gamma \in \mathbb{Z}_2 \subset \mathbb{Q}_2$  such that  $457 = \gamma^2$ ; then  $(11, 4\gamma)$  is a point on the curve over  $\mathbb{Q}_2$ . For the next few primes, we shall repeatedly use the result from lectures that, if  $p \neq 2$  and  $|a|_p = 1$ , then  $a$  is a square in  $\mathbb{Q}_p$  iff it is a square mod  $p$ . Now, note that  $-8, -34, -34, -8, 10, -8$  are quadratic residues modulo  $p = 3, 5, 7, 11, 13, 17$ , respectively, and so there exists  $\gamma \in \mathbb{Q}_p$  such that  $\gamma^2 = -8, -34, -34, -8, 10, -8$ , respectively; this gives the  $\mathbb{Q}_p$ -rational points on the curve:  $(1, \gamma), (0, \gamma/2), (0, \gamma/2), (1, \gamma), (4, \gamma/2), (1, \gamma)$ , respectively. Hence  $\mathcal{C}$  has  $\mathbb{Q}_p$ -rational points for all primes up to and including  $p = 17$  [in fact, it was only necessary to do here  $p = 2, 3, 5, 7, 11, 17$  here].

Let  $p$  be any prime  $p \geq 13$  and  $p \neq 17$ . Our curve  $\mathcal{C}$  (after multiplying both sides by 2) is birationally equivalent to  $V^2 = 2X^4 - 34$ , where  $V = 2Y$ ; note that  $2X^4 - 34$  has discriminant  $-2^{11}17^3$ . Recall Theorem 1.15 from lectures, that any curve  $y^2 = Q(x)$ , where  $Q(x) = f_4x^4 + \dots + f_0$  has nonzero discriminant over  $\mathbb{F}_p$ , has at least  $p-1-2\sqrt{p} > 4$  affine points over  $\mathbb{F}_p$ , and so at least 5 affine points. At most 4 of these can have  $2x^4 - 34 \equiv 0 \pmod{p}$ , and so there exists  $x_0 \in \mathbb{Z}$  such that  $2(x_0^4 - 17)$  is a nonzero quadratic residue mod  $p$ . As above, there exists  $\gamma \in \mathbb{Q}_p$  such that  $\gamma^2 = 2(x_0^4 - 17)$ , and so  $(x_0, \gamma/2)$  is a  $\mathbb{Q}_p$ -rational point on our original curve  $\mathcal{C}$ . Hence  $\mathcal{C}$  has points in  $\mathbb{R}$  and every  $\mathbb{Q}_p$ .

*Alternative method for the above parts:* Note that for  $p \nmid 2\Delta$  (that is:  $p \neq 2, 17$ , so that  $\tilde{\mathcal{C}}$  is still an elliptic curve mod  $p$ ) and  $p \geq 5$ , we have  $\#\tilde{\mathcal{C}}(\mathbb{F}_p) \geq p + 1 - 2\sqrt{p} > 1$ , and so the number of affine points is  $> 0$ ; these are non-singular, since  $\tilde{\mathcal{C}}$  is still an elliptic curve mod  $p$ . Also, by a theorem from lectures, any non-singular point on  $\tilde{\mathcal{C}}(\mathbb{F}_p)$  lifts to a point on  $\mathcal{C}(\mathbb{Q}_p)$ ; also, any affine non-singular point on  $\tilde{\mathcal{C}}(\mathbb{F}_p)$  lifts to an affine point on  $\mathcal{C}(\mathbb{Q}_p)$  (since the point at infinity on  $\mathcal{C}(\mathbb{Q}_p)$  maps to the point at infinity on  $\tilde{\mathcal{C}}(\mathbb{F}_p)$  under the reduction map mod  $p$ ). This shows the existence of such  $x, y \in \mathbb{Q}_p$  for all  $p$  except  $p = 2, 3$  and bad primes, so only still need to consider  $p = 2, 3, 17$ . Also, note that for  $p = 3$  there is the non-singular affine point  $(1, 1)$  on  $\tilde{\mathcal{C}}(\mathbb{F}_3)$ , and for  $p = 17$  there is the non-singular affine point  $(1, 3)$  on  $\tilde{\mathcal{C}}(\mathbb{F}_{17})$ , and as before these must lift to affine points  $(x, y) \in \mathcal{C}(\mathbb{Q}_p)$ . Over  $\mathbb{R}$ , the curve has the point  $(4, \sqrt{239/2})$ . In  $\mathbb{Q}_2$ , let  $x = 11$ . Then  $11^4 - 17 = 2^5 \cdot 457$ ; but 457 is an integer congruent to 1 mod 8, and so, from lectures, there exists  $\gamma \in \mathbb{Z}_2 \subset \mathbb{Q}_2$  such that  $457 = \gamma^2$ ; then  $(11, 4\gamma)$  is a point on the curve over  $\mathbb{Q}_2$ . This completes the alternative method of showing that  $\mathcal{C}$  has points in  $\mathbb{R}$  and every  $\mathbb{Q}_p$ .

Now, imagine that  $\mathcal{C}$  had a  $\mathbb{Q}$ -rational point; that is, imagine that there exist  $X, Y \in \mathbb{Q}$  such that  $2Y^2 = X^4 - 17$ . Let  $X = t/r$ , where  $r, t \in \mathbb{Z}$  and  $\gcd(r, t) = 1$ . Then  $2(r^2Y)^2 = t^4 - 17r^4 \in \mathbb{Z}$ . For any prime  $p$ , this gives that  $|2|_p |r^2Y|_p^2 \leq 1$ . When  $p \neq 2$ , we have  $|2|_p = 1$  and so  $|r^2Y|_p^2 \leq 1$  and so  $|r^2Y|_p \leq 1$ . When  $p = 2$ , we have  $2^{-1}|r^2Y|_2^2 \leq 1$ ; but this still gives  $|r^2Y|_2 \leq 1$  [since  $|r^2Y|_2 > 1$  would mean  $|r^2Y|_2 \geq 2^1$  and so  $2^{-1}|r^2Y|_2^2 \geq 2^{-1}2^2 > 1$ ]. Overall, we have shown that  $|r^2Y|_p \leq 1$  for all  $p$ ; since also  $r^2Y \in \mathbb{Q}$ , this gives that  $r^2Y \in \mathbb{Z}$ ; let us say:  $s = r^2Y \in \mathbb{Z}$ . Therefore  $r, s, t \in \mathbb{Z}$  satisfy  $2s^2 = t^4 - 17r^4$  and  $\gcd(r, t) = 1$ .

Let  $q$  be any prime such that  $q|s$ . Note that  $q \neq 17$  [since if  $17|s$  then our equation  $2s^2 = t^4 - 17r^4$  would force  $17|t$ , and so  $17^2|2s^2$  and  $17^2|t^4$ , which would give  $17^2|17r^4$ , forcing  $17|r$ , which would contradict  $\gcd(r, t) = 1$ ]. Note also that we must not then have  $q|r$  [since if  $q|s$  and  $q|r$ , then our equation  $2s^2 = t^4 - 17r^4$  would force  $q|t$ , which would contradict  $\gcd(r, t) = 1$ ]. Reducing our equation modulo  $q$  gives that  $0 \equiv t^4 - 17r^4$  and so  $(t^2/r^2)^2 \equiv 17 \pmod{q}$  [note that division by  $r^2$  is allowable mod  $q$  since  $r$  is not divisible by  $q$ ]. Hence 17 is a nonzero quadratic residue mod  $q$ . For  $q \neq 2$ , quadratic reciprocity then allows us to deduce that  $q$  is a quadratic residue mod 17 [since  $17 \equiv 1 \pmod{4}$ ]. Furthermore, one can check directly that 2 is a quadratic residue mod 17, since  $2 \equiv 6^2 \pmod{17}$ . Overall, we have shown that every prime  $q$  dividing  $s$  must be a quadratic residue mod 17, and we can take  $s > 0$  (if necessary, replacing  $s$  by  $-s$ ), so that  $s$  is the product of its prime factors. It follows that  $s$  itself must be a quadratic residue mod 17, since it is a product of quadratic residues mod 17. Hence

$s^2$  is a nonzero fourth power [also known as a *quartic residue*] mod 17. Note also that reducing our equation mod 17 gives  $2s^2 \equiv t^4 \pmod{17}$ , so that  $2s^2$  is also a nonzero fourth power mod 17. Since  $s, 2s^2$  are both nonzero fourth powers mod 17, it follows that  $2 = (2s^2)/s^2$  is also a fourth power mod 17. On the other hand, one can compute directly that  $0^4, 1^4, 2^4, 3^4, 4^4, 5^4, 6^4, 7^4, 8^4, 9^4, 10^4, 11^4, 12^4, 13^4, 14^4, 15^4, 16^4$  are congruent mod 17 to: 0, 1, 16, 13, 1, 13, 4, 4, 16, 16, 4, 4, 13, 1, 13, 16, 1, respectively, so that in fact 2 is not a fourth power mod 17. This contradiction shows that our original curve  $\mathcal{C}$  has no  $\mathbb{Q}$ -rational points.

**5.** Since  $p \equiv 2 \pmod{3}$ , we have  $\gcd(3, p-1) = 1$ , and so there exist  $\lambda, \mu \in \mathbb{Z}$  such that  $3\lambda + (p-1)\mu = 1$ . For any  $x, y \in \mathbb{F}_p^*$ , we have  $x^{p-1} = y^{p-1} = 1$  [by Fermat's Little Theorem], and so:

$$\begin{aligned} x^3 = y^3 &\Rightarrow (x^3)^\lambda \cdot 1^\mu = (y^3)^\lambda \cdot 1^\mu \Rightarrow (x^3)^\lambda \cdot (x^{p-1})^\mu = (y^3)^\lambda \cdot (y^{p-1})^\mu \\ &\Rightarrow x^{3\lambda+(p-1)\mu} = y^{3\lambda+(p-1)\mu} \Rightarrow x = y. \end{aligned}$$

Thus the map  $x \mapsto x^3$  is injective on  $\mathbb{F}_p^*$  and so is a bijection.

Now, let  $a \in \mathbb{Z}$  be such that  $p \nmid a$ . From above, there must exist  $x_0 \in \mathbb{Z}$  such that  $x_0^3 \equiv a \pmod{p}$ ; clearly  $p \nmid x_0$ . Let  $f(x) = x^3 - a$ . Then  $|f(x_0)|_p = |x_0^3 - a|_p < 1$ , since  $x_0^3 - a \equiv 0 \pmod{p}$ . Also,  $|f'(x_0)|_p = |3x_0^2|_p = 1$ , since  $|3|_p = 1$  and  $p \nmid x_0$ . Hence  $|f(x_0)|_p < |f'(x_0)|_p^2$ , and so by Hensel's Lemma, there exists  $x \in \mathbb{Z}_p$  with  $f(x) = 0$ , that is,  $x^3 = a$ , as required.

**6.** Recall the standard fact about resultants (stated in Section 0, the preliminary reading), that there exist polynomials  $p(x), q(x) \in R[x]$  such that:  $p(x)f(x) + q(x)g(x) = \text{Res}(f(x), g(x))$ . Since the discriminant  $D$  is the resultant of  $f(x)$  and  $f'(x)$ , there must exist polynomials  $p(x), q(x) \in R[x]$  such that  $p(x)f(x) + q(x)f'(x) = D$ , and so:  $p(a_0)f(a_0) + q(a_0)f'(a_0) = D$ . Since  $|a_0| \leq 1$ , and since  $p(x), q(x), f(x), f'(x) \in R[x]$ , we must have  $|p(a_0)|, |f(a_0)|, |q(a_0)|, |f'(a_0)| \leq 1$  and so  $|D| \leq \max(|p(a_0)f(a_0)|, |q(a_0)f'(a_0)|) \leq 1$ , which also implies  $|D^2| = |D|^2 \leq |D|$ . Now, we are given that  $|f(a_0)| < |D|^2$  and so  $|p(a_0)f(a_0)| \leq |f(a_0)| < |D|^2 \leq |D|$ . Hence  $|D| \neq |p(a_0)f(a_0)|$ , and so  $|q(a_0)f'(a_0)| = |D - p(a_0)f(a_0)| = \max(|D|, |p(a_0)f(a_0)|) = |D|$  [using the fact that, if  $|u| \neq |v|$  then  $|u \pm v| = \max(|u|, |v|)$ ]. Since also  $|q(a_0)| \leq 1$ , this gives that  $|f'(a_0)| \geq |D|$ . Finally,  $|f(a_0)| < |D|^2 \leq |f'(a_0)|^2$ , and so  $f(X)$  has a root  $a \in R$  by Hensel's Lemma.

**7.** Let  $f(X) = 5X^3 - 7X^2 + 3X + 6$ , so that  $f'(X) = 15X^2 - 14X + 3$ . Let  $x_0 = a_0 = 1$ . Then  $|f(x_0)|_7 = |7|_7 = 7^{-1}$  and  $|f'(x_0)|_7 = |4|_7 = 1$ . Then Hensel's Lemma tells us that there exists  $\alpha \in \mathbb{Z}_7$  with  $|\alpha - 1|_7 < 1$ , and the proof tells us to construct a sequence  $x_{n+1} = x_n - f(x_n)/f'(x_n)$  which will converge to  $\alpha$ , with each  $f(x_n) \equiv 0 \pmod{7^{n+1}}$ , so that  $a = x_3$  will satisfy  $|\alpha - a|_7 \leq 7^{-4}$ . Now compute  $x_1 = x_0 - f(x_0)/f'(x_0) = 1 - 7/4 = -3/4 \equiv 36 \pmod{7^2}$ . Then compute  $x_2 = x_1 - f(x_1)/f'(x_1) = 152494/6313 \equiv 36 \pmod{7^3}$ . Finally compute  $x_3 = x_2 - f(x_2)/f'(x_2) = 152494/6313 \equiv 379 \pmod{7^4}$ . As a check, note that  $f(x_3) = f(379) = 271195351 = 7^4 \cdot 112951 \equiv 0 \pmod{7^4}$ , and all  $|f(x_n)|_7 \leq 7^{-4}$  for  $n \geq 3$ . So, for all  $n \geq 3$ , the process  $x_{n+1} = x_n - f(x_n)/f'(x_n)$  will not affect  $x_n \pmod{7^4}$ ; that is,  $x_n \equiv 379 \pmod{7^4}$  [and so  $|x_n - 379|_7 \leq 7^{-4}$ ] for all  $n \geq 3$ . Since  $x_n \rightarrow \alpha$  in  $\mathbb{Z}_7$  it follows that  $|\alpha - 379|_7 \leq 7^{-4}$ , as required.

## Elliptic Curves. Solutions to Sheet 5.

1. We first (as suggested in the hint) show:

**Lemma (\*).** For any  $\alpha = a + b\sqrt{d} \in \mathbb{Q}_p(\sqrt{d})$ ,  $|\alpha|_p \leq 1 \Rightarrow |\alpha + 1|_p \leq 1$ .

**Proof.** Assume  $|\alpha|_p \leq 1$ , so that  $|a^2 - b^2d|_p \leq 1$ . The result is trivial for  $b = 0$ , so we can assume  $b \neq 0$ . Let:

$$f(x) = (x - (a + b\sqrt{d}))(x - (a - b\sqrt{d})) = x^2 - 2ax + a^2 - b^2d.$$

Imagine  $|2a|_p > 1$ . Then  $g(x) = f(x)/(2a) = \frac{x^2}{2a} - x + \frac{a^2 - b^2d}{2a}$  would be defined over  $\mathbb{Z}_p$ , with  $|g(0)|_p = |\frac{a^2 - b^2d}{2a}|_p < 1 = |-1|_p^2 = |g'(0)|_p^2$ , so by Hensel's Lemma, there would exist a root in  $\mathbb{Z}_p$ , contradicting the fact that  $a + b\sqrt{d}, a - b\sqrt{d} \notin \mathbb{Z}_p$  (since  $d$  is non-square and  $b \neq 0$ ).

