Honour Moderations: Linear Algebra
Problem Sheet 6

Michaelmas Term 2004

1. (a) Decide which of the following mappings $T : \mathbb{R}^3 \to \mathbb{R}^3$ are linear transformations:

(i) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ 0 \end{pmatrix}$;

(ii) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} |x| \\ -z \\ 0 \end{pmatrix}$;

(iii) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-1 \\ x \\ y \end{pmatrix}$;

(iv) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ y-2 \\ 4y \end{pmatrix}$.

(b) Let $V = M_{n\times n}(\mathbb{R})$, and let X be an arbitrary but fixed matrix in V. Define $T : V \to V$ by $T(A) = AX + XA$, where A is any matrix in V. Show that T is a linear transformation of V.

2. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 - x_3 \\ x_1 + x_4 \end{pmatrix}.$$

Describe $\operatorname{im} T$ and find a basis for it.

3. Let V be a vector space of dimension $n \geq 1$. If $T : V \to V$ is a linear transformation, prove that the following statements are equivalent:

(i) $\operatorname{im} T = \ker T$;

(ii) $T^2 = 0$, n is even and $\operatorname{rank} T = \frac{1}{2}n$.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Show that $\operatorname{im} T^2 \subseteq \operatorname{im} T$ and that $\ker T \subseteq \ker T^2$. Prove the equivalence of the following three statements:

(a) $\mathbb{R}^3 = \ker T \oplus \operatorname{im} T$;

(b) $\ker T = \ker T^2$;

(c) $\operatorname{im} T = \operatorname{im} T^2$.

[We write $\mathbb{R}^3 = \ker T \oplus \operatorname{im} T$ if $V = \ker T + \operatorname{im} T$ and $\ker T \cap \operatorname{im} T = \{0\}$.]

5. Describe the kernel and image of each of the following linear transformations, and in each case give the rank and nullity of the transformation:

(a) $T : \mathbb{R}^4 \to \mathbb{R}^3$ is given by $T(\mathbf{x}) = A\mathbf{x}$ (if $\mathbf{x}$ is a column vector in $\mathbb{R}^4$), where A is the matrix

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 0 & 3 & -2 & 0 \end{pmatrix}$$

(b) V is the vector space of all polynomials in x of degree $\leq n$, and $T : V \to V$ is given by differentiation with respect to x.

(c) $V = M_{n\times n}(\mathbb{R})$, and $T : V \to \mathbb{R}$ is given by $T(A) = \operatorname{tr} A = \sum_{i=1}^{n} a_{i,i}$.

Optional:

6. Let V be an n-dimensional vector space and let S and T be linear transformations on V. Prove that

$$\text{nullity}\,(ST) \leq \text{nullity}\,(S) + \text{nullity}\,(T).$$

If $S^n = 0$ but $S^{n-1} \neq 0$, then determine $\text{nullity}\,(S)$.

G.A.S.