Problem sheet (To be done in week 6)

Linear Algebra I, Dr A Henke, MT 2007

Student Number: 654321
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The points of this online-homework will be recorded. For every correct answer, you get 1 point; for every wrong answer, you get -1 points. For every question you decide not to answer (indicated by the option -) you get 0 points. Submit your answers via the web-interface.

1	Given are subspaces U_i of the vector space $\mathbb{R}[X]$: $$U_1 = \mathrm{Span}\{X, X^2, 2X^2 - X\}, \quad U_2 = \mathrm{Span}\{1, X^2, 2X^2 - X + 1\}, \quad U_3 = \mathrm{Span}\{X^2, X^2 + 1\}.$$ Answer the following questions. Note that multiple answers are possible. Do not forget to unclick the option - if you like this problem to be graded.	
	For which $i \in \{1,2,3\}$ is $X^4 + X^2 + 1 \in U_i$?	☐ 1 / ☐ 2 / ☐ 3 / ☐ none of these
	For which $i \in \{1,2,3\}$ is $\dim U_i = 2$?	☐ 1 / ☐ 2 / ☐ 3 / ☐ none of these
	For which $i \in \{1,2,3\}$ is $U_i = \mathrm{Span}\{X, X^2, 2X^2 + 1\}$?	☐ 1 / ☐ 2 / ☐ 3 / ☐ none of these
2	Which dimension has the following vector space V?	
	$V = \mathrm{Span}\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\}$;	________
	$V = \mathrm{Span}\{(1, a, a^2) \mid a \in \mathbb{R}\}$;	________
	V is the set of all vectors (x, y, z) with $x, y, z \in \mathbb{R}$ and $x + 2y + 3z = 0$;	________
3	Let V be a vector space and let S, T be subsets of V and X, Y, Z subspaces of V. Decide whether the following statements are correct.	
	$\mathrm{Span}(S \cup T) \supseteq \mathrm{Span}(S) + \mathrm{Span}(T)$;	◯ Yes / ◯ No
	if $Y \subseteq X$, then $Y + (X \cap Z) \supseteq X \cap (Y + Z)$;	◯ Yes / ◯ No
	$\mathrm{Span}(S \cap T) \supseteq \mathrm{Span}(S) \cap \mathrm{Span}(T)$;	◯ Yes / ◯ No
4	Recall that $\mathbb{R}^{\mathbb{R}}$ denotes the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Given are $\mathbb{R}$-vector spaces V and W. In each case, decide whether the given map $\varphi : V \to W$ is linear.	
	$V := \mathbb{M}_{1\times 3}(\mathbb{R})$, $W := \mathbb{M}_{1\times 3}(\mathbb{R})$, $\varphi : (x, y, z) \mapsto (\lvert x + y\rvert, -z, 0)$	◯ Yes / ◯ No
	$V := \mathbb{M}_{2\times 3}(\mathbb{R})$, $W := \mathbb{M}_{1\times 3}(\mathbb{R})$, $\varphi : M \mapsto (1, 2) \cdot M$	◯ Yes / ◯ No
	$V := \mathbb{R}^{\mathbb{R}}$, $W := \mathbb{R}^{\mathbb{R}}$, $\varphi : f \mapsto f - f$	◯ Yes / ◯ No
5	Let V, W and U be vector spaces and $\varphi : V \to W$ and $\psi : W \to U$ linear maps. Are the following statements correct?	
	Let $v_1 \neq v_2$ be elements of V such that $\varphi(v_1) = \varphi(v_2)$, then $\{v_1, v_2\}$ in V are linearly dependent.	◯ Yes / ◯ No
	Let $v_1 \neq v_2$ be elements of V such that $\varphi(v_1) = \varphi(v_2) \neq 0$, then $\{v_1, v_2\}$ in V are linearly independent.	◯ Yes / ◯ No
	Let $v_1 \neq v_2$ be elements of V such that $\psi(\varphi(v_1)) = \psi(\varphi(v_2)) \neq 0$, then $\{\varphi(v_1), \varphi(v_2)\}$ in W are linearly dependent.	◯ Yes / ◯ No

Please submit your written solutions to the following problems to your college tutor. For this written part of your homework, the deadline set by your college tutor applies.	
6	Consider the vector space $\mathbb{R}^4$. Let $$\begin{aligned} U_1 &:= \text{Span}\{(1,1,3,2),(0,1,0,2)\}, \\ U_2 &:= \text{Span}\{(2,2,4,0),(2,-1,3,1),(2,1,1,1)\}. \end{aligned}$$ (a) Find a basis B of $U_1 \cap U_2$. (b) Use the exchange procedure by Steinitz to get a basis B_i of U_i with $B \subseteq B_i$ for $i = 1,2$. (c) Prove that $U_1 + U_2 = \mathbb{R}^4$.
7	Let V be a vector space of dimension n over $\mathbb{R}$. (a) Prove that for each r such that $0 \leq r \leq n$, V contains a subspace of dimension r. (b) Let U, W be subspaces of V with $U \subseteq W$. Show that there exists a subspace W' in V such that $W \cap W' = U$ and $W + W' = V$.
8	Let $S : U \to V$ and $T : U \to V$ be linear transformations between vector spaces U, V, let $\lambda \in \mathbb{R}$. Show that λT and $S + T$ are linear transformations.
9	(Optional.) A magical square is a table with nine digits with the following properties: the sum of all numbers in each row, and in each column, and in each diagonal is equal. This number is called the magical number. For example, 4 3 8 9 5 1 2 7 6 and the magical number is 15, the number in the center of the square is 5. Consider the set of all magical squares with entries from the set of real numbers $\mathbb{R}$. (a) Show that the magical squares form a vector space. (b) Show that the magical number is always three times the number in the center of the square. (c) Find a basis of the vector space of magical squares and determine its dimension.