Hence  $|2a|_p \leq 1$ . So:

$$|\alpha + 1|_p = |a + 1 + b\sqrt{d}|_p = |(a + 1)^2 - b^2d|_p^{1/2} = |(a^2 - b^2d) + 2a + 1|_p^{1/2} \leq 1 \quad (\text{since } |a^2 - b^2d|_p, |2a|_p, |1|_p \leq 1), \text{ completing the proof of Lemma (*).}$$

For any  $\alpha = a + b\sqrt{d} \in \mathbb{Q}_p(\sqrt{d})$ , let  $\bar{\alpha}$  denote  $a - b\sqrt{d}$ ; also, let:

$$\alpha_1 = a_1 + b_1\sqrt{d}, \alpha_2 = a_2 + b_2\sqrt{d} \in \mathbb{Q}_p(\sqrt{d}), \text{ where } a, b, a_1, b_1, a_2, b_2 \in \mathbb{Q}_p.$$

Note that  $|\alpha|_p = |a^2 - b^2d|_p^{1/2} \geq 0$ , and that:

$$|\alpha|_p = 0 \iff a^2 - b^2d = 0 \iff a = b = 0 \iff \alpha = 0 \quad (\text{since } d \text{ is non-square}).$$

$$|\alpha_1\alpha_2|_p = |\alpha_1\alpha_2\bar{\alpha}_1\bar{\alpha}_2|_p^{1/2} = |\alpha_1\bar{\alpha}_1\alpha_2\bar{\alpha}_2|_p^{1/2} = |\alpha_1\bar{\alpha}_1|_p^{1/2}|\alpha_2\bar{\alpha}_2|_p^{1/2} = |\alpha_1|_p|\alpha_2|_p.$$

Also, without loss of generality,  $|\alpha_1|_p \leq |\alpha_2|_p$ , so that  $|\frac{\alpha_1}{\alpha_2}|_p \leq 1$ . Then:

$$|\alpha_1 + \alpha_2|_p = |\frac{\alpha_1}{\alpha_2} + 1|_p|\alpha_2|_p \leq [\text{by Lemma (*)}] |\alpha_2|_p = \max(|\alpha_1|_p, |\alpha_2|_p).$$

Hence,  $|\cdot|_p$  is a non-Archimedean valuation on  $\mathbb{Q}_p(\sqrt{d})$ . Finally, note that when  $\alpha = a$  [with  $b = 0$ ], our definition of  $|\cdot|_p$  on  $\mathbb{Q}_p(\sqrt{d})$  gives  $|\alpha|_p = |a^2 - 0^2d|_p^{1/2} = |a^2|_p^{1/2} = |a|_p$ , which agrees with the usual  $|\cdot|_p$  on  $\mathbb{Q}_p$ , so that our non-Archimedean valuation on  $\mathbb{Q}_p(\sqrt{d})$  extends that on  $\mathbb{Q}_p$ .

2. In projective form, the point is:  $(-64/25, 59/125, 1)$ . The coordinate with largest 5-adic value is  $59/125$ , and on dividing all coordinates through by this, we can also represent the point as:  $(-320/59, 1, 125/59)$ , which is in 5-adic standard form (that is, 1 is the maximum of the 5-adic valuations of the coordinates). Reducing mod 5 gives  $(0, 1, 0)$  on  $\tilde{\mathcal{E}}$ , which is the point at infinity.

3. Suppose that  $(x, y)$  on  $\mathcal{E}$  reduces mod  $p$  to  $(0, 0)$  on  $\tilde{\mathcal{E}}$ . Then  $|x|_p, |y|_p < 1$ , indeed  $|x|_p, |y|_p \leq p^{-1}$  (since any  $p$ -adic value is one of:  $\dots, p^{-2}, p^{-1}, 1, p^1, p^2, \dots$ , so that if a  $p$ -adic value is  $< 1$  then it must be  $\leq p^{-1}$ ), so that  $|x|_p^3 \leq p^{-3}$ . But  $|p|_p = p^{-1}$  and so  $|x|_p^3 \neq |p|_p$  which means that  $|x^3 + p|_p = \max(|x|_p^3, |p|_p) = p^{-1}$ . Since  $x, y$  satisfy  $y^2 = x^3 + p$ , we must therefore have:  $|y|_p^2 = p^{-1}$ , which contradicts the fact that  $|y|_p^2 = p^{2r}$  for some integer  $r$ .

4.

(a). Let  $\mathcal{E} : Y^2 = X^3 + p^2$ . Then the point  $(0, p)$  on  $\mathcal{E}$  reduces modulo  $p$  to the cusp  $(0, 0)$  on  $\tilde{\mathcal{E}} : Y^2 = X^3$ , which is another way of saying that  $(0, 0)$  on  $\tilde{\mathcal{E}}$  lifts to the point  $(0, p)$  in  $\mathcal{E}(\mathbb{Q}_p)$ .

(b). Question 3 is an example of this.

(c). Let  $\mathcal{E} : Y^2 = X^3 + X^2 + p^2$ . Then the point  $(0, p)$  on  $\mathcal{E}$  reduces modulo  $p$  to the node  $(0, 0)$  on  $\tilde{\mathcal{E}} : Y^2 = X^3 + X^2$ , which is another way of saying that  $(0, 0)$  on  $\tilde{\mathcal{E}}$  lifts to the point  $(0, p)$  in  $\mathcal{E}(\mathbb{Q}_p)$ .

(d). Let  $\mathcal{E} : Y^2 = X^3 + X^2 + p$  and  $\tilde{\mathcal{E}} : Y^2 = X^3 + X^2$ . Then the node  $(0, 0)$  on  $\tilde{\mathcal{E}}$  does not lift to any point in  $\mathcal{E}(\mathbb{Q}_p)$  by the same argument as in Question 2.

5. Take  $F(X, Y) = X + Y + tXY^p$ , clearly non-commutative. One only has to check associativity.

$$\begin{aligned} F(F(X, Y), Z) &= F(X + Y + tXY^p, Z) = X + Y + tXY^p + Z + t(X + Y + tXY^p)Z^p \\ &= X + Y + tXY^p + Z + tXZ^p + tYZ^p, \end{aligned}$$

since the remaining term  $t^2XY^pZ^p \in I = t^2\mathbb{F}_p[t]$  and so  $t^2XY^pZ^p = 0$  in  $R = \mathbb{F}_p[t]/I$ . Also:

$$\begin{aligned} F(X, F(Y, Z)) &= X + F(Y, Z) + tXF(Y, Z)^p = X + Y + Z + tYZ^p + tX(Y + Z + tYZ^p)^p \\ &= X + Y + Z + tYZ^p + tX(Y^p + Z^p + (tYZ^p)^p), \end{aligned}$$

since all other terms in the binomial expansion of  $(Y + Z + tYZ^p)^p$  are divisible by  $p$  and so are equal to 0 in  $R$ , which has characteristic  $p$ . But  $(tYZ^p)^p = t^p(t^{p-2}Y^pZ^{p^2}) \in I$  and so  $(tYZ^p)^p = 0$  in  $R$ , so that the above are the same, giving associativity, as required.

6. Recall from lectures that the first step in deriving the formal group of an elliptic curve is to write the equation  $\mathcal{E} : Y^2 = X^3 + AX + B$  as:  $\mathcal{E}' : w = f(z, w) = z^3 + Aw^2z + Bw^3$ , where  $z = -x/y, w = -1/y$ . We then Inductively define  $f_n(z, w)$  by:  $f_1(z, w) = f(z, w)$  and  $f_{m+1}(z, w) = f_m(z, f(z, w))$  and define

$$w(z) = \lim_{m \rightarrow \infty} f_m(z, 0) \in \mathbb{Z}[A, B][[z]],$$

which satisfies  $w(z) = f(z, w(z))$ . The terms of  $f_1(z, w) = z^3 + Aw^2z + Bw^3$  all have weighted degree  $\equiv 3$  [where we give  $z$  weight 1,  $w$  weight 3,  $A$  weight  $-4$  and  $B$  weight  $-6$ ]. Suppose that this is also true of  $f_m(z, w)$ ; then  $f_{m+1}(z, w) = f_m(z, z^3 + Aw^2z + Bw^3)$ , where  $z$  has weighted degree 1,  $z^3 + Aw^2z + Bw^3$  is homogeneous of weighted degree 3, and so the same will be true of  $f_{m+1}$ . So, by induction, all terms of every  $f_m$  and so all terms of  $w(z)$  will have weighted degree 3, and so  $w(z) = \sum c_n z^n$ , where  $c_n = 0$  unless  $n \equiv 3 \pmod{4}$ . We now perform the addition  $(z_1, w_1) + (z_2, w_2)$ . As usual, we first write the line  $z = \lambda w + \mu$  through the points, given by  $\lambda = (w(z_1) - w(z_2))/(z_1 - z_2)$  and  $\mu = (z_1 w(z_2) - z_2 w(z_1))/(z_1 - z_2)$ , both in  $\mathbb{Z}[A, B][[z_1, z_2]]$ . Letting  $z_1$  and  $z_2$  each have weight 1, the numerator  $w(z_1) - w(z_2)$  is homogeneous of weighted degree 3, and so  $\lambda$  is homogeneous of weighted degree 2. Similarly,  $\mu$  is homogeneous of weighted degree 3. As in lectures, substituting  $w = \lambda z + \mu$  into  $\mathcal{E}'$  gives  $\lambda z + \mu = z^3 + A(\lambda z + \mu)^2 z + B(\lambda z + \mu)^3$ , and so:

$$(1 + A\lambda^2 + B\lambda^3)z^3 + (2A\lambda\mu + 3B\lambda^2\mu)z^2 + \dots = 0.$$

Let  $(z_3, w(z_3))$  be the third point of intersection of  $\mathcal{E}'$  and the line  $z = \lambda w + \mu$ , so that  $z_1, z_2, z_3$  are the roots of the above cubic, giving that  $z_1 + z_2 + z_3 = -(\text{coeff of } z^2)/(\text{coeff of } z^3)$ , so:

$$z_3 = -z_1 - z_2 - \frac{2A\lambda\mu + 3B\lambda^2\mu}{1 + A\lambda^2 + B\lambda^3} \in \mathbb{Z}[A, B][[z_1, z_2]],$$

since the denominator is of the form  $1 + \phi(z)$ , where  $\phi(z)$  has no constant term [and so is an invertible power series, with  $1/(1 + \phi(z)) = 1 - \phi(z) + \phi(z)^2 + \dots$ ]. The sum  $(z_1, w_1) + (z_2, w_2) + (z_3, w_3) =$  the identity, and so  $(z_1, w_1) + (z_2, w_2) = -(z_3, w_3)$ . Negation  $(x, y) \mapsto (x, -y)$  induces  $(z, w) \mapsto (-z, -w)$  [since  $z = -x/y, w = -1/y$ ], so that the  $z$ -coordinate of  $(z_1, w_1) + (z_2, w_2)$  is given by  $F_{\mathcal{E}}(z_1, z_2)$ , where:

$$F_{\mathcal{E}}(z_1, z_2) = z_1 + z_2 + (\text{ terms of degree } \geq 2) \in \mathbb{Z}[A, B][[z_1, z_2]].$$

But note that the expression for  $z_3$  above consists of the terms  $z_1, z_2$ , both homogeneous of weighted degree 1, and the fraction whose numerator  $2A\lambda\mu + 3B\lambda^2\mu$  is homogeneous of weighted degree 1, and whose denominator  $1 + A\lambda^2 + B\lambda^3$  is homogeneous of weighted degree 0. Therefore the final power series giving the formal group must be homogeneous of weighted degree 1.

For the case when our curve is of the form  $Y^2 = X^3 + AX$  (so that  $B = 0$ ), each term of the formal group must be an integer multiple of  $A^k z_1^{n_1} z_2^{n_2}$ . Since the weighted degree is 1 and since  $A$  has weight  $-4$ , we see that the degree purely in  $z_1, z_2$  must be  $\equiv 1 \pmod{4}$ , as required.

When our curve is of the form  $Y^2 = X^3 + B$  (so that  $A = 0$ ) the fact that  $B$  has weight  $-6$  similarly gives that each term of the formal group has degree  $\equiv 1 \pmod{6}$  in  $z_1, z_2$ .

## Elliptic Curves. Solutions to Sheet 6.

1. In all of the following, we use the result from lectures that (when the coefficients of  $\mathcal{E}$  are in  $\mathbb{Z}$ )  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$  is isomorphic to a subgroup of  $\tilde{\mathcal{E}} \bmod p$ , where  $p \neq 2$  is a prime not dividing the discriminant. Note that this typically gives a much faster way of computing  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$  than the Nagell-Lutz result. N.B. (a),(b),(c) are about the level that could be asked as part of a 3-hour exam.

(a). There are the obvious points  $P = (0, 1)$  of order 3 and  $Q = (-1, 0)$  of order 2 in  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$ , which generate a subgroup of  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$  of size 6 (namely, the 6 points  $mP + nQ$  for  $0 \leq m \leq 2$  and  $0 \leq n \leq 1$ ). Further,  $\Delta = 4A^3 + 27B^2 = 27$ , so we can reduce modulo any prime except 2 (which must always be avoided) and 3. Over  $\mathbb{F}_5$ , there are only six points:  $\mathbf{o}, (0, \pm 1), (2, \pm 3), (4, 0)$ . So, we conclude that  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$  has size at most 6, which means that it consists of precisely the  $C_6 = C_2 \times C_3$  group of points we have found already.

Note that, if  $P = (0, 1)$  and  $Q = (-1, 0)$ , then the complete list of torsion points is:  $\mathbf{o}, P = (0, 1), 2P = (0, -1), Q = (-1, 0), P + Q = (2, -3), 2P + Q = (2, 3)$ .

(b). Here, the (birational over  $\mathbb{Q}$ ) transformation  $(X, Y) \mapsto (X - 1, Y)$  takes the given curve to:  $Y^2 = X(X + 1)(X - 1) = X^3 - X$ , which has discriminant  $-4$ , and so we can reduce modulo the prime 3. Over  $\mathbb{F}_3$  there are the points:  $\mathbf{o}, (0, 0), (1, 0), (2, 0)$  giving that  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$  has size at most 4. But, in fact,  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$  contains  $\mathbf{o}, (0, 0), (-1, 0), (1, 0)$ , which means the this  $C_2 \times C_2$  group gives all of  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$ . [Alternatively, even without using a birational transformation to the form  $Y^2 = X^3 + AX + B$ , we could just work entirely with the given equation of the curve, noting that the original cubic  $X(X - 1)(X - 2)$  has no repeated roots mod 3, so that  $Y^2 = X(X - 1)(X - 2)$  is an elliptic curve mod 3, and then noting that the only points are:  $\mathbf{o}, (0, 0), (1, 0), (2, 0)$ .]

(c). The (birational over  $\mathbb{Q}$ ) transformation  $(X, Y) \mapsto (3^2X, 3^3Y)$  takes the given curve to:  $Y^2 = X^3 + 1$  which we have already seen in part (a) to have a  $C_2 \times C_3$  group as its torsion group.

Note that, if  $P = (0, 1/27)$  and  $Q = (-1/9, 0)$  then the complete list of torsion points is given by:  $\mathbf{o}, P = (0, 1/27), 2P = (0, -1/27), Q = (-1/9, 0), P + Q = (2/9, -1/9), 2P + Q = (2/9, 1/9)$ .

(d). [optional] The discriminant of  $X^3 - 219X + 1654$  is  $\Delta = 4(-219)^3 + 27 \cdot 1654^2 = 31850496 = 2^{17}3^5$ , so we need only avoid the primes 2, 3 (alternatively, it would be sufficient to compute that  $\Delta \not\equiv 0 \pmod{5}$ , since that is all we shall require). Reducing modulo 5 gives the nine points:  $\mathbf{o}, (0, \pm 2), (1, \pm 1), (2, \pm 2), (3, \pm 2)$ , and so  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$  has size at most 9. A short integer search for possible  $x$ -coordinates reveals the point  $P = (11, 24)$  of order 9, as can be checked by computing  $P, 2P, 3P, \dots, 9P = \mathbf{o}$ . Hence,  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$  is just the  $C_9$  group consisting of these points.

