Honour Moderations: Linear Algebra
Problem Sheet 2

Michaelmas Term 2004

1. If A and B are two matrices, we say that A and B *commute* if $AB = BA$. Now let A be a 2×2 matrix.

(a) Show that A commutes with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ if and only if A is diagonal.

(b) Show that A commutes with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if A is diagonal.

(c) Which 2×2 matrices A commute with $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$?

(d) Deduce that A commutes with *all* 2×2 matrices if and only if $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ for some scalar λ.

2. (a) Decide the independence or dependence of the following subsets of $\mathbb{R}^3$:

(i) the vectors $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$

(ii) the vectors $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$

(iii) the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$

[Let S be a non-empty subset of the vector space V. A vector $v \in V$ is said to be linearly dependent "on S" if there exist vectors $v_1, \ldots, v_k$ in S such that $v = \Sigma_{i=1}^{k} a_i v_i$ for some $a_1, \ldots, a_k \in \mathbb{R}$. The set $\mathrm{Sp}(S)$ is the set of vectors in V which are linearly dependent on S; $\mathrm{Sp}(S)$ is known as the subspace of V *spanned by* S.]

(b) Describe the subspace of $\mathbb{R}^3$ spanned by

(i) the two vectors $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$;

(ii) the three vectors $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(c) Describe $\mathrm{Sp}(S)$ in the following cases in $M_{2\times 2}(\mathbb{R})$:

(i) $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$;

(ii) $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$;

(iii) $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

3. In each of the following cases, **either** give a careful proof that V is a vector space over $\mathbb{R}$, **or** give a reason why it is not:

(a) V is the set of all polynomials over $\mathbb{R}$ which have a non-zero constant term, with the usual addition of polynomials and the usual scalar multiplication.

(b) V is the set of all functions $f : X \to \mathbb{R}$ (for some fixed set X), and if $f, g \in V$, $\alpha \in \mathbb{R}$, then the functions $f+g$, αf are defined by setting

$$(f+g)(x) = f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x).$$

(c) V is the set of all symmetric $n \times n$ matrices over $\mathbb{R}$.

(d) V is the set of all skew-symmetric $n \times n$ matrices over $\mathbb{R}$.

(e) V is the set of all invertible $n \times n$ matrices over $\mathbb{R}$ (that is, the set of all matrices A such that A^{-1} exists).

4. Determine which of the following subsets of $\mathbb{R}^n$ are subspaces:

(a) All vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ such that $x_1 = 1$;

(b) All vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ such that $x_1 + 2x_2 = 0$;

(c) All vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ such that $x_1 + x_2 + \cdots + x_n = 1$;

(d) All vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ such that $x_1^2 = x_2$.

5. If A is a real $m \times n$ matrix, prove that the solutions of the system $A\mathbf{x} = \mathbf{0}$ form a subspace of $\mathbb{R}^n$.

Optional:
6. Let ω be a complex cube root of 1 ($\omega \neq 1$). Prove that $1 + \omega + \omega^2 = 0$. Letting A be the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

find A^2 and A^{-1}.

G.A.S.