Problem sheet (To be done in week 3)

Linear Algebra I, Dr A Henke, MT 2007

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Deadline for online homework: Sat Oct 20 18:00:00 2007
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From now on your online submission will be marked, and the points will be recorded. For every correct answer, you get 1 point; for every wrong answer, you get -1 points. For every question you decide not to answer (indicated by the option -) you get 0 points. Submit your answers via the web-interface. Before the deadline you are able to change your solutions if necessary and re-submit this changed solution again; the last submission counts.

1 Which of the following matrices are symmetric, are skew-symmetric, are orthogonal?

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, E = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$

$F = \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, H = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$

Is D orthogonal?	◯ Yes / ◯ No
Is F symmetric?	◯ Yes / ◯ No
Is E orthogonal?	◯ Yes / ◯ No

2 Let A and B be two square matrices of the same size with A symmetric and B skew-symmetric. Determine which of the following matrices are symmetric and which are skew-symmetric.

Is B^2 symmetric?	◯ Yes / ◯ No
Is $B^T(A^T + A)B$ skew-symmetric?	◯ Yes / ◯ No
Is $AB + BA$ symmetric?	◯ Yes / ◯ No

3 Let C and D be orthogonal square matrices of the same size with real entries. Let λ be some real number with $\lambda \neq 1$. Decide whether the following statements are always correct.

If the determinant of a matrix is one, then the matrix is orthogonal.	◯ Yes / ◯ No
The matrix C^{-1} is orthogonal.	◯ Yes / ◯ No
The matrix $\lambda \cdot C$ is orthogonal.	◯ Yes / ◯ No

4 All matrices occuring in this problem have real entries and all matrix products and sums are defined. We write I for the identity matrix and 0 for the zero matrix (of suitable sizes). Are the following statements correct?

Let A, B be matrices with $AB = 0$. Then $BA = 0$.	◯ Yes / ◯ No
Let A be an invertible matrix with $A^2 = A$. Then $A = I$.	◯ Yes / ◯ No
Let B be a matrix with $B^3 = B - I$. Then $B^{-1} = I - B^2$.	◯ Yes / ◯ No

5 Let I_n be the $n \times n$ identity matrix and let 0_n be the $n \times n$ zero matrix. Are the following statements about matrices correct?

Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$ and $B \in \mathbb{M}_{n\times m}(\mathbb{R})$ with $A \cdot B = I_m$, then $B \cdot A = I_n$.	◯ Yes / ◯ No
Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$ and $B \in \mathbb{M}_{n\times m}(\mathbb{R})$ with $A \cdot B \cdot A = A$. Then $A \cdot B = I_m$.	◯ Yes / ◯ No
Assume $A \in \mathbb{M}_{2\times 2}(\mathbb{R})$ with $A^2 = I_2$, then $A = \pm I_2$.	◯ Yes / ◯ No

Please submit your written solutions to the following problems to your college tutor. For this written part of your homework, the deadline set by your college tutor applies.

6	For each $a \in \mathbb{R}$, define the matrix $A(a)$ by $$A(a) = \begin{pmatrix} 1 & a & \frac{1}{2}a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}.$$ Show that for all $a, b \in \mathbb{R}$ we have $A(a+b) = A(a)A(b)$. Deduce that each matrix $A(a)$ is invertible.
7	Let A, B be square matrices of the same size. Show that the following statements are true. (a) If A is invertible then the inverse is unique. (b) If A, B are invertible then AB is also invertible.
8	Let A and B be two matrices such that AB and BA are defined and of the same size. We say that A and B commute with respect to multiplication if $AB = BA$. Now let A be a 2×2 matrix with entries in $\mathbb{R}$. (a) Show that A commutes with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ if and only if A is diagonal. (b) Show that A commutes with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if A is diagonal. (c) Which 2×2 matrices A commute with $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$? (d) Deduce that A commutes with all 2×2 matrices if and only if $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$. (For those of you who found this problem too easy: Find all $n \times n$ matrices which commute with any matrix $A \in \mathbb{M}_n(\mathbb{R})$ for fixed $n \in \mathbb{N}$. Justify your answer.)
9	(Optional.) Show that a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse if and only if $ad - bc \neq 0$. Find A^{-1}.