Note that, if  $P = (11, 24)$ , then the complete list of torsion points is:  $\mathbf{o}, P = (11, 24), 2P = (-13, 48), 3P = (3, -32), 4P = (35, -192), 5P = (35, 192), 6P = (3, 32), 7P = (-13, -48), 8P = (11, -24)$ .

2. Let  $\phi : \mathcal{C}(\mathbb{Q}) \rightarrow \mathcal{D}(\mathbb{Q})$  be the usual isogeny, which is a group homomorphism with kernel:  $\mathbf{o}, (0, 0)$ , and let  $\hat{\phi} : \mathcal{D}(\mathbb{Q}) \rightarrow \mathcal{C}(\mathbb{Q})$  be the usual dual isogeny, which is also a group

homomorphism with kernel:  $\mathbf{o}, (0, 0)$ . Now, let  $\mathcal{C}_{\text{odd tors}}(\mathbb{Q})$  be the set of torsion elements of  $\mathcal{C}(\mathbb{Q})$  which have odd order. First check that  $\mathcal{C}_{\text{odd tors}}(\mathbb{Q})$  is a subgroup: (1) The identity  $\mathbf{o}$  has order 1, which is odd, so  $\mathbf{o} \in \mathcal{C}_{\text{odd tors}}(\mathbb{Q})$ , (2) If  $P, Q$  have odd torsion orders  $m, n$ , respectively, then  $mn(P + Q) = n(mP) + m(nQ) = \mathbf{o}$  and so the order of  $P + Q$  divides  $mn$ , giving that the order of  $P + Q$  is odd, i.e.  $P + Q \in \mathcal{C}_{\text{odd tors}}(\mathbb{Q})$ , (3) The order of  $-P$  is the same as the order of  $P$ , so  $P \in \mathcal{C}_{\text{odd tors}}(\mathbb{Q}) \Rightarrow -P \in \mathcal{C}_{\text{odd tors}}(\mathbb{Q})$ . Similarly define  $\mathcal{D}_{\text{odd tors}}(\mathbb{Q})$ , a subgroup of  $\mathcal{D}(\mathbb{Q})$ .

Now, consider any  $P \in \mathcal{C}_{\text{odd tors}}(\mathbb{Q})$ . Then  $P$  must have odd order  $m$ , say. Let  $R = \phi(P)$ . Then  $mR = m\phi(P) = \phi(mP) = \phi(\mathbf{o}) = \mathbf{o}$ , which means that the order of  $R$  divides  $m$ , and so the order of  $R$  is odd; that is:  $R \in \mathcal{D}_{\text{odd tors}}(\mathbb{Q})$ . Hence,  $\phi$  gives a map from  $\mathcal{C}_{\text{odd tors}}(\mathbb{Q})$  to  $\mathcal{D}_{\text{odd tors}}(\mathbb{Q})$ , which certainly satisfies the homomorphism property  $\phi(P + Q) = \phi(P) + \phi(Q)$  for all  $P, Q \in \mathcal{C}_{\text{odd tors}}(\mathbb{Q})$ , since  $\phi$  satisfies this property on the larger set  $\mathcal{C}(\mathbb{Q})$ . Furthermore, the kernel of  $\phi : \mathcal{C}_{\text{odd tors}}(\mathbb{Q}) \rightarrow \mathcal{D}_{\text{odd tors}}(\mathbb{Q})$  must only contain  $\mathbf{o}$  [since  $(0, 0) \notin \mathcal{C}_{\text{odd tors}}(\mathbb{Q})$ ], and so  $\phi : \mathcal{C}_{\text{odd tors}}(\mathbb{Q}) \rightarrow \mathcal{D}_{\text{odd tors}}(\mathbb{Q})$  is an injection [any homomorphism with trivial kernel is an injection]. We have therefore established that there is an injective homomorphism (namely  $\phi$ ) from  $\mathcal{C}_{\text{odd tors}}(\mathbb{Q})$  to  $\mathcal{D}_{\text{odd tors}}(\mathbb{Q})$ , which implies that  $\mathcal{C}_{\text{odd tors}}(\mathbb{Q})$  is isomorphic to a subgroup of  $\mathcal{D}_{\text{odd tors}}(\mathbb{Q})$ , namely the subgroup  $\phi(\mathcal{C}_{\text{odd tors}}(\mathbb{Q}))$  [note, since  $\mathcal{C}_{\text{odd tors}}(\mathbb{Q}), \mathcal{D}_{\text{odd tors}}(\mathbb{Q})$  are finite, it follows from this that  $\#\mathcal{C}_{\text{odd tors}}(\mathbb{Q}) = \#\phi(\mathcal{C}_{\text{odd tors}}(\mathbb{Q})) \leq \#\mathcal{D}_{\text{odd tors}}(\mathbb{Q})$ ]. (We have just used here the general fact that if  $\theta$  is a homomorphism from group  $G$  to group  $H$ , then  $G/\ker\theta$  is isomorphic to  $\text{im}\theta$ , which is the same as  $\theta(G)$ ; when  $\theta$  is injective,  $\ker\theta$  contains only the identity element and  $G/\ker\theta$  can be replaced by  $G$ ). Applying the same argument to  $\hat{\phi} : \mathcal{D}_{\text{odd tors}}(\mathbb{Q}) \rightarrow \mathcal{C}_{\text{odd tors}}(\mathbb{Q})$  gives that  $\mathcal{D}_{\text{odd tors}}(\mathbb{Q})$  is isomorphic to a subgroup of  $\mathcal{C}_{\text{odd tors}}(\mathbb{Q})$ , namely  $\hat{\phi}(\mathcal{D}_{\text{odd tors}}(\mathbb{Q}))$  [and consequently  $\#\mathcal{D}_{\text{odd tors}}(\mathbb{Q}) = \#\hat{\phi}(\mathcal{D}_{\text{odd tors}}(\mathbb{Q})) \leq \#\mathcal{C}_{\text{odd tors}}(\mathbb{Q})$ ]. So, all of  $\mathcal{C}_{\text{odd tors}}(\mathbb{Q}), \mathcal{D}_{\text{odd tors}}(\mathbb{Q}), \hat{\phi}(\mathcal{D}_{\text{odd tors}}(\mathbb{Q})), \phi(\mathcal{C}_{\text{odd tors}}(\mathbb{Q}))$  must have the same number of elements. Combining the fact that  $\phi(\mathcal{C}_{\text{odd tors}}(\mathbb{Q}))$  is a subgroup of  $\mathcal{D}_{\text{odd tors}}(\mathbb{Q})$  and the fact that  $\#\phi(\mathcal{C}_{\text{odd tors}}(\mathbb{Q})) = \#\mathcal{D}_{\text{odd tors}}(\mathbb{Q})$  gives that  $\phi(\mathcal{C}_{\text{odd tors}}(\mathbb{Q}))$  must be equal to  $\mathcal{D}_{\text{odd tors}}(\mathbb{Q})$ . But  $\mathcal{C}_{\text{odd tors}}(\mathbb{Q})$  is isomorphic to  $\phi(\mathcal{C}_{\text{odd tors}}(\mathbb{Q}))$  which is equal to  $\mathcal{D}_{\text{odd tors}}(\mathbb{Q})$ ; hence  $\mathcal{C}_{\text{odd tors}}(\mathbb{Q})$  is isomorphic to  $\mathcal{D}_{\text{odd tors}}(\mathbb{Q})$ , as required.

**3.** The preimages of  $(0, 0)$  under  $\hat{\phi}$  are given by the points of order 2 on  $\mathcal{D}$  distinct from  $(0, 0)$ , namely  $Q_1 = ((-a_1 + \sqrt{a_1^2 - 4b_1})/2, 0) = (a + 2\sqrt{b}, 0)$  and  $Q_2 = ((-a_1 - \sqrt{a_1^2 - 4b_1})/2, 0) = (a - 2\sqrt{b}, 0)$ . Recall the standard map from lectures  $q : \mathcal{D}(\mathbb{Q}) \rightarrow \mathbb{Q}^*/(\mathbb{Q}^*)^2$  which takes  $\mathbf{o} \rightarrow 1$ ,  $(0, 0) \rightarrow b_1$ , and otherwise takes  $(u, v) \rightarrow u$ . We know from lectures that this map has kernel precisely  $\phi(\mathcal{C}(\mathbb{Q}))$ . Now, the multiplication by 2 map on  $\mathcal{C}(\mathbb{Q})$  is  $\hat{\phi} \circ \phi$ , and so  $(0, 0) \in 2\mathcal{C}(\mathbb{Q})$  iff either  $Q_1$  or  $Q_2$  is a member of  $\phi(\mathcal{C}(\mathbb{Q})) \iff q(Q_1) = 1$  or  $q(Q_2) = 1 \iff a + 2\sqrt{b}$  or  $a - 2\sqrt{b} = 1 \in \mathbb{Q}^*/(\mathbb{Q}^*)^2 \iff a + 2\sqrt{b}$  or  $a - 2\sqrt{b} \in (\mathbb{Q}^*)^2 \iff b = m^2$  and  $a + 2m = n^2$ , for some  $m, n \in \mathbb{Q}$ ; but in fact  $m, n$  must be in  $\mathbb{Z}$  since  $a, b \in \mathbb{Z}$ .

**4.** First let us suppose that our elliptic curve  $Y^2 = X^3 + AX + B$  has a point  $(x_0, y_0)$  of order 4. Performing the birational transformation  $(X, Y) \mapsto (X - x_0, Y - y_0)$  [which corresponds to replacing the variables  $X, Y$  by  $X + x_0, Y + y_0$ ] maps  $(x_0, y_0)$  to  $(0, 0)$  on the birationally equivalent curve of the form:  $Y^2 + 2y_0Y + y_0^2 = (\text{monic cubic in } X)$ , and so

is of the form:  $Y^2 + 2y_0Y = X^3 + f_2X^2 + f_1X$  [after absorbing  $-y_0^2$  into the cubic in  $X$ ]. The fact that there is no constant term  $f_0$  is a consequence of the fact that our new curve contains the point  $(0,0)$ . We perform the further birational transformation  $(X, Y) \mapsto (X, Y - f_1X/(2y_0))$  [which corresponds to replacing the variable  $Y$  by  $Y + f_1X/(2y_0)$ ] which maps to the new birationally equivalent curve  $Y^2 + bXY + aY = X^3 + cX^2$ , where  $b = f_1/y_0, a = 2y_0, c = f_2 - (f_1/(2y_0))^2$ . Note that, for any  $r$ , the birational transformation  $(X, Y) \mapsto (r^2X, r^3Y)$  [which corresponds to replacing the variables  $X, Y$  by  $X/r^2, Y/r^3$ ] maps to the birationally equivalent curve:  $Y^2 + brXY + ar^3Y = X^3 + cr^2X^2$ . Choosing  $r = c/a$  forces the coefficients of  $Y$  and  $X^2$  both to be  $c^3/a^2$ . So, let  $r = c/a, v = c^3/a^2$  and  $w = br = b^2/c$ . Note that  $y_0 \neq 0$ , since  $(x_0, y_0)$  is not of order 2, and so  $a = 2y_0 \neq 0$ . Also,  $c \neq 0$  (since if  $c = 0$  then  $Y = 0$  would meet the curve  $Y^2 + bXY + aY = X^3 + cX^2$  3 times at  $(0,0)$  making  $(0,0)$  of order 3, and so  $(x_0, y_0)$  of order 3 on the original curve, a contradiction). Hence, it was legitimate to have divided by  $a$  and  $c$ . Our curve is now of the form:

$$\mathcal{E} : Y^2 + wXY + vY = X^3 + vX^2,$$

and  $(0,0)$  is our point of order 4. So far, we have only used the fact that our original curve contained a rational point  $(x_0, y_0)$  [not of order 2 or 3], which we have mapped to  $(0,0)$ . We now use the fact that  $(0,0)$  is of order 4 on  $\mathcal{E}$  to see what condition this gives on  $v, w$ . First perform the addition  $(0,0) + (0,0)$ . The line tangent to  $\mathcal{E}$  at  $(0,0)$  is  $Y = 0$ . Substituting this into  $\mathcal{E}$  gives:  $X^3 + vX^2$ , which has roots  $0, 0, -v$ , and so  $(0,0), (0,0), (-v,0)$  are the three points of intersection of  $Y = 0$  and  $\mathcal{E}$ , so that  $(0,0) + (0,0) + (-v,0) = \mathbf{o}$ , giving:  $2(0,0) = -(-v,0)$ . Now, since  $\mathbf{o}$  is the point at infinity, the negative of  $(-v,0)$  is the other point on  $\mathcal{E}$  with  $X$ -coordinate  $-v$ . Substituting  $X = -v$  into  $\mathcal{E}$  gives:  $Y^2 - vwY + vY = 0$  and so:  $Y(Y - v(w - 1))$ , which has roots  $0, v(w - 1)$ . So, the negative of  $(-v,0)$  is  $(-v, v(w - 1))$ , giving:  $2(0,0) = (-v, v(w - 1))$ . Now,  $(0,0)$  is of order 4 iff  $2(0,0)$  is of order 2 iff  $2(0,0)$  equals its own inverse (since it is not the identity). But  $(-v, v(w - 1))$  is the inverse of  $(-v,0)$  and so equals its own inverse iff  $v(w - 1) = 0$ . Now, we cannot have  $v = 0$ , since then  $\mathcal{E}$  would be singular and so would not be an elliptic curve. So,  $(0,0)$  having order 4 on  $\mathcal{E}$  is equivalent to  $w = 1$ . Substituting  $w = 1$  into  $\mathcal{E}$  gives the required form.

## Elliptic Curves. Solutions to Sheet 7.

1.

(a). Let  $\mathcal{C} : Y^2 = X(X^2 + aX + b) = X(X^2 + 2X + 3)$ , where  $a = 2, b = 3$ , and isogenous curve  $\mathcal{D} : Y^2 = X(X^2 + a_1X + b_1) = X(X^2 - 4X - 8)$ , where  $a_1 = -4, b_1 = -8$ , with the usual isogeny  $\phi : \mathcal{C}(\mathbb{Q}) \rightarrow \mathcal{D}(\mathbb{Q}) : (x, y) \mapsto (y^2/x^2, y - 3y/x^2)$ , and dual isogeny  $\hat{\phi} : \mathcal{D}(\mathbb{Q}) \rightarrow \mathcal{C}(\mathbb{Q}) : (u, v) \mapsto (\frac{1}{4}v^2/u^2, \frac{1}{8}(v + 8v/u^2))$ .

The map  $q : \mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q})) \rightarrow \mathbb{Q}^*/(\mathbb{Q}^*)^2 : (u, v) \mapsto u$ , for  $(u, v) \neq (0, 0)$ , with  $q : (0, 0) \mapsto b_1$  and  $q : \mathbf{o} \mapsto 1$ , is an injection with  $\text{im} q$  contained in  $\{d : d \text{ is square free and } d|b_1\} = \{\pm 1, \pm 2\}$ . Also,  $\mathbf{o} \mapsto 1$ ,  $(0, 0) \mapsto -8 = -2$ , so that  $\{1, -2\} \subset \text{im} q \subset \{\pm 1, \pm 2\}$ .

There is only one coset to check, represented by  $-1$ , say. We know that  $-1 \in \text{im} q$  iff there are integers  $\ell, m, n$ , not all 0, and with  $\gcd(\ell, m) = 1$ , such that:  $(-1) \cdot \ell^4 + a_1 \ell^2 m^2 + (b_1/(-1)) \cdot m^4 = n^2$  that is:  $-\ell^4 - 4\ell^2 m^2 + 8m^4 = n^2$ . Rewrite as:  $-(\ell^2 + 2m^2)^2 + 12m^4 = n^2$ . Reducing modulo 3 gives  $-(\ell^2 + 2m^2)^2 \equiv n^2 \pmod{3}$ . If  $n$  were coprime to 3, then this would give:  $((\ell^2 + 2m^2)/n)^2 = -1$  in  $\mathbb{F}_3$ , contradicting the fact that  $-1$  is not a quadratic residue modulo 3. So,  $3|n$  and so  $3|(\ell^2 + 2m^2)$  also. This means that  $9|(\ell^2 + 2m^2)^2$  and  $9|n^2$ , which can be combined with  $-(\ell^2 + 2m^2)^2 + 12m^4 = n^2$  to give:  $9|12m^4$  and so  $3|m$ . Combining  $3|m$  with  $3|(\ell^2 + 2m^2)$  gives that  $3|\ell$ . This contradicts the fact that  $\gcd(\ell, m) = 1$ . Hence our equation is impossible in  $\mathbb{Q}_3$ , and so impossible in  $\mathbb{Q}$ . Hence  $-1 \notin \text{im} q$ .

