Honour Moderations: Linear Algebra
Problem Sheet 3

Hilary Term 2005

1. An $n \times n$ matrix $A = (a_{ij})$ is called *orthogonal* if $AA^T = I$. Show that if A is orthogonal then $\det A = \pm 1$.

2. Calculate the determinants of the five matrices

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ 3 & 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 1 & 5 \\ 4 & 2 & 3 \end{pmatrix}, \quad AB^2, \quad A+B, \quad AB+A^2.$$

Which of these matrices are invertible?

3. Show that

$$\det \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \alpha+\beta & \alpha+\gamma \\ 1 & \beta+\alpha & 0 & \beta+\gamma \\ 1 & \gamma+\alpha & \gamma+\beta & 0 \end{pmatrix} = -4(\alpha\beta + \beta\gamma + \gamma\alpha).$$

What is the value of this when α, β, γ are the three roots of the equation $x^3 - 1 = 0$?

4. Using elementary row operations, compute

$$\det \begin{pmatrix} 1 & 2 & 3 & 0 \\ 5 & 0 & 2 & 1 \\ -1 & 1 & 0 & 3 \\ 2 & 1 & 3 & -2 \end{pmatrix}.$$

5.

(i) Let $B = (b_{ij})$ be an upper triangular $n \times n$ matrix, so that $b_{ij} = 0$ if $i > j$. Show that $\det B = \prod_{i=1}^{n} b_{ii}$.

(ii) Let A be an $n \times n$ matrix over $\mathbb{R}$, and P a non-singular $n \times n$ matrix over $\mathbb{R}$. Show that $\det A = \det(P^{-1}AP)$.

(iii) Let A be an $n \times n$ matrix over $\mathbb{R}$. Show that $\det(\text{adj}A) = (\det A)^{n-1}$.

6. For what values of x is the following matrix singular?

$$\begin{pmatrix} 3-x & 2 & 2 \\ 1 & 4-x & 1 \\ -2 & -4 & -1-x \end{pmatrix}$$

Supplementary Questions

7. Let $D = (\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2$, and $S_r = \alpha^r + \beta^r + \gamma^r$ for $r = 0, 1, 2, \ldots$. By considering $X = \begin{pmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{pmatrix}$ and its transpose X^T, or otherwise, prove that

$$\det \begin{pmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix} = D.$$

Now suppose that α, β, γ are the zeros of the equation $x^3 + px + q = 0$. Verify that $S_1 = 0$, $S_2 = -2p$, $S_3 = -3q$, $S_4 = 2p^2$, and hence show that $D = -4p^3 - 27q^2$. [**Hint:** try not to use too much heavy algebraic manipulation; remember that each of α, β, γ satisfies $x^3 = -px - q$ and $x^4 = -px^2 - qx$.]

8. Let $A = (a_{ij})$ be a 4×4 matrix of integers such that $a_{ij} = -a_{ji}$ for $1 \leqslant i, j \leqslant 4$. Show that $\det(A)$ is the square of an integer.

G.A.S.