**Honour Moderations: Linear Algebra**
**Problem Sheet 4**

**Michaelmas Term 2004**

1. $U$ comes from $A$ by subtracting row 1 from row 3:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(i) Find bases for the two column spaces;

(ii) Find bases for the two row spaces;

(iii) Find bases for the two null spaces.

2. Let $M_{n\times n}(\mathbb{R})$ be the vector space of $n \times n$ matrices over $\mathbb{R}$. Show that the set $W$ of symmetric matrices in $M_{n\times n}(\mathbb{R})$ is a subspace of $M_{n\times n}(\mathbb{R})$. (Remember that $A \in W$ if and only if $A^T = A$.) Find

(i) a basis for $M_{n\times n}(\mathbb{R})$;

(ii) a basis for $W$.

3. Solve the following systems of equations using Gaussian elimination:

(a)
$$\begin{aligned} x + 2y - 4z &= -4 \\ 2x + 5y - 9z &= -10 \\ 3x - 2y + 3z &= 11. \end{aligned}$$

(b)
$$\begin{aligned} x + 2y - 3z &= -1 \\ -3x + y - 2z &= -7 \\ 5x + 3y - 4z &= 2. \end{aligned}$$

(c)
$$\begin{aligned} x + 2y - 3z &= 1 \\ 2x + 5y - 8z &= 4 \\ 3x + 8y - 13z &= 7. \end{aligned}$$

4. (a) Find the echelon form of $A$ where

$$A = \begin{pmatrix} 2 & -2 & 2 & 1 \\ -3 & 6 & 0 & -1 \\ 1 & -7 & 10 & 2 \end{pmatrix}.$$

(b) Find the reduced echelon form of $A$ where

$$A = \begin{pmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{pmatrix}.$$

5. Find the inverses of the following matrices using elementary row operations :

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 2 \\ 3 & 3 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

**Optional:**

6. Let $A$ be an $n \times n$ matrix over $\mathbb{R}$.

(a) Prove that $A$ is invertible if and only if its transpose $A^T$ is invertible.

(b) Prove that $A$ is invertible if and only if the only solution to the simultaneous equations $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. (Here $\mathbf{x}$ denotes a column vector in $\mathbb{R}^n$.)

(c) The $n \times n$ *Van der Monde matrix* is the matrix $A$ defined by

$$A = \begin{pmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^{n-1} \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & {x_n}^2 & \cdots & {x_n}^{n-1} \end{pmatrix}$$

where $x_1, x_2, \ldots, x_n$ are distinct real numbers. Show that $A$ is invertible.

[Hint: let $\mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$ be a solution to the simultaneous equations $A\mathbf{a} = \mathbf{0}$. Show that $x_1, x_2, \ldots, x_n$ are all roots of the polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$.]

**G.A.S.**