We conclude that  $\text{im} q = \{1, -2\}$ , and so  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q}))$  is generated by  $(0, 0)$ .

The map  $\hat{q} : \mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q})) \rightarrow \mathbb{Q}^*/(\mathbb{Q}^*)^2 : (x, y) \mapsto x$ , for  $(x, y) \neq (0, 0)$ , with  $\hat{q} : (0, 0) \mapsto b$  and  $\hat{q} : \mathbf{o} \mapsto 1$ , is an injection with  $\text{im} \hat{q}$  contained in  $\{d : d \text{ is square free and } d|b\} = \{\pm 1, \pm 3\}$ . Also,  $\mathbf{o} \mapsto 1$  and  $(0, 0) \mapsto 3$ , so that  $\{1, 3\} \subset \text{im} \hat{q} \subset \{\pm 1, \pm 3\}$ .

There is only one coset to check, represented by  $-1$ , say. We know that  $-1 \in \text{im} \hat{q}$  iff there are integers  $\ell, m, n$ , not all 0, and with  $\gcd(\ell, m) = 1$ , such that:  $(-1) \cdot \ell^4 + a \ell^2 m^2 + (b/(-1)) \cdot m^4 = n^2$ ; that is:  $-\ell^4 + 2\ell^2 m^2 - 3m^4 = n^2$ . Rewrite as:  $-(\ell^2 - m^2)^2 - 2m^4 = n^2$ . This is impossible in  $\mathbb{R}$  (the left hand side is  $\leq 0$  and the right hand side is  $\geq 0$ , and equality only occurs when  $\ell^2 - m^2 = m^4 = n^2 = 0$ , implying  $\ell = m = n = 0$ , which is not allowed). Hence  $-1 \notin \text{im} \hat{q}$ .

We conclude that  $\text{im} \hat{q} = \{1, 3\}$ , and so  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q}))$  is generated by  $(0, 0)$ .

Finally, since multiplication by 2 in  $\mathcal{C}(\mathbb{Q})$  is  $\hat{\phi} \circ \phi$ , we have that  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is generated by: generators for  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q}))$  [namely:  $(0, 0)$ ] together with the images under  $\hat{\phi}$  of generators for  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q}))$  [namely,  $\hat{\phi}((0, 0)) = \mathbf{o}$ ]. Conclusion:  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is generated by  $(0, 0)$ , and so is isomorphic to  $C_2$ . We also know that  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is isomorphic to  $\mathcal{C}_{\text{tors}}(\mathbb{Q})/2\mathcal{C}_{\text{tors}}(\mathbb{Q}) \times C_2^r$ , which is isomorphic to  $\mathcal{C}(\mathbb{Q})[2] \times C_2^r$ , where  $\mathcal{C}(\mathbb{Q})[2]$  is the 2-torsion group and  $r$  is the rank. The 2-torsion points on  $\mathcal{C} : Y^2 = X(X^2 + 2X + 3)$  are  $\mathbf{o}$  together with the points of the form  $(x, 0)$ , where  $x$  is a root of  $X(X^2 + 2X + 3)$ , that is:  $(0, 0)$ ,  $(-1 + \sqrt{-2}, 0)$  and  $(-1 - \sqrt{-2}, 0)$ , of which only  $\mathbf{o}$  and  $(0, 0)$  are in  $\mathcal{C}(\mathbb{Q})[2]$ , giving that  $\mathcal{C}(\mathbb{Q})[2]$  is isomorphic to  $C_2$ . Combining this with the facts (already found) that  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is isomorphic both to  $C_2$  and to  $\mathcal{C}(\mathbb{Q})[2] \times C_2^r$ , give that  $\mathcal{C}(\mathbb{Q})$  has rank 0.

(b). Let  $\mathcal{C} : Y^2 = X(X^2 + aX + b) = X(X^2 + 14X + 1)$ , where  $a = 14, b = 1$ , and isogenous curve  $\mathcal{D} : Y^2 = X(X^2 + a_1X + b_1) = X(X^2 - 28X + 192)$ , where  $a_1 = -28, b_1 = 192$ , with the usual isogeny  $\phi : \mathcal{C}(\mathbb{Q}) \rightarrow \mathcal{D}(\mathbb{Q}) : (x, y) \mapsto (y^2/x^2, y - y/x^2)$ , and dual isogeny  $\hat{\phi} : \mathcal{D}(\mathbb{Q}) \rightarrow \mathcal{C}(\mathbb{Q}) : (u, v) \mapsto (\frac{1}{4}v^2/u^2, \frac{1}{8}(v - 192v/u^2))$ .

The map  $q : \mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q})) \mapsto \mathbb{Q}^*/(\mathbb{Q}^*)^2 : (u, v) \mapsto u$ , for  $(u, v) \neq (0, 0)$ , with  $q : (0, 0) \mapsto b_1$  and  $q : \mathbf{o} \mapsto 1$ , is an injection with  $\text{im}q$  contained in  $\{d : d \text{ is square free and } d|b_1\} = \{\pm 1, \pm 2, \pm 3, \pm 6\}$ . Also,  $\mathbf{o} \mapsto 1$ ,  $(0, 0) \mapsto 192 = 3$  and the obvious point  $(8, 16) \mapsto 2$  [N.B. when searching for a point in  $\mathcal{D}(\mathbb{Q})$  which might map to 2 under  $q$ , one need only try points with  $x$ -coordinate equal to 2 modulo squares, e.g. 2, 8, 1/2, 18, etc, so one should find the point  $(8, 16)$  quickly; also, it's a good idea at the outset just to look for "obvious" members of  $\mathcal{D}(\mathbb{Q})$  with  $x$  coordinates being integers in the range from  $-10$  to  $10$ ; doing this at the outset would also reveal the point  $(8, 16)$ .] This means that  $1, 3, 2 \in \text{im}q$ ; but  $\text{im}q$  is a group, so  $6 \in \text{im}q$  also, and indeed we can just take:  $(0, 0) + (8, 16) = (24, -48)$ , which maps to 6 under  $q$ . Hence,  $\{1, 2, 3, 6\} \subset \text{im}q \subset \{\pm 1, \pm 2, \pm 3, \pm 6\}$ .

There is only one coset to check, represented by  $-1$ , say. We know that  $-1 \in \text{im}q$  iff there are integers  $\ell, m, n$ , not all 0, and with  $\gcd(\ell, m) = 1$ , such that:  $(-1) \cdot \ell^4 + a_1 \ell^2 m^2 + (b_1/(-1)) \cdot m^4 = n^2$  that is:  $-\ell^4 - 28\ell^2 m^2 - 192m^4 = n^2$ . We can see that the LHS is  $\leq 0$  and the RHS is  $\geq 0$ , and there is equality iff  $\ell = m = n = 0$ , a contradiction. Hence  $-1 \notin \text{im}q$ . We conclude that  $\text{im}q = \{1, 2, 3, 6\}$  and that  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q})) = \{\mathbf{o}, (0, 0), (8, 16), (24, -48)\}$ , and so  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q}))$  is generated by  $(0, 0)$  and  $(8, 16)$ .

The map  $\hat{q} : \mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q})) \mapsto \mathbb{Q}^*/(\mathbb{Q}^*)^2 : (u, v) \mapsto u$ , for  $(u, v) \neq (0, 0)$ , with  $\hat{q} : (0, 0) \mapsto a_1^2 - 4b_1 = b$  and  $\hat{q} : \mathbf{o} \mapsto 1$ , is an injection with  $\text{im}\hat{q}$  contained in  $\{d : d \text{ is square free and } d|b\} = \{\pm 1\}$ . Also,  $\mathbf{o} \mapsto 1$  and  $(0, 0) \mapsto 1$ , so that  $\{1\} \subset \text{im}\hat{q} \subset \{\pm 1\}$ .

There is only one coset to check, represented by  $-1$ , say. We know that  $-1 \in \text{im}\hat{q}$  iff there are integers  $\ell, m, n$ , not all 0, and with  $\gcd(\ell, m) = 1$ , such that:  $(-1) \cdot \ell^4 + a \ell^2 m^2 + (b/(-1)) \cdot m^4 = n^2$ ; that is:  $-\ell^4 + 14\ell^2 m^2 - m^4 = n^2$ . Rewrite as:  $-(\ell^2 - 7m^2)^2 + 48m^4 = n^2$ . Reducing modulo 3 gives:  $-(\ell^2 - 7m^2)^2 \equiv n^2 \pmod{3}$ . If  $n$  were coprime to 3, then this would give:  $((\ell^2 - 7m^2)/n)^2 = -1$  in  $\mathbb{F}_3$ , contradicting the fact that  $-1$  is not a quadratic residue modulo 3. So,  $3|n$  and so  $3|(\ell^2 - 7m^2)$ , also. Hence 9 divides  $(\ell^2 - 7m^2)^2$  and  $n^2$  and so must divide  $48m^4$ ; but 48 is only divisible by 3 (not by 9) so that 3 must divide  $m^4$ ; hence 3 divides  $m$ . Combining this with the fact (already found) that  $3|(\ell^2 - 7m^2)$  gives that  $3|\ell$  also. We've shown that 3 divides all of  $\ell, m, n$ , a contradiction. Hence  $-1 \notin \text{im}\hat{q}$ .

We conclude that  $\text{im}\hat{q} = \{1\}$ , and so  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q}))$  contains only  $\mathbf{o}$ . [N.B.  $(0, 0)$  should not also be included as a separate member of  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q}))$  even though it is a rational point;  $(0, 0)$  maps to 1 under  $\hat{q}$ , and so  $(0, 0) \in \hat{\phi}(\mathcal{D}(\mathbb{Q}))$ ; that is  $(0, 0) = \mathbf{o}$  in  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q}))$ ; the same comment applies to the obvious point  $(1, 4)$ ; in general, for each distinct member  $r$  of  $\text{im}\hat{q}$ , you should only include exactly one point in  $\mathcal{C}(\mathbb{Q})$  which maps to  $r$ ].

Finally, since multiplication by 2 in  $\mathcal{C}(\mathbb{Q})$  is  $\hat{\phi} \circ \phi$ , we have that  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is generated by: generators for  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q}))$  [namely:  $\mathbf{o}$ ] together with the images under  $\hat{\phi}$  of generators for  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q}))$  [namely,  $\hat{\phi}((0, 0)) = \mathbf{o}$  and  $\hat{\phi}((8, 16)) = (1, -4)$ ]. Conclusion:  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is generated by  $(1, -4)$ , and so is the group  $C_2$ . Now  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is isomor-

phic to  $\mathcal{C}_{\text{tors}}(\mathbb{Q})/2\mathcal{C}_{\text{tors}}(\mathbb{Q}) \times C_2^{\text{rank}}$ , and  $\mathcal{C}_{\text{tors}}(\mathbb{Q})/2\mathcal{C}_{\text{tors}}(\mathbb{Q})$  is isomorphic to the 2-torsion group of  $\mathcal{C}_{\text{tors}}(\mathbb{Q})$  which is  $C_2$  (consisting only of  $\mathbf{o}$  and  $(0,0)$ ), so that  $\mathcal{C}_{\text{tors}}(\mathbb{Q})/2\mathcal{C}_{\text{tors}}(\mathbb{Q})$  is isomorphic to  $C_2$ . Conclusion:  $\text{rank} = 0$ . [If this seems surprising, then note that  $(1,4), (1,-4)$  are points of order 4 in  $\mathcal{C}(\mathbb{Q})$ ].

**2.** We are given that  $A, +$  is an Abelian group, and that  $h : A \rightarrow \mathbb{R}_{\geq 0}$  satisfies:

(I) There exists a constant  $C$ , independent of  $P, Q$ , such that

$$|h(P+Q) + h(P-Q) - 2h(P) - 2h(Q)| \leq C, \text{ for all } P, Q \in A,$$

(II) For any  $B \in \mathbb{R}$ , the set  $\{P \in A : h(P) \leq B\}$  is finite.

From (I) we obtain:  $h(P+Q) + h(P-Q) - 2h(P) - 2h(Q) \leq C$  and so (since  $h(P-Q) \geq 0$ ):  $h(P+Q) \leq h(P+Q) + h(P-Q) \leq 2h(P) + 2h(Q) + C \leq 2h(P) + C_1(Q)$ , where  $C_1(Q) = 2h(Q) + C$ . Hence Property (1) in the definition of height function is satisfied.

Letting  $Q = P$  in (I), we obtain:

$$|h(2P) + h(e) - 4h(P)| \leq C, \quad (*)$$

where  $e$  denotes the identity element of the group  $A$ . This gives:  $h(2P) + h(e) - 4h(P) \geq -C$ , and so:  $h(2P) \geq 4h(P) - C_2$ , where  $C_2 = C + h(e)$ . Hence Property (2) in the definition of height function is satisfied. Furthermore, (II) is the same as Property (3) in the definition of height function. Hence all 3 required properties are satisfied, giving that  $h$  is a height function, as required.

Replacing  $P, Q$  in (I) with  $2P, P$ , respectively, gives:

$$|h(3P) - 2h(2P) - h(P)| \leq C.$$

Multiplying  $(*)$  by 2 gives:

$$|2h(2P) + 2h(e) - 8h(P)| \leq 2C.$$

These last two equations then give:

$$\begin{aligned} |h(3P) - 9h(P)| &= |h(3P) - 2h(2P) - h(P) + 2h(2P) + 2h(e) - 8h(P) - 2h(e)| \\ &\leq |h(3P) - 2h(2P) - h(P)| + |2h(2P) + 2h(e) - 8h(P)| + 2|h(e)| \leq C_3, \end{aligned}$$

where  $C_3 = 3C + 2|h(e)|$  (which is independent of  $P$ ).

**3.** In all of the following, each step multiplies numbers  $\leq N$  (followed by a possible reduction modulo  $N$ ), and so we are guaranteed that everything can be done on a 9-digit calculator, since  $N^2$  has only 9 digits.

**(a).** First compute (modulo  $N = 10481$ ):  $2^1 \equiv 2$ ,  $2^2 \equiv 4$ ,  $2^4 \equiv 16$ ,  $2^8 \equiv 256$ ,  $2^{16} \equiv 2650$ ,  $2^{32} \equiv 230$  (where each of these was obtained by squaring the previous one, and reducing modulo  $N$ ). Now, we write 46 in base 2:  $46 = 2 + 4 + 8 + 32$  and so  $2^{46} \equiv 2^2 2^4 2^8 2^{32} \equiv 4 \cdot 16 \cdot 256 \cdot 230 \equiv 64 \cdot 6475 \equiv 5641$  modulo  $N$ , so that  $2^{46} - 1 \equiv 5640$  modulo  $N$ .

Now, compute  $\text{gcd}(5640, N)$  by Euclid's Algorithm:  $10481 = 1 \cdot 5640 + 4841$ ;  $5640 = 1 \cdot 4841 + 799$ ;  $4841 = 6 \cdot 799 + 47$ ,  $799 = 17 \cdot 47 + 0$ . So, 47 is a factor of  $N$ . Compute  $10481/47 = 223$ , giving the factorisation  $N = 10481 = 47 \cdot 223$ .

