Problem sheet (To be done in week 7)

Linear Algebra I, Dr A Henke, MT 2007

Student Number: 654321
Deadline for online homework: Sat Nov 17 18:00:00 2007
This file has been created: Wed Oct 10 16:41:24 2007

The points of this online-homework will be recorded. For every correct answer, you get 1 point; for every wrong answer, you get -1 points. For every question you decide not to answer (indicated by the option -) you get 0 points. Submit your answers via the web-interface.

1	Is there a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ with the requested properties?	
	$n = 4$, $m = 4$ and $\mathrm{Ker}(f) = \mathrm{Span}\{(1,0,0,0),(0,1,0,0)\}$.	◯ Yes / ◯ No
	$n = 3$, $m = 3$ and $\mathrm{Im}(f) = \mathbb{R}^3$.	◯ Yes / ◯ No
	$n = 3$, $m = 3$ and $\mathrm{Ker}(f) = \{0\}$ and $\mathrm{Im}(f) = \mathbb{R}^3$.	◯ Yes / ◯ No
2	Determine the following dimensions.	
	Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear map of rank 2. Then the dimension of the kernel of f is:	________
	Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be given by $T(x) = Ax$ for x a column vector in $\mathbb{R}^4$, and where A is the matrix $\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 0 & 3 & -2 & 0 \end{pmatrix}$. Then the rank of T is:	________
	Let $f : \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+2y \\ 3x-z \end{pmatrix}$. Then the rank of f is:	________
3	Let $V := M_{2,3}(\mathbb{R})$, $W := M_{2,2}(\mathbb{R})$ and $F : V \to W$ the following $\mathbb{R}$-linear map: $$F : V \longrightarrow W \quad , \quad M \longmapsto M \cdot A \quad , \quad \text{with } A = \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} \in M_{3,2}(\mathbb{R}).$$ Choose the bases $$\mathcal{B} := \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$ of V and $$\mathcal{C} := \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \right)$$ of W. Determine the matrix $M^{\mathcal{B}}_{\mathcal{C}}(F)$ of F with respect to the above bases, and enter the requested coefficients.	
	The entry in the 4. row and the 2. column of $M^{\mathcal{B}}_{\mathcal{C}}(F)$ is	________
	The entry in the 1. row and the 3. column of $M^{\mathcal{B}}_{\mathcal{C}}(F)$ is	________
	The entry in the 4. row and the 5. column of $M^{\mathcal{B}}_{\mathcal{C}}(F)$ is	________
4	Let V and W be two finite dimensional vector spaces and $\varphi : V \to W$ a linear map. Moreover, let $\mathcal{B} := (v_1, \ldots, v_n)$ be an (ordered) basis of V and $\mathcal{C} := (w_1, \ldots, w_m)$ be an (ordered) basis of W and $M^{\mathcal{B}}_{\mathcal{C}}(\varphi)$ the matrix corresponding to φ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$.	
	Let $\mathcal{B}' = (v_n, v_{n-1}, \ldots, v_1)$, then $M^{\mathcal{B}'}_{\mathcal{C}}(\varphi)$ is obtained from $M^{\mathcal{B}}_{\mathcal{C}}(\varphi)$ by writing the same rows in reversed order.	◯ Yes / ◯ No
	Let $\mathcal{B}' = (v_1, \ldots, v_{n-2}, v_n, v_{n-1})$, then $M^{\mathcal{B}'}_{\mathcal{C}}(\varphi)$ is obtained from $M^{\mathcal{B}}_{\mathcal{C}}(\varphi)$ by swapping the first two rows.	◯ Yes / ◯ No

	Let $C' = (w_1 + w_2, w_2, w_3, \ldots, w_m)$, then $M^{\mathcal{B}}_{C'}(\varphi)$ is obtained from $M^{\mathcal{B}}_{C}(\varphi)$ by adding the second row to the first row.	◯ Yes / ◯ No
5	Let V and W be (finitely generated) vector spaces and $\varphi : V \to W$ a linear map. Which of the following statements is correct?	
	Assume b_1 and b_2 are elements of V with $\{b_1, b_2\}$ linearly independent. If φ is injective, then $\{\varphi(b_1), \varphi(b_2)\}$ is linearly independent.	◯ Yes / ◯ No
	Assume that b_1 and b_2 are elements of V such that $\{\varphi(b_1), \varphi(b_2)\}$ is linearly independent, then (b_1, b_2) is linearly independent.	◯ Yes / ◯ No
	Assume that for every basis $\{b_1, \ldots, b_n\}$ of V we have that $\{\varphi(b_1), \ldots, \varphi(b_n)\}$ is a basis of W, then φ is an isomorphism.	◯ Yes / ◯ No
Please submit your written solutions to the following problems to your college tutor. For this written part of your homework, the deadline set by your college tutor applies.		
6	Let U, V, W be vector spaces over $\mathbb{R}$, and $T : U \to V$ and $S : V \to W$ be linear transformations. (a) Show that the composition $S \circ T : U \to W$ is linear. (b) Show that if the inverse map S^{-1} of S exists, then $S^{-1} : W \to V$ is linear.	
7	Let V be a vector space of dimension $n \geq 1$. If $T : V \to V$ is a linear transformation, prove that the following statements are equivalent: (a) $im(T) = ker(T)$; (b) $T^2 = 0$, n is even and $rk(T) = \frac{1}{2}n$.	
8	Let V be an n-dimensional vector space and let S and T be linear transformations on V. (a) Prove that $\text{nullity}(ST) \leq \text{nullity}(S) + \text{nullity}(T)$. (b) If $S^n = 0$ but $S^{n-1} \neq 0$, then determine $\text{nullity}(S)$.	
9	(Optional.) Let $n \in \mathbb{N}$. Consider the vector space $\mathbb{R}_n[x]$ of polynomials of degree at most n. Let $B_n = \{1, x, \ldots, x^n\}$. Define $D_n : \mathbb{R}_n[x] \to \mathbb{R}_{n-1}[x]$ by $f \mapsto f'$, where f' denotes the first derivative of f. (a) Show that D_n is linear. (b) Determine $M^{B_n}_{B_{n-1}}(D_n)$. (c) Show that there is a linear map $I_n : \mathbb{R}_{n-1}[x] \to \mathbb{R}_n[x]$ with $D_n \circ I_n = id$ and determine $M^{B_{n-1}}_{B_n}(I_n)$.	