Elliptic Curves. Sheet 3. To be handed in during 4th Week.

1. Let K be a field with non-Archimedean valuation $|\ |$.

(a). For any $x, y \in K$ show that, if $|x| \neq |y|$ then $|x \pm y| = \max(|x|, |y|)$.
(b). If $x_1, \ldots, x_n \in K$ and if there exists ℓ such that $|x_\ell| > |x_i|$ for all $i \neq \ell$, then show that $|x_1 + \ldots + x_n| = |x_\ell|$.

(c). Suppose that $s_n \to s$ in $K, |\ |$. Show that $|s_n| \to |s|$ in $\mathbb{R}, |\ |_\infty$. When $s \neq 0$, show that there exists N such that, for all $n > N$, $|s_n| = |s|$.

2.
(a). Find: $|3/50|_5$, $|3/50|_3$, $|3/50|_7$, $d_5(2/3, 1/5)$, $d_7(2/3, 1/5)$, $d_{11}(2/3, 1/5)$.

(b). Describe $|3/7|_p$ for all p. What is the product $\prod |3/7|_i$, taken over $i = p$, for all primes p, and $i = \infty$? Given any $x \in \mathbb{Q}$ $(x \neq 0)$, what is $\prod |x|_i$?

3. Which of the following are convergent in $\mathbb{Q}_5$?
(**a**). $1/5^n$. (**b**). n. (**c**). $n!$ (**d**). $3 + 10^n$. (**e**). $\sum_0^\infty 10^n$. (**f**). $\sum_0^\infty 7^n$.

4. For each p, m, r, either find an $x \in \mathbb{Z}$ such that $|x^2 - r|_p \leqslant p^{-m}$ or show that no such x exists.

(a). $p = 5, r = -1, m = 4$. **(b).** $p = 3, r = 7/8, m = 7$. **(c).** $p = 5, r = 5/4, m = 4$.

5. Find the 7-adic expansion of each of: 200 and 3/14. Determine the member of $\mathbb{Q}$ expressed by the 5-adic expansion $2, \overline{34}$.

6. Let $x \in \mathbb{Q}$. Show that $x \in \mathbb{Z} \iff \big(x \in \mathbb{Z}_p \text{ for all } p\big)$.

The following question is compulsory for students taking the MSc in MFoCS (Mathematics and the Foundations of Computer Science). For everyone else, it is optional.

7. Show that $|n!|_p = p^{-M}$ where $M = \sum_{i=1}^\infty \left[\frac{n}{p^i}\right]$ (where $[x]$ denotes the greatest integer $\leqslant x$). Let K be a field containing $\mathbb{Q}_p$, let $|\ |$ be a non-Archimedean valuation on K which extends $|\ |_p$, and assume that K is complete with respect to this valuation. For any $x \in K$, show that $\exp_p(x) = \sum_{n=0}^\infty \frac{x^n}{n!}$ converges if and only if $|x| < p^{-\frac{1}{p-1}}$. When $K = \mathbb{Q}_p$ $(p \neq 2)$, show that $\exp_p(x)$ converges if any only if $|x|_p < 1$. When $K = \mathbb{Q}_2$, show that $\exp_2(x)$ converges if any only if $|x|_2 < \frac{1}{2}$.