**(b).** The line tangent to  $\mathcal{E}$  at  $P = (5, 11)$  has slope  $y'$  given by  $2yy' = 3x^2 - 1$ , with  $x = 5, y = 11$ ; that is, the slope is  $74/22 = 37/11$ . This tangent line also goes through  $(5, 11)$  and so has equation:  $Y = (37/11)X - 64/11$ . The  $x$ -coordinate of  $2P$  is therefore  $(37/11)^2 - (5 + 5) = 159/121$ . [It will turn out not to be necessary to evaluate this mod  $N$ , although if this is done using EA, then it is 7364], and the  $y$ -coordinate is:

$-((37/11) \cdot (159/121) - (64/11)) = 1861/1331$  [again, although unnecessary in this example, this can be computed by EA to be:  $6679 \bmod N$ ], so that  $Q = 2P = (159/121, 1861/1331)$ . We now wish to compute  $3P = P + Q$ , and so again the first step is to find the line joining  $P$  and  $Q$ . This has slope given by  $(1861/1331 - 11)/(159/121 - 5) = 6930/2453$ , and so we need to compute  $6930/2453$  (modulo  $N = 10481$ ), for which the first step is to find the inverse of  $2453$  (modulo  $N = 10481$ ). Using Euclid's Algorithm:  $10481 = 4 \cdot 2453 + 669$ ;  $2453 = 3 \cdot 669 + 446$ ;  $669 = 1 \cdot 446 + 223$ ;  $446 = 2 \cdot 223 + 0$ . So, we cannot find the inverse of  $2453$  (modulo  $N = 10481$ ), and this step has given us a factor  $223$  of  $N$ . As before, compute  $10481/223 = 47$ , giving the factorisation  $N = 10481 = 47 \cdot 223$ .

(c). Since  $N = 47 \cdot 223$ , we have  $\phi(N) = 46 \cdot 222 = 10212$ . Compute the gcd of  $d = 4085$  and  $\phi(N)$ , we see:  $10212 = 2 \cdot 4085 + 2042$ ;  $4085 = 2 \cdot 2042 + 1$ , so that  $\gcd(10212, 4085) = 1$ . Reversing the steps:  $1 = 4085 - 2 \cdot 2042 = 4085 - 2 \cdot (10212 - 2 \cdot 4085) = 5 \cdot 4085 - 2 \cdot 10212$ . Hence,  $5$  is the inverse of  $4085$  modulo  $10212$ . The decoding operation is therefore  $X \mapsto X^5 \bmod N$ . Computing  $6012^5 = (6012^2)^2 \cdot 6012 \equiv 5656^2 \cdot 6012 \equiv 2324 \cdot 6012 \equiv 715$  (modulo  $N = 10481$ ). Also:  $3236^5 = (3236^2)^2 \cdot 3236 \equiv 1177^2 \cdot 3236 \equiv 1837 \cdot 3236 \equiv 1805$  (modulo  $N = 10481$ ). The decoded message is therefore: 0715, 1805; that is: GORE.

# Elliptic Curves. Solutions to Sheet 8.

## Solutions to the Exam-style Questions.

### Question 1.

(i). Let  $x_0 = a_0 = 2$ . Then  $|x_0^2 + 3|_7 = 7^{-1}$ . Look for  $x_1 = a_0 + 7a_1$  such that  $|x_1^2 + 3|_7 \leq 7^{-2}$ . This is satisfied if:  $(2 + 7a_1)^2 \equiv -3 \pmod{7^2} \iff 2 \cdot 2 \cdot 7a_1 \equiv -3 - 2^2 \pmod{7^2} \iff 4a_1 \equiv -1 \pmod{7} \iff a_1 \equiv 5 \pmod{7}$ . So, now define:  $a_1 = 5$  and  $x_1 = a_0 + 7a_1 = 37$ . Check that indeed  $|x_1^2 + 3|_7 = |1372|_7 = 7^{-3} < 7^{-1}$ . **[5 marks]**

(ii). Compute  $1, \bar{2} = 1 + 2p + 2p^2 + 2p^3 + \dots = 1 + 2p(1 + p + p^2 + \dots) = 1 + 2p/(1 - p) = (1 + p)/(1 - p)$ . So,  $1, \bar{2} = -9/8 \iff (1 + p)/(1 - p) = -9/8 \iff 8(1 + p) = -9(1 - p) \iff p = 17$ . **[3 marks]**

(iii). We are given that  $p \equiv 1 \pmod{3}$ . Then  $p \neq 2$ , so  $p$  is odd, and so satisfies either  $p \equiv 1 \pmod{4}$  or  $p \equiv 3 \pmod{4}$ . In the first case, we have that  $\left(\frac{-3}{p}\right) = \left(\frac{3}{p}\right)\left(\frac{-1}{p}\right) = \left(\frac{p}{3}\right)\left(\frac{-1}{3}\right)$  [by quadratic reciprocity]  $= \left(\frac{1}{p}\right)\left(\frac{-1}{3}\right) = 1 \cdot 1$  [using the result from notes that  $\left(\frac{-1}{p}\right) = 1$  when  $p \equiv 1 \pmod{4}$ ]. In the second case,  $\left(\frac{-3}{p}\right) = \left(\frac{3}{p}\right)\left(\frac{-1}{p}\right) = -\left(\frac{p}{3}\right)\left(\frac{-1}{p}\right)$  [by quadratic reciprocity]  $= -\left(\frac{1}{3}\right)\left(\frac{-1}{p}\right) = -1 \cdot (-1)$  [using the result from notes that  $\left(\frac{-1}{p}\right) = 1$  when  $p \equiv 1 \pmod{4}$ ]. In either case, we have that  $\left(\frac{-3}{p}\right) = 1$ , as required. We may apply the general rule from lectures (which follows easily from Hensel's Lemma) that, if  $p \neq 2$  and  $|a|_p = 1$ , then  $a$  is a square in  $\mathbb{Q}_p^*$  iff  $a$  is a quadratic residue in  $\mathbb{F}_p$ . Hence  $-3$  is a square in  $\mathbb{Q}_p^*$ . **[6 marks]**

(iv). For  $p \equiv 2 \pmod{3}$ , note that 3 is coprime to  $p - 1$ , and so there exist  $\lambda, \mu \in \mathbb{Z}$  such that  $(p - 1)\lambda + 3\mu = 1$ . Now, suppose that  $v^3 = w^3$  in  $\mathbb{F}_p^*$ . Then  $v^{3\mu} = w^{3\mu}$ . Also  $(p - 1)\lambda$  is a multiple of the order of the group  $\mathbb{F}_p^*$  and so  $v^{(p-1)\lambda} = w^{(p-1)\lambda}$  (since both are equal to 1). Multiplying these last two equations gives:  $v^{(p-1)\lambda + 3\mu} = w^{(p-1)\lambda + 3\mu}$ , and so  $v = w$ . We have shown that  $v^3 = w^3$  implies  $v = w$ , and so the map  $v \mapsto v^3$  is injective, and hence surjective (and hence bijective), from  $\mathbb{F}_p^*$  to  $\mathbb{F}_p^*$ . Since also  $0 \mapsto 0^3$ , the map  $v \mapsto v^3$  must also be a bijection from  $\mathbb{F}_p$  to  $\mathbb{F}_p$ . For any  $d \in \mathbb{Z}$  not divisible by 3, it follows that there exists  $x_0 \in \mathbb{Z}$  such that  $x_0^3 \equiv d \pmod{p}$ . This is the same as  $|f(x_0)|_p < 1$ , where  $f(x) = x^3 - d$ . But from the fact that  $d$  is not divisible by  $p$  and  $x_0^3 \equiv d \pmod{p}$  it follows that  $x_0$  is not divisible by  $p$ , and so  $|x_0|_p = 1$ , giving that  $|f'(x_0)|_p = |3x_0^2|_p = 1$ . Therefore,  $|f(x_0)|_p < |f'(x_0)|_p^2$  and so, by Hensel's Lemma, there exists a solution in  $\mathbb{Q}_p$  to  $x^3 - d$ ; that is,  $d$  is a cube in  $\mathbb{Q}_p$ , for any  $d \in \mathbb{Z}$  not divisible by 3.

We have already established in (c) that  $X^2 + 3 = 0$  has a solution  $\mathbb{Q}_p$  for every  $p \equiv 1 \pmod{3}$ , and we have already established (immediately above), since the given  $q$  is not divisible by 3, that  $X^3 - q = 0$  has a solution  $\mathbb{Q}_p$  for every  $p \equiv 2 \pmod{3}$ . For  $p = 3$ , note that we are given  $q \equiv 1 \pmod{27}$ , so that  $|f(x_0)|_3 \leq 3^{-3}$ , where  $x_0 = 1$  and  $f(x) = x^3 - q$ . Also,  $|f'(x_0)|_3 = |3 \cdot 1^2| = 3^{-1}$ , so that  $|f(x_0)|_3 < |f'(x_0)|_3^2$ . By Hensel's Lemma, it follows that there exists a solution in  $\mathbb{Q}_3$  to  $X^3 - q = 0$ . Finally note that, since  $q > 0$ , there exists a solution to  $X^3 - q = 0$  in  $\mathbb{R}$ . Hence there is a solution to  $(X^2 + 3)(X^3 - q) = 0$  in  $\mathbb{R}$  and every  $\mathbb{Q}_p$ , as required. **[11 marks]**

**Question 2.**

(i). Let  $\mathcal{C}$  be the elliptic curve  $Y^2 = X^3 + 3$ . The discriminant of  $X^3 + 3$  is  $4 \cdot 0^3 + 27 \cdot 3^2 = 3^5$ , and so  $\tilde{\mathcal{C}} : Y^2 = X^3 + 3$  is an elliptic curve for all  $p \neq 2, 3$ . The elements of  $\tilde{\mathcal{C}}(\mathbb{F}_5)$  are:  $\mathbf{o}, (1, \pm 2), (2, \pm 1), (3, 0)$ , and so  $\tilde{\mathcal{C}}(\mathbb{F}_5)$  has order 6. The elements of  $\tilde{\mathcal{C}}(\mathbb{F}_7)$  are:  $\mathbf{o}, (1, \pm 2), (2, \pm 2), (3, \pm 3), (4, \pm 2), (5, \pm 3), (6, \pm 3)$ , and so  $\tilde{\mathcal{C}}(\mathbb{F}_7)$  has order 13. Since the torsion subgroup of  $\mathcal{C}(\mathbb{Q})$  injects into both of these groups, its order must divide  $\gcd(6, 13) = 1$  and so equal 1. Hence the torsion subgroup of  $\mathcal{C}(\mathbb{Q})$  is precisely  $\{\mathbf{o}\}$ . [5 marks]

(ii). Let  $\mathcal{D}$  be the elliptic curve  $Y^2 = X^3 + 4X$ . Then  $\mathbf{o}$  and  $(0, 0)$  [which has order 2] are both members of the torsion subgroup. Furthermore, consider the point  $(2, 4) \in \mathcal{D}(\mathbb{Q})$ ; the tangent to  $\mathcal{D}$  at  $(2, 4)$  has slope  $(3 \cdot 2^2 + 4)/(2 \cdot 4) = 2$  and has equation  $Y = 2X$ . So, the  $x$ -coordinate of  $2(2, 4)$  is  $2^2 - 2 - 2 = 0$ , with corresponding  $y$ -coordinate 0, giving:  $2(2, 4) + (0, 0) = \mathbf{o}$  and so  $2(2, 4) = -(0, 0) = (0, 0)$ . Hence  $4(2, 4) = 2(0, 0) = \mathbf{o}$ ; similarly  $4(2, -4) = \mathbf{o}$ . We have found 4 torsion elements:  $\mathbf{o}, (0, 0), (2, 4), (2, -4)$ , and so the torsion subgroup is of order at least 4. The discriminant of  $X^3 + 4X$  is  $4 \cdot 4^3 + 27 \cdot 0^2 = 2^8$ , and so  $\tilde{\mathcal{D}} : Y^2 = X^3 + 4X$  is an elliptic curve for all  $p \neq 2$ . The elements of  $\tilde{\mathcal{D}}(\mathbb{F}_3)$  are:  $\mathbf{o}, (0, 0), (2, \pm 1)$ , and so  $\tilde{\mathcal{D}}(\mathbb{F}_3)$  has order 4. Since the torsion subgroup of  $\mathcal{D}(\mathbb{Q})$  injects into this groups, its order must divide 4. Hence the torsion subgroup of  $\mathcal{D}(\mathbb{Q})$  consists precisely of the 4 torsion elements already found, namely:  $\{\mathbf{o}, (0, 0), (2, \pm 4)\}$ . [6 marks]

(iii). Let  $\mathcal{F}$  be the elliptic curve  $Y^2 = X(X+1)(X+n^2) = X^3 + (n^2+1)X^2 + n^2X$ . The tangent to  $\mathcal{D}$  at  $(n, n(n+1))$  has slope  $(3n^2 + 2(n^2+1)n + n^2)/(2n(n+1)) = n+1$ , and so equation  $Y = (n+1)X + c$  for some  $c$ ; the line must pass through  $(n, n(n+1))$  so that  $c = 0$  and the equation of the tangent is:  $Y = (n+1)X$ . The  $x$ -coordinate of  $P = 2(n, n(n+1))$  is  $(n+1)^2 - (n^2+1) - 2n = 0$ , with  $y$ -coordinate  $(n+1) \cdot 0 = 0$ . Hence  $2(n, n(n+1)) + (0, 0) = \mathbf{o}$ , so that  $2(n, n(n+1)) = -(0, 0) = (0, 0)$ , giving  $4(n, n(n+1)) = \mathbf{o}$ . So  $(n, n(n+1))$  is torsion, as must be  $(n, -n(n+1)) = -(n, n(n+1))$ . Replacing every occurrence of  $n$  with  $-n$  gives that the tangent to  $\mathcal{D}$  at  $(-n, n(n-1))$  has equation  $Y = (-n+1)X$  and  $2(-n, n(n-1)) = (0, 0)$ ; we deduce, as before, that  $(-n, \pm n(n-1))$  are torsion elements. Therefore there are at least 8 members of the torsion subgroup of  $\mathcal{F}(\mathbb{Q})$ , namely:  $\mathbf{o}, (0, 0), (-1, 0), (-n^2, 0)$  and the above 4 points of order 4. Finally, let  $n \equiv 2$  (modulo 5), so that  $n^2 \equiv 4$  (modulo 5). Then  $\tilde{\mathcal{F}}$  modulo 5 is:  $Y^2 = X(X+1)(X+4)$ , and the cubic has no repeated roots, so that  $f$  has good reduction at 5. The elements of  $\tilde{\mathcal{F}}(\mathbb{F}_5)$  are:  $\mathbf{o}, (0, 0), (1, 0), (2, \pm 1), (3, \pm 2), (4, 0)$ , and so  $\tilde{\mathcal{F}}(\mathbb{F}_5)$  has order 8. Since the torsion subgroup of  $\mathcal{F}(\mathbb{Q})$  injects into this groups, its order must divide 8. Hence the torsion subgroup of  $\mathcal{F}(\mathbb{Q})$  consists precisely of the 8 torsion elements already found, namely:  $\{\mathbf{o}, (0, 0), (-1, 0), (-n^2, 0), (n, \pm n(n+1)), (-n, \pm n(n-1))\}$ . [7 marks]

(iv). The discriminant of  $X^3 - k^2X + k^3$  is  $4 \cdot (-k^2)^3 + 27 \cdot (k^3)^2 = 23k^6$ . If  $(x, y)$  is a  $\mathbb{Q}$ -rational torsion point on  $\mathcal{E} : Y^2 = X^3 - k^2X + k^3$ , then we know by Nagell-Lutz that  $x, y \in \mathbb{Z}$  with  $y^2 | 23k^6$  and so  $|y|^2 \leq 23|k|^6$ ; that is:  $|y| \leq \sqrt{23}|k|^3 = (4.79583 \dots)|k|^3 \leq 5|k|^3$ .

We cannot have  $k = 0$  since then the discriminant would be 0 and  $\mathcal{E}$  would not be an elliptic curve. Since  $k \in \mathbb{Z}$ , we have  $|k| \geq 1$  and so:  $|k|^2 \leq |k|^4$  and  $|k|^3 \leq |k|^6$ . Imagine  $|x| > 3|k|^2$ . Then:  $|x^2 - k^2| \geq |x^2| - |k^2| > 9|k|^4 - |k^2| \geq 9|k|^4 - |k^4| = 8|k|^4$ , so:  $|y|^2 = |x(x^2 - k^2) + k^3| \geq |x(x^2 - k^2)| - |k^3| = |x||x^2 - k^2| - |k^3| > 3|k|^2 8|k|^4 - |k|^6 = 23|k|^6$ , contradicting  $|y|^2 \leq 23|k|^6$  (above). Hence  $|x| \leq 3|k|^2$ , as required. [7 marks]

### Question 3.

(i). Let  $r \in \mathbb{Q}^*/(\mathbb{Q}^*)^2, r \in \text{im } q, r \in \mathbb{Z}, r$  square free. We want to prove that  $r|b_1$ . Suppose  $r = q(u, v)$ , where  $(u, v) \in \mathcal{D}(\mathbb{Q})$ , which must exist since  $r \in \text{im } q$ . Then:  $r = q(u, v) = u = u^2 + a_1u + b_1$  in  $\mathbb{Q}^*/(\mathbb{Q}^*)^2$  [since  $u(u^2 + a_1u + b_1) = v^2$ ]. So,  $r, u, u^2 + a_1u + b_1$  are all the same modulo squares, which means we can write:

$$u^2 + a_1u + b_1 = rs^2, \quad u = rt^2, \quad \text{for some } s, t \in \mathbb{Q}.$$

Hence:  $(rt^2)^2 + a_1(rt^2) + b_1 = rs^2$ . Let  $t = \ell/m$ , where  $\ell, m \in \mathbb{Z}$  and  $\gcd(\ell, m) = 1$ . Then:  $r^2\ell^4/m^4 + a_1r\ell^2/m^2 + b_1 = rs^2$ , and so:  $r^2\ell^4 + a_1r\ell^2m^2 + b_1m^4 = r(m^2s)^2$ . Now,  $a_1, b_1, r, \ell, m \in \mathbb{Z}$ , so the LHS of this last equation is in  $\mathbb{Z}$ , and so the RHS is also in  $\mathbb{Z}$ ; that is:  $r(m^2s)^2 \in \mathbb{Z}$ . Since  $r$  is square free, we must therefore have  $m^2s \in \mathbb{Z}$ . Define:  $n = m^2s \in \mathbb{Z}$ . Then our equation becomes:

$$r^2\ell^4 + a_1r\ell^2m^2 + b_1m^4 = rn^2, \quad \text{for some } \ell, m, n \in \mathbb{Z}, \gcd(\ell, m) = 1. \quad (\dagger)$$

We want to show that  $r|b_1$ , and we know that  $r$  is square free. It is sufficient to show, for any prime  $p$ , that  $p|r \Rightarrow p|b_1$ .

Imagine  $p|r$  and  $p \nmid b_1$ , for some prime  $p$ . Then  $p|r^2\ell^4, a_1r\ell^2m^2, rn^2$  and so by  $(\dagger)$ ,  $p|b_1m^4$ , which in turn gives:  $p|m$  [since  $p \nmid b_1$ ]. Hence, since now  $p|r$  and  $p|m$ , we have:  $p^2|r^2\ell^4, a_1r\ell^2m^2, b_1m^4$ , and so by  $(\dagger)$ ,  $p^2|rn^2$ , which in turn gives:  $p|n$  [since  $r$  is square free]. Hence, since now  $p|r, m, n$ , we have:  $p^3|a_1r\ell^2m^2, b_1m^4, rn^2$ , and so by  $(\dagger)$ ,  $p^3|r^2\ell^4$ , which in turn gives:  $p|\ell$  [since  $r$  is square free]. This is a contradiction, since  $p|\ell$  and  $p|m$  but  $\gcd(\ell, m) = 1$ .

The above assumption that  $p|r$  and  $p \nmid b_1$  led to a contradiction, and so it is impossible for any prime  $p$  to satisfy  $p|r$  and  $p \nmid b_1$ . This is the same as saying that  $p|r \Rightarrow p|b_1$  for any prime  $p$ . Since  $r$  is square free, we conclude that  $r|b_1$ , as required [13 marks]

(ii) Let  $\mathcal{C} : Y^2 = X(X^2 + aX + b) = X(X^2 + 2X + 3)$ , where  $a = 2, b = 3$ , and isogenous curve  $\mathcal{D} : Y^2 = X(X^2 + a_1X + b_1) = X(X^2 - 4X - 8)$ , where  $a_1 = -4, b_1 = -8$ , with the usual isogeny  $\phi : \mathcal{C}(\mathbb{Q}) \rightarrow \mathcal{D}(\mathbb{Q}) : (x, y) \mapsto (y^2/x^2, y - 3y/x^2)$ , and dual isogeny  $\hat{\phi} : \mathcal{D}(\mathbb{Q}) \rightarrow \mathcal{C}(\mathbb{Q}) : (u, v) \mapsto (\frac{1}{4}v^2/u^2, \frac{1}{8}(v + 8v/u^2))$ .

The map  $q : \mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q})) \rightarrow \mathbb{Q}^*/(\mathbb{Q}^*)^2 : (u, v) \mapsto u$ , for  $(u, v) \neq (0, 0)$ , with  $q : (0, 0) \mapsto b_1$  and  $q : \mathbf{o} \mapsto 1$ , is an injection with  $\text{im } q$  contained in  $\{d : d \text{ is square free and } d|b_1\} = \{\pm 1, \pm 2\}$ . Also,  $\mathbf{o} \mapsto 1, (0, 0) \mapsto -8 = -2$ , so that  $\{1, -2\} \subset \text{im } q \subset \{\pm 1, \pm 2\}$ .

There is only one coset to check, represented by  $-1$ , say. We know that  $-1 \in \text{im } q$  iff there are integers  $\ell, m, n$ , not all 0, and with  $\gcd(\ell, m) = 1$ , such that:  $(-1) \cdot \ell^4 + a_1\ell^2m^2 + (b_1/(-1)) \cdot m^4 = n^2$  that is:  $-\ell^4 - 4\ell^2m^2 + 8m^4 = n^2$ . Rewrite as:  $-(\ell^2 + 2m^2)^2 + 12m^4 = n^2$ . Reducing modulo 3 gives  $-(\ell^2 + 2m^2)^2 \equiv n^2 \pmod{3}$ . If  $n$  were coprime to 3, then this would give:  $((\ell^2 + 2m^2)/n)^2 = -1$  in  $\mathbb{F}_3$ , contradicting the fact that  $-1$  is not a quadratic residue modulo 3. So,  $3|n$  and so  $3|(\ell^2 + 2m^2)$  also. This means that  $9|(\ell^2 + 2m^2)^2$  and  $9|n^2$ , which can be combined with  $-(\ell^2 + 2m^2)^2 + 12m^4 = n^2$  to give:  $9|12m^4$  and so  $3|m$ . Combining  $3|m$  with  $3|(\ell^2 + 2m^2)$  gives that  $3|\ell$ . This contradicts the fact that

$\gcd(\ell, m) = 1$ . Hence our equation is impossible in  $\mathbb{Q}_3$ , and so impossible in  $\mathbb{Q}$ . Hence  $-1 \notin \text{im} q$ .

We conclude that  $\text{im} q = \{1, -2\}$ , and so  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q}))$  is generated by  $(0, 0)$ .

The map  $\hat{q} : \mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q})) \rightarrow \mathbb{Q}^*/(\mathbb{Q}^*)^2 : (x, y) \mapsto x$ , for  $(x, y) \neq (0, 0)$ , with  $\hat{q} : (0, 0) \mapsto b$  and  $\hat{q} : \mathbf{o} \mapsto 1$ , is an injection with  $\text{im} \hat{q}$  contained in  $\{d : d \text{ is square free and } d|b\} = \{\pm 1, \pm 3\}$ . Also,  $\mathbf{o} \mapsto 1$  and  $(0, 0) \mapsto 3$ , so that  $\{1, 3\} \subset \text{im} \hat{q} \subset \{\pm 1, \pm 3\}$ .

There is only one coset to check, represented by  $-1$ , say. We know that  $-1 \in \text{im} \hat{q}$  iff there are integers  $\ell, m, n$ , not all 0, and with  $\gcd(\ell, m) = 1$ , such that:  $(-1) \cdot \ell^4 + a\ell^2 m^2 + (b/(-1)) \cdot m^4 = n^2$ ; that is:  $-\ell^4 + 2\ell^2 m^2 - 3m^4 = n^2$ . Rewrite as:  $-(\ell^2 - m^2)^2 - 2m^4 = n^2$ . This is impossible in  $\mathbb{R}$  (the left hand side is  $\leq 0$  and the right hand side is  $\geq 0$ , and equality only occurs when  $\ell^2 - m^2 = m^4 = n^2 = 0$ , implying  $\ell = m = n = 0$ , which is not allowed). Hence  $-1 \notin \text{im} \hat{q}$ .

We conclude that  $\text{im} \hat{q} = \{1, 3\}$ , and so  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q}))$  is generated by  $(0, 0)$ .

Finally, since multiplication by 2 in  $\mathcal{C}(\mathbb{Q})$  is  $\hat{\phi} \circ \phi$ , we have that  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is generated by: generators for  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q}))$  [namely:  $(0, 0)$ ] together with the images under  $\hat{\phi}$  of generators for  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q}))$  [namely,  $\hat{\phi}((0, 0)) = \mathbf{o}$ ]. Conclusion:  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is generated by  $(0, 0)$ , and so is isomorphic to  $C_2$ . We also know that  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is isomorphic to  $\mathcal{C}_{\text{tors}}(\mathbb{Q})/2\mathcal{C}_{\text{tors}}(\mathbb{Q}) \times C_2^r$ , which is isomorphic to  $\mathcal{C}(\mathbb{Q})[2] \times C_2^r$ , where  $\mathcal{C}(\mathbb{Q})[2]$  is the 2-torsion group and  $r$  is the rank. The 2-torsion points on  $\mathcal{C} : Y^2 = X(X^2 + 2X + 3)$  are  $\mathbf{o}$  together with the points of the form  $(x, 0)$ , where  $x$  is a root of  $X(X^2 + 2X + 3)$ , that is:  $(0, 0)$ ,  $(-1 + \sqrt{-2}, 0)$  and  $(-1 - \sqrt{-2}, 0)$ , of which only  $\mathbf{o}$  and  $(0, 0)$  are in  $\mathcal{C}(\mathbb{Q})[2]$ , giving that  $\mathcal{C}(\mathbb{Q})[2]$  is isomorphic to  $C_2$ . Combining this with the facts (already found) that  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is isomorphic both to  $C_2$  and to  $\mathcal{C}(\mathbb{Q})[2] \times C_2^r$ , give that  $\mathcal{C}(\mathbb{Q})$  has rank 0.

[12 marks]

#### Question 4.

(i). First compute (modulo  $N = 10001$ ):  $2^1 \equiv 2$ ,  $2^2 \equiv 4$ ,  $2^4 \equiv 16$ ,  $2^8 \equiv 256$ ,  $2^{16} \equiv 5530$ ,  $2^{32} \equiv 7843$ ,  $2^{64} \equiv 6499$  (where each of these was obtained by squaring the previous one, and reducing modulo  $N$ ). Now, we write 68 in base 2:  $68 = 4 + 64$  and so  $2^{68} \equiv 2^4 2^{64} \equiv 16 \cdot 6499 \equiv 3974$  modulo  $N$ , so that  $2^{68} - 1 \equiv 3973$  modulo  $N$ .

Now, compute  $\gcd(3973, N)$  by Euclid's Algorithm:  $10001 = 2 \cdot 3973 + 2055$ ;  $3973 = 1 \cdot 2055 + 1918$ ;  $2055 = 1 \cdot 1918 + 137$ ,  $1918 = 14 \cdot 137 + 0$ . So, 137 is a factor of  $N$ . Compute  $10001/137 = 73$ , giving the factorisation  $N = 10001 = 73 \cdot 137$ . [5 marks]

(ii). Since  $N = 73 \cdot 137$ , we have  $\phi(N) = 72 \cdot 136 = 9792$ . Computing the gcd of  $d = 4451$  and  $\phi(N)$ , we see:  $9792 = 2 \cdot 4451 + 890$ ;  $4451 = 5 \cdot 890 + 1$ , so that  $\gcd(9792, 4451) = 1$ . Reversing the steps:  $1 = 4451 - 5 \cdot 890 = 4451 - 5 \cdot (9792 - 2 \cdot 4451) = -5 \cdot 9792 + 11 \cdot 4451$ . Hence, 11 is the inverse of 4451 modulo 9792. The decoding operation is therefore  $X \mapsto X^{11} \bmod N$ . Computing  $6847^{11} = ((6847^2)^2)^2 \cdot (6847^2 \cdot 6847) \equiv (6722^2)^2 \cdot (6722 \cdot 6847) \equiv 766^2 \cdot 932 \equiv 6698 \cdot 932 \equiv 1912$ . (modulo  $N = 10001$ ). Also:  $2577^{11} = ((2577^2)^2)^2 \cdot (2577^2 \cdot 2577) \equiv (265^2)^2 \cdot (265 \cdot 2577) \equiv 218^2 \cdot 2837 \equiv 7520 \cdot 2837 \equiv 2107$  (modulo  $N = 10001$ ). The decoded message is therefore: 1912, 2107; that is: SLUG. [5 marks]

(iii). We are given that  $A/2A$  is finite, so let  $A/2A = S = \{Q_1, \dots, Q_r\} \subset A$ . Let  $P$  be any element of  $A$ . Then  $P = Q_{i_1}$  in  $A/2A$  for some  $Q_{i_1} \in S$  and so we can write:  $P = 2P_1 + Q_{i_1}$ , for some  $P_1 \in A$ . Inductively, continue to write:  $P_1 = 2P_2 + Q_{i_2}, P_2 = 2P_3 + Q_{i_3}, \dots$ , where each  $P_j \in A$  and each  $Q_{i_j} \in S$ . Now:

$$h(P_j) \leq \frac{1}{4}(h(2P_j) + C_2) \text{ [by (2)]} = \frac{1}{4}(h(P_{j-1} - Q_{i_j}) + C_2) \leq \frac{1}{4}(2h(P_{j-1}) + C'_1 + C_2) \text{ [by (1)], where:}$$

$$C'_1 = \max\{C_1(-Q) : Q \in S\}. \text{ So, if } h(P_{j-1}) > (C'_1 + C_2)/2 \text{ then } h(P_j) < \frac{1}{4}(2h(P_{j-1}) + 2h(P_{j-1})) = h(P_{j-1}).$$

Imagine that  $h(P) > (C'_1 + C_2)/2$  and  $h(P_j) > (C'_1 + C_2)/2$  for all  $j$ . Then  $h(P), h(P_1), h(P_2), \dots$  would be a strictly decreasing sequence, giving infinitely many distinct members of  $A$  with height  $\leq h(P)$ , which would contradict (3). This contradiction shows that there must exist an  $n$  such that  $h(P_n) \leq (C'_1 + C_2)/2$ . So, we can write:  $P = 2P_1 + Q_{i_1} = 2(2P_2 + Q_{i_2}) + Q_{i_1} = \dots$ , and after  $n$  steps  $P$  will be written as a linear combination of  $P_n$  and members of  $S$ . Let  $T = \{Q \in A : h(Q) \leq (C'_1 + C_2)/2\}$ . We have shown (since  $P_n \in T$ ) that any  $P \in A$  is a linear combination of members of  $S \cup T$ . Furthermore,  $T$  is finite, by (3). In conclusion:  $A$  is generated by the finite set  $S \cup T$ , and so is finitely generated. **[10 marks]**

(iv). Suppose that  $h(P) > \frac{1}{3}C_2$ . Then  $C_2 < 3h(P)$  and so by (2):  $h(2P) \geq 4h(P) - C_2 > 4h(P) - 3h(P) = h(P)$ . Inductively:  $h(P) < h(2P) < h(4P) < \dots$ , so that  $P, 2P, 4P, \dots$  are all distinct, which forces  $P$  to have infinite order (since otherwise  $P$  would generate a finite group). Hence, if  $P$  is a torsion element then  $h(P) \leq \frac{1}{3}C_2$ . **[5 marks]**

## Solutions to the 2006 Examination Questions.

### Question 1.

(i) [Proof from lectures]. Define  $f_j(x)$  by:

$$f(x+y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \dots,$$

so that  $f_0(x) = f(x), f_1(x) = f'(x)$ . Define  $b_0 = -f(a_0)/f'(a_0)$ . By (\*),  $|b_0| < 1$ .

Define  $a_1 = a_0 + b_0 = a_0 - f(a_0)/f'(a_0)$ . Then:

$$\begin{aligned} |f'(a_1) - f'(a_0)| &= |f'(a_0 + b_0) - f'(a_0)| = |(\text{poly in } a_0)b_0 + (\text{poly in } a_0)b_0^2 + \dots| \\ &\leq |b_0| < |f'(a_0)|, \end{aligned}$$

so that  $|f'(a_1)| = |f'(a_0)|$ .

$$\begin{aligned} \text{Also, } |f(a_1)| &= |f(a_0 + b_0)| = |f_0(a_0) + f_1(a_0)b_0 + f_2(a_0)b_0^2 + \dots| \\ &= |f_2(a_0)b_0^2 + \dots| \text{ [since } f_0(a_0) + f_1(a_0)b_0 = 0] \\ &\leq \max_{j \geq 2} |f_j(a_0)| |b_0|^j \leq |b_0|^2 = \frac{|f(a_0)|^2}{|f'(a_0)|^2} = \rho |f(a_0)| < |f(a_0)|, \end{aligned}$$

where  $\rho = \frac{|f(a_0)|}{|f'(a_0)|^2} < 1$ .

Summarising:  $|f'(a_1)| = |f'(a_0)|$  and  $|f(a_1)| \leq \rho |f(a_0)| < |f(a_0)|$ , where  $\rho = \frac{|f(a_0)|}{|f'(a_0)|^2} < 1$ .

Given  $a_n \in R$ , define  $b_n = -f(a_n)/f'(a_n)$  and  $a_{n+1} = a_n + b_n = a_n - f(a_n)/f'(a_n)$ .

Assume, as induction hypothesis, that:

$$|f'(a_n)| = \dots = |f'(a_1)| = |f'(a_0)| \text{ and } |f(a_n)| \leq \rho |f(a_{n-1})| \leq \dots \leq \rho^n |f(a_0)|. \quad (1)$$

Then, as above:  $|f'(a_{n+1})| = \dots = |f'(a_1)| = |f'(a_0)|$ .

Then  $|f(a_{n+1})| \leq |b_n|^2$  [justified as for the case  $n = 0$  above]

$$= \frac{|f(a_n)|^2}{|f'(a_n)|^2} = \frac{|f(a_n)|^2}{|f'(a_0)|^2} \quad [\text{by (1), the induction hypothesis}]$$

$$\leq \frac{|f(a_0)|^2}{|f'(a_0)|^2} |f(a_0)| \quad [\text{since } |f(a_n)| \leq |f(a_0)| \text{ by (1), the induction hypothesis}]$$

hypothesis]

$$= \rho |f(a_0)| \leq \rho^{n+1} |f(a_0)| \quad [\text{by (1), the induction hypothesis}].$$

By induction,  $\forall n$ ,  $|f'(a_n)| = |f'(a_0)|$  and  $|f(a_n)| \leq \rho^n |f(a_0)|$  which  $\rightarrow 0$  as  $n \rightarrow \infty$ . (2)

Now,  $|b_n| = |f(a_n)|/|f'(a_n)| = |f(a_n)|/|f'(a_0)| \rightarrow 0$ , so [by the theorem from lectures that a series converges in a non-Archimedean field iff its terms converge to 0]:

$$a_n = a_0 + b_0 + b_1 + \dots + b_n \text{ converges to } a, \text{ say.}$$

By continuity of polynomials,  $f(a) = \lim f(a_n) = 0$  [by (2)].

Furthermore:  $|a - a_0| = |\sum b_n| \leq \max |b_n| = \max \frac{|f(a_n)|}{|f'(a_n)|} = \max \frac{|f(a_n)|}{|f'(a_0)|} = \frac{|f(a_0)|}{|f'(a_0)|}$ , as required.

Imagine that  $\hat{a} \neq a$  also satisfied  $f(\hat{a}) = 0$  and  $|\hat{a} - a_0| \leq |f(a_0)|/|f'(a_0)|$ . Let  $\hat{b} = \hat{a} - a$ .

$$\text{Then } 0 = f(\hat{a}) - f(a) = f(a + \hat{b}) - f(a) = \hat{b}f_1(a) + \hat{b}^2f_2(a) + \dots \quad (3)$$

But  $|\hat{b}| = |\hat{a} - a_0 + a_0 - a| \leq \max(|\hat{a} - a_0|, |a - a_0|) \leq |f(a_0)|/|f'(a_0)|$

$$< |f'(a_0)| = |f_1(a_0)| = |f_1(a)| \quad [\text{by (1) and continuity of } |f'(x)|].$$

This gives  $|\hat{b}^j f_j(a)| \leq |\hat{b}^j| \leq |\hat{b}^2| < |\hat{b}f_1(a)|$  for  $j \geq 2$ , so that the leading term of the sum in (3) has valuation strictly greater than the valuations of the other terms, which is inconsistent with the sum being 0. Hence  $a$  is unique. **[10 marks]**

(ii) [They have seen a variation on an exercise sheet, although this one is substantially harder]. For  $p = 2$ , can see the equation becomes  $-y^2 \equiv 2 \pmod{4}$ , which is impossible, since  $y \equiv 0, 1, 2, 3$  give  $y^2 \equiv 0, 1, 0, 1$ , respectively. Hence, there are no such  $x, y \in \mathbb{Z}_2$ . For  $p = 3$ , take  $x = 1$ , in which case we want  $y \in \mathbb{Z}_3$  such that  $3y^2 = -6$ , that is:  $f(y) = y^2 + 2 = 0$ . But  $a_0 = 1$  gives  $|f(a_0)|_3 = 3^{-1}$  and  $|f'(a_0)| = 1$ , so by Hensel's Lemma, there exists  $y \in \mathbb{Z}_3$  which is a root of  $f$ . For  $p = 5$ , take  $x = 2$ , in which case we want  $y \in \mathbb{Z}_5$  such that  $g(y) = 3y^2 - 22 = 0$ . But  $a_0 = 2$  gives  $|g(a_0)|_5 = 5^{-1}$  and  $|g'(a_0)| = 1$ , so by Hensel's Lemma, there exists  $y \in \mathbb{Z}_5$  which is a root of  $g$ . For  $p = 7$ , take  $y = 0$ , in which case we want  $x \in \mathbb{Z}_7$  such that  $h(x) = (4x^3 - 10)/3 = 0$ . But  $a_0 = -1$  gives  $|h(a_0)|_7 = 7^{-1}$  and  $|h'(a_0)| = 1$ , so by Hensel's Lemma, there exists  $x \in \mathbb{Z}_7$  which is a root of  $h$ . For  $p \geq 11$ , the curve is an elliptic curve mod  $p$  [since the only bad primes are 2, 3, 5] and  $\#\tilde{\mathcal{E}}(\mathbb{F}_p) \geq p + 1 - 2\sqrt{p} > 5$ . At most 4 of these are 2-torsion, so there exists  $x_0$  such that  $h(x_0)$  is a nonzero residue mod  $p$ ; applying Hensel's Lemma to  $y^2 - h(x_0)$  gives the existence of a corresponding  $y \in \mathbb{Z}_p$ . In summary, there exist such  $x, y \in \mathbb{Z}_p$  for all  $p$  except 2. **[7 marks]**

(iii) [Unseen]. We may apply the general rule from lectures (which follows easily from Hensel's Lemma) that, if  $p \neq 2$  and  $|\alpha|_p = 1$ , then  $\alpha$  is a square in  $\mathbb{Q}_p^*$  iff  $\alpha$  is a quadratic residue in  $\mathbb{F}_p$ . An element  $\alpha \in (\mathbb{Q}_p^*)^2$  iff (both  $|\alpha|_p = p^r$  for  $r$  even and the leading digit  $a_r$  of  $\alpha$  is a quadratic residue mod  $p$ ) [N.B. if  $|\alpha|_p = p^r$  for  $r$  odd, then  $\alpha \neq \beta^2$  for any  $\beta$ , since  $|\beta|_p = p^s$  for some integer  $s$ ]. Let  $H_1 = (\mathbb{Q}_p^*)^2$ , and fix a quadratic nonresidue

mod 5, say  $\gamma$ . Let  $H_2 = \gamma H_1$ ,  $H_3 = p H_1$ ,  $H_4 = \gamma p H_1$ . Since any nonresidue  $= \gamma(\text{residue})$  and vice versa, these give all the cases when ( $a_r$  is a residue and  $r$  is even) [H1], ( $a_r$  is a nonresidue and  $r$  is even) [H2], ( $a_r$  is a residue and  $r$  is odd) [H3], ( $a_r$  is a nonresidue and  $r$  is odd) [H4], which partition  $\mathbb{Q}_p^*$ . Hence there are precisely 4 distinct elements in  $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ , when  $p \neq 2$ .

Similarly, 1, 3, 5, 7 are all distinct in  $(\mathbb{Q}_2^*)^2$ , since for  $|\alpha|_2 = 1$ , Hensel's Lemma gives that  $\alpha \in (\mathbb{Q}_2^*)^2$  iff  $\alpha \equiv 1 \pmod{8}$ . Furthermore, any  $\alpha \equiv 1$  is equal to precisely one of 1, 3, 5, 7 in  $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ . For any  $\alpha$ , if  $|\alpha|_2 = 2^r$  with  $r$  even, then  $\alpha = \alpha \cdot 2^r = 1, 3, 5$  or  $7$  in  $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ , since  $|\alpha \cdot 2^r|_2 = 1$ . If  $r$  is odd, then  $\alpha/2 = 2^r \alpha = 1, 3, 5$  or  $7$  in  $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ , and so  $\alpha = 2, 6, 10$  or  $14$  in  $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ . So, there are 8 elements when  $p = 2$ , represented, say, by 1, 3, 5, 7, 2, 6, 10, 14 (generated by 3, 5, 2). **[8 marks]**

**Question 2.** Let  $R$  be any ring, and let  $F, G$  be formal groups over  $R$ .

(i) [*Proof from lectures*]. Let  $P(T) = F_X(0, T)^{-1}$ , so that  $\omega(T) = P(T)dT$ . Note that  $F_X(0, T) = 1 + \dots$  is invertible, so that  $P(T)$  is indeed a member of  $R[[T]]$ . Furthermore,  $P(0) = 1$ , so that it is normalised.

We need to show that  $\omega$  is an invariant differential; that is,  $\omega \circ F(T, S) = \omega(T)$ , which is equivalent to:  $P(F(T, S))F_X(T, S) = P(T)$  so, in our case, it is sufficient to show:

$$F_X(0, F(T, S))^{-1} F_X(T, S) = F_X(0, T)^{-1},$$

which is true iff:

$$F_X(0, F(T, S)) = F_X(T, S)F_X(0, T).$$

But this last statement is immediate from differentiating  $F(U, F(T, S)) = F(F(U, T), S)$  [associativity] with respect to  $U$  to get:  $F_X(U, F(T, S)) = F_X(F(U, T), S)F_X(U, T)$  and setting  $U = 0$ . Hence  $\omega$  is an invariant differential.

Suppose that  $\hat{\omega} = Q(T)dT \in R[[T]]dT$  is also an invariant differential, so that  $Q(T)$  satisfies  $Q(F(T, S))F_X(T, S) = Q(T)$ . Substituting  $T = 0$  gives  $Q(S)F_X(0, S) = Q(0)$ , so that  $Q(S) = Q(0)F_X(0, S)^{-1}$ . It follows that  $\hat{\omega} = a\omega$ , where  $a = Q(0)$ . **[9 marks]**

(ii) [*Proof from lectures*]. First, note that  $\omega_G \circ f(F(T, S)) = \omega_G(G(f(T), f(S))) = \omega_G \circ f(T)$ , so that  $\omega_G \circ f$  is an invariant differential on  $G$ . From part (a), it follows that  $\omega_G \circ f = a\omega_F$ , for some  $a \in R$ . Since  $\omega_F, \omega_G$  are normalised,  $(1 + \dots)df(T) = a(1 + \dots)dT$ , and so  $(1 + \dots)f'(T)dT = a(1 + \dots)dT$ ; equating constant terms gives  $a = f'(0)$ , as required.

Let  $\omega$  be the normalised invariant differential on  $F$ . Since  $[p](T) = pT + \dots$ , it satisfies  $[p]'(0) = p$ . Applying the previous result to  $[p]$ , a homomorphism from  $F$  to itself, gives:  $\omega \circ [p] = [p]'(0)\omega = p\omega$ , and so

$$p\omega(T) = \omega \circ [p](T) = (1 + \dots)d([p](T)) = (1 + \dots)[p]'(T)dT.$$

Hence  $[p]'(T) \in pR_T$ . Each term  $a_n T^n$  in  $[p](T)$  must then satisfy  $p|na_n$  and so  $p|n$  or  $p|a_n$ , as required. **[8 marks]**

(iii) [*Unseen*]. Let  $\mathcal{E}$  be the given curve, which is of the form  $Y^2 = X^3 - M^2X + N^2$ , where  $M = m^2 + 1$  and  $N = 3n$ . First note that for  $m \equiv 0, 1, 2 \pmod{3}$ ,  $M \equiv$

$1, 2, 2 \pmod{3}$ , so that  $M$  is never divisible by 3. Hence  $\Delta = 4(-M^2)^3 + 27(N^2)^2 = -4M^6 + 27N^4$  is never 0, and so the curve is non-singular and is an elliptic curve. It has the obvious point  $(0, N)$ . Let  $P = 2(0, N) \in \mathcal{E}(\mathbb{Q})$ . The tangent to  $\mathcal{E}$  at  $(0, N)$  has slope  $(3 \cdot 0^2 - M^2)/(2 \cdot N) = -M^2/2N$ , and so the  $x$ -coordinate of  $P = 2(0, N)$  is  $(-M^2/2N)^2 - 2 \cdot 0$ , which is not an integer, since the numerator is not divisible by 3, but the denominator  $N = 3n$  is divisible by 3. Therefore  $P$  is not a torsion point (using the result from lectures that any torsion point in  $\mathcal{E}(\mathbb{Q})$  must have both coordinates in  $\mathbb{Z}$ ), and so must be of infinite order, giving that  $\mathcal{E}(\mathbb{Q})$  is infinite, as required. **[8 marks]**

### Question 3.

(i) [Seen similar on an exercise sheet]. The line tangent to  $\mathcal{E}$  at  $P = (1, 1)$  has slope  $y'$  given by  $2yy' = 3x^2 + 7$ , with  $x = 1, y = 1$ ; that is, the slope is  $10/2 = 5$ . This tangent line also goes through  $(1, 1)$  and so has equation:  $Y = 5X - 4$ . The  $x$ -coordinate of  $2P$  is therefore  $5^2 - (1 + 1) = 23$ , and the  $y$ -coordinate is:  $-(5 \cdot 23 - 4) = -111 \equiv 1406$ , so that  $Q = 2P \equiv (23, 1406)$  (modulo  $N = 1517$ ). We now wish to compute  $4P = 2Q = 2(23, 1406)$ , and so the first step is to find the line tangent to  $\mathcal{E}$  at  $Q$ . This has slope  $y'$  given by  $2yy' = 3x^2 + 7$ , with  $x = 23, y = 1406$ , that is,  $y' = 1594/2812$ . In order to compute this modulo  $N = 1517$ , we must first find the inverse of 2812 modulo 1517. Using Euclid's Algorithm:  $2812 = 1 \cdot 1517 + 1295$ ,  $1517 = 1 \cdot 1295 + 222$ ,  $1295 = 5 \cdot 222 + 185$ ,  $222 = 1 \cdot 185 + 37$ ,  $185 = 5 \cdot 37 + 0$ . So, we cannot find the inverse of 2812 (modulo  $N = 1517$ ), and this step has given us a proper factor 37 of  $N$ . **[8 marks]**

(ii) [Seen similar on an exercise sheet]. The elliptic curve has  $\Delta = 4A^3 + 27B^2 = 4 \cdot (-2)^3 + 27 \cdot 0^2 = -32$ , and so has good reduction at all primes  $p$  except  $p = 2$ . Taking  $p = 3$ , we find that  $\mathcal{E}(\mathbb{F}_3) = \{\mathbf{o}, (0, 0), (2, 1), (2, 2)\}$ , and since  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$  is isomorphic to a subgroup of  $\mathcal{E}(\mathbb{F}_3)$ , this gives that  $\#\mathcal{E}_{\text{tors}}(\mathbb{Q}) \mid 4$ . Taking  $p = 5$ , we find that  $\mathcal{E}(\mathbb{F}_5) = \{\mathbf{o}, (0, 0), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1), (3, 4), (4, 1), (4, 4)\}$ , and since  $\mathcal{E}_{\text{tors}}(\mathbb{Q})$  is isomorphic to a subgroup of  $\mathcal{E}(\mathbb{F}_5)$ , this gives that  $\#\mathcal{E}_{\text{tors}}(\mathbb{Q}) \mid 10$ . Hence,  $\#\mathcal{E}_{\text{tors}}(\mathbb{Q}) \mid \gcd(4, 10) = 2$ . However, there are the obvious torsion points:  $\mathbf{o}$  and  $(0, 0)$  and so  $\#\mathcal{E}_{\text{tors}}(\mathbb{Q}) = \{\mathbf{o}, (0, 0)\}$ . **[7 marks]**

(iii) [Unseen]. Using  $2YY' = 3X^2$ , we see that the slope of the curve at any  $(x, y)$  is  $3x^2/(2y)$ , and so the tangent to the curve at  $(x, y)$  is the line:  $Y = \left(\frac{3x^2}{2y}\right)X + c$ , for some  $c$ . The  $x$ -coordinate of the third point of intersection is then:  $\left(\frac{3x^2}{2y}\right)^2 - x - x = \frac{9x^4 - 8xy^2}{4y^2} = \frac{9x^4 - 8x(x^3 + k)}{4(x^3 + k)} = \frac{x(x^3 - 8k)}{4(x^3 + k)}$ . Furthermore  $3(x, y) = \mathbf{o} \iff 2(x, y) = -(x, y) = (x, -y) \iff \left(\frac{x(x^3 - 8k)}{4(x^3 + k)}, ?\right) = (x, -y) \iff \frac{x(x^3 - 8k)}{4(x^3 + k)} = x$  [the  $\Rightarrow$  of the last step is trivial; for  $\Leftarrow$ , note that if  $2(x, y)$  has that same  $x$ -coordinate as  $(x, y)$  then  $2(x, y) = (x, -y)$  or  $(x, y)$ ; but the latter would imply  $(x, y) = \mathbf{o}$ , which we have excluded]. Note further that this last condition on  $x$  is satisfied when  $x(x^3 - 8k) = 4x(x^3 + k)$ , that is:  $x(x^3 + 4k) = 0$ . This gives on  $\mathcal{E}_k(\mathbb{C})$  the points  $(0, \sqrt{k})$  and  $(-\sqrt[3]{4k}, i\sqrt{3k})$ . But it is impossible for both  $\sqrt{k}$  and  $i\sqrt{3k}$  to be in  $\mathbb{Q}$  (indeed they cannot both be in  $\mathbb{R}$ ), so that at least one of these points of order 3 is not in  $\mathcal{E}_k(\mathbb{Q})$ . **[10 marks]**

**Question 4.**

(i) [Seen similar on an Exercise Sheet]. Let  $\mathcal{C} : Y^2 = X(X^2 + aX + b) = X(X^2 + 3X + 5)$ , where  $a = 3, b = 5$ , and isogenous curve  $\mathcal{D} : Y^2 = X(X^2 + a_1X + b_1) = X(X^2 - 6X - 11)$ , where  $a_1 = -2a = -6, b_1 = a^2 - 4b = -11$ , with the usual isogeny  $\phi : \mathcal{C}(\mathbb{Q}) \rightarrow \mathcal{D}(\mathbb{Q}) : (x, y) \mapsto (y^2/x^2, y - 5y/x^2)$ , and dual isogeny  $\hat{\phi} : \mathcal{D}(\mathbb{Q}) \rightarrow \mathcal{C}(\mathbb{Q}) : (u, v) \mapsto (\frac{1}{4}v^2/u^2, \frac{1}{8}(v + 11v/u^2))$ .

The map  $q : \mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q})) \hookrightarrow \mathbb{Q}^*/(\mathbb{Q}^*)^2 : (u, v) \mapsto u$ , for  $(u, v) \neq (0, 0)$ , with  $q : (0, 0) \mapsto b_1$  and  $q : \mathbf{o} \mapsto 1$ , is an injection with  $\text{im}q$  contained in  $\{d : d \text{ is square free and } d|b_1\} = \{\pm 1, \pm 11\}$ . Also,  $\mathbf{o} \mapsto 1, (0, 0) \mapsto -11$  and the obvious point  $(-1, 2) \mapsto -1$ , so that  $1, -11, -1 \in \text{im}q$ ; but  $\text{im}q$  is a group, so  $11 \in \text{im}q$  also, and indeed we can just take:  $(0, 0) + (-1, 2) = (11, 22)$ , which maps to 11 under  $q$ . Hence,  $\{\pm 1, \pm 11\} \subset \text{im}q \subset \{\pm 1, \pm 11\}$ , that is  $\text{im}q = \{\pm 1, \pm 11\}$ , and so  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q})) = \{\mathbf{o}, (0, 0), (-1, 2), (11, 22)\}$ , that is,  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q}))$  is generated by  $(0, 0), (-1, 2)$ .

The map  $\hat{q} : \mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q})) \hookrightarrow \mathbb{Q}^*/(\mathbb{Q}^*)^2 : (u, v) \mapsto u$ , for  $(u, v) \neq (0, 0)$ , with  $\hat{q} : (0, 0) \mapsto a_1^2 - 4b_1 = b$  and  $\hat{q} : \mathbf{o} \mapsto 1$ , is an injection with  $\text{im}\hat{q}$  contained in  $\{d : d \text{ is square free and } d|b\} = \{\pm 1, \pm 5\}$ . Also,  $\mathbf{o} \mapsto 1$  and  $(0, 0) \mapsto 5$ , so that  $\{1, 5\} \subset \text{im}\hat{q} \subset \{\pm 1, \pm 5\}$ .

There is only one coset to check, represented by  $-1$ , say. We know that  $-1 \in \text{im}\hat{q}$  iff there are integers  $\ell, m, n$ , not all 0, and with  $\gcd(\ell, m) = 1$ , such that:  $(-1) \cdot \ell^4 + a\ell^2m^2 + (b/(-1)) \cdot m^4 = n^2$ ; that is:  $-\ell^4 + 3\ell^2m^2 - 5m^4 = n^2$ . Multiply both sides by 4 and rewrite as:  $-(2\ell^2 - 3m^2)^2 - 11m^4 = 4n^2$ . We can see that the LHS is  $\leq 0$  and the RHS is  $\geq 0$ , and the equation is satisfied iff  $(2\ell^2 - 3m^2) = 11m^4 = 4n^2 = 0$ . From this we see that  $m = n = 0$ , which when combined with  $2\ell^2 - 3m^2 = 0$  gives that  $\ell = 0$ , also, which is a contradiction. Hence  $-1 \notin \text{im}\hat{q}$ .

We conclude that  $\text{im}\hat{q} = \{1, 5\}$ , and so  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q}))$  is generated by  $(0, 0)$ .

Finally, since multiplication by 2 in  $\mathcal{C}(\mathbb{Q})$  is  $\hat{\phi} \circ \phi$ , we have that  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is generated by: generators for  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q}))$  [namely:  $(0, 0)$ ] together with the images under  $\hat{\phi}$  of generators for  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q}))$  [namely,  $\hat{\phi}((0, 0)) = \mathbf{o}$  and  $\hat{\phi}((-1, 2)) = (1, 3)$ ]. Conclusion:  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is generated by  $(0, 0)$  and  $(1, 3)$  and so is the group  $C_2 \times C_2$ . Now  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is isomorphic to  $\mathcal{C}_{\text{tors}}(\mathbb{Q})/2\mathcal{C}_{\text{tors}}(\mathbb{Q}) \times C_2^{\text{rank}}$ , and  $\mathcal{C}_{\text{tors}}(\mathbb{Q})/2\mathcal{C}_{\text{tors}}(\mathbb{Q})$  is isomorphic to the 2-torsion group of  $\mathcal{C}_{\text{tors}}(\mathbb{Q})$  which is  $C_2$  (consisting only of  $\mathbf{o}$  and  $(0, 0)$ ), so that  $\mathcal{C}_{\text{tors}}(\mathbb{Q})/2\mathcal{C}_{\text{tors}}(\mathbb{Q})$  is isomorphic to  $C_2$ . Conclusion: rank = 1. [13 marks]

(ii) [Unseen]. Let  $\mathcal{C} : Y^2 = X(X^2 + aX + b) = X(X^2 + p^2)$ , where  $a = 0, b = p^2$ , and isogenous curve  $\mathcal{D} : Y^2 = X(X^2 + a_1X + b_1) = X(X^2 - 4p^2)$ , where  $a_1 = -2a = 0, b_1 = a^2 - 4b = -4p^2$ , with the usual isogeny  $\phi : \mathcal{C}(\mathbb{Q}) \rightarrow \mathcal{D}(\mathbb{Q}) : (x, y) \mapsto (y^2/x^2, y - p^2y/x^2)$ , and dual isogeny  $\hat{\phi} : \mathcal{D}(\mathbb{Q}) \rightarrow \mathcal{C}(\mathbb{Q}) : (u, v) \mapsto (\frac{1}{4}v^2/u^2, \frac{1}{8}(v + 4p^2v/u^2))$ .

The map  $q : \mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q})) \hookrightarrow \mathbb{Q}^*/(\mathbb{Q}^*)^2 : (u, v) \mapsto u$ , for  $(u, v) \neq (0, 0)$ , with  $q : (0, 0) \mapsto b_1$  and  $q : \mathbf{o} \mapsto 1$ , is an injection with  $\text{im}q$  contained in  $\{d : d \text{ is square free and } d|b_1\} = \{\pm 1, \pm 2, \pm p, \pm 2p\}$ . Also,  $\mathbf{o} \mapsto 1, (0, 0) \mapsto -1, (2p, 0) \mapsto 2p, (-2p, 0) \mapsto -2p$ , so that  $\{\pm 1, \pm 2p\} \subset \text{im}q \subset \{\pm 1, \pm 2, \pm p, \pm 2p\}$ . There is only one coset to check, represented by  $-2$ , say. We know that  $-2 \in \text{im}q$  iff there are integers  $\ell, m, n$ , not all 0, and with  $\gcd(\ell, m) = 1$ , such that:  $(-2) \cdot \ell^4 + a_1\ell^2m^2 + (b_1/(-2)) \cdot m^4 = n^2$ ; that is:  $-2\ell^4 + 2p^2m^4 = n^2$ . Since  $p \equiv 5 \pmod{8}$ ,  $(\frac{-2}{p}) = -1$ , and so  $p|\ell, p|n$ . Let  $\ell = p\ell'$ ,

$n = pn'$ . Then  $-2p^2(\ell')^4 + 2m^4 = (n')^2$ . Since also  $\left(\frac{2}{p}\right) = -1$ , this implies  $p|m$ ,  $p|n'$ . This gives a contradiction, since  $p|\ell$ ,  $p|m$  and  $\gcd(\ell, m) = 1$ . Hence  $-2 \notin \text{im}q$ , giving that  $\text{im}q = \{\pm 1, \pm 2p\}$ , and  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q})) = \{\mathbf{o}, (0, 0), (2p, 0), (-2p, 0)\}$ .

The map  $\hat{q} : \mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q})) \mapsto \mathbb{Q}^*/(\mathbb{Q}^*)^2 : (u, v) \mapsto u$ , for  $(u, v) \neq (0, 0)$ , with  $\hat{q} : (0, 0) \mapsto a_1^2 - 4b_1 = b$  and  $\hat{q} : \mathbf{o} \mapsto 1$ , is an injection with  $\text{im}\hat{q}$  contained in  $\{d : d \text{ is square free and } d|b\} = \{\pm 1, \pm p\}$ . Also,  $\mathbf{o} \mapsto 1$  and  $(0, 0) \mapsto p^2 = 1$ , so that  $\{1\} \subset \text{im}\hat{q} \subset \{\pm 1, \pm p\}$ .

We know that  $-1 \in \text{im}\hat{q}$  iff there are integers  $\ell, m, n$ , not all 0, and with  $\gcd(\ell, m) = 1$ , such that:  $(-1) \cdot \ell^4 + a\ell^2m^2 + (b/(-1)) \cdot m^4 = n^2$ ; that is:  $-\ell^4 - p^2m^4 = n^2$ , clearly impossible in  $\mathbb{R}$ . Hence  $-1 \notin \text{im}\hat{q}$ . Similarly,  $-p \notin \text{im}\hat{q}$ . We know that  $p \in \text{im}\hat{q}$  iff there are integers  $\ell, m, n$ , not all 0, and with  $\gcd(\ell, m) = 1$ , such that:  $p \cdot \ell^4 + a\ell^2m^2 + (b/p) \cdot m^4 = n^2$ ; that is:  $p\ell^4 + pm^4 = n^2$ , and so  $p|n$ . Letting  $n = pn'$ , the equation becomes  $\ell^4 + m^4 = p^2(n')^2$ , which in turn implies  $p|\ell$ ,  $p|m$ , since  $-1 \not\equiv a^4$  for any  $a$  [as can be seen by taking both sides of  $-1 \equiv a^4$  to the power of  $(p-1)/4$ ]. This again contradicts  $\gcd(\ell, m) = 1$ , and so  $p \notin \hat{q}$ .

We conclude that  $\text{im}\hat{q} = \{1\}$ , and so  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q})) = \{\mathbf{o}\}$ .

Finally, since multiplication by 2 in  $\mathcal{C}(\mathbb{Q})$  is  $\hat{\phi} \circ \phi$ , we have that  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is generated by: generators for  $\mathcal{C}(\mathbb{Q})/\hat{\phi}(\mathcal{D}(\mathbb{Q}))$  [namely:  $\mathbf{o}$ ] together with the images under  $\hat{\phi}$  of generators for  $\mathcal{D}(\mathbb{Q})/\phi(\mathcal{C}(\mathbb{Q}))$  [namely,  $\hat{\phi}((0, 0)) = \mathbf{o}$ , and  $\hat{\phi}((2p, 0)) = (0, 0)$ ]. Conclusion:  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is generated by  $(0, 0)$ , and so is the group  $C_2$ . Now  $\mathcal{C}(\mathbb{Q})/2\mathcal{C}(\mathbb{Q})$  is isomorphic to  $\mathcal{C}_{\text{tors}}(\mathbb{Q})/2\mathcal{C}_{\text{tors}}(\mathbb{Q}) \times C_2^{\text{rank}}$ , and  $\mathcal{C}_{\text{tors}}(\mathbb{Q})/2\mathcal{C}_{\text{tors}}(\mathbb{Q})$  is isomorphic to the 2-torsion group of  $\mathcal{C}_{\text{tors}}(\mathbb{Q})$  which is  $C_2$  (consisting only of  $\mathbf{o}$  and  $(0, 0)$ ), so that  $\mathcal{C}_{\text{tors}}(\mathbb{Q})/2\mathcal{C}_{\text{tors}}(\mathbb{Q})$  is isomorphic to  $C_2$ . Conclusion:  $\text{rank} = 0$ . **[12 marks]**