Nonlinear systems. Specimen Finals questions

I

1. Describe what is meant by an *equilibrium point* and its (linear) *stability* for the set of differential equations

$$\dot{x} = f(x), \quad x \in \mathbf{R}^n,$$

and explain how the stability can be calculated.

Show that the Lorenz equations

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= (r - z)x - y, \\ \dot{z} &= xy - z, \end{aligned}$$

with σ, $b > 0$, have a pitchfork bifurcation at the origin when $r = 1$, and a Hopf bifurcation from the non-zero steady states when

$$r = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}.$$

I 2. Define what is meant by saddle-node, transcritical, pitchfork, and Hopf bifurcations for a differential equation

$$\dot{x} = f(x, \mu), \quad x \in \mathbf{R}^n.$$

Give criteria for the existence of each, and illustrate the first three using specific examples.

Show that the Rayleigh equation

$$\ddot{y} - \mu\dot{y} + \frac{1}{3}\dot{y}^3 + y = 0$$

has a Hopf bifurcation at $\mu = 0$, and use perturbation methods to find its amplitude.

I 3. Show that if x satisfies

$$\dot{x} = Ax + g_r(x) + O(r+1)$$

where $x \in \mathbf{R}^n$ and g_r is a polynomial containing terms of degree $r \geq 2$, and $O(r+1)$ signifies terms of degree $r+1$ or higher, then the substitution $x = y + p_r(y)$, with p_r a polynomial of degree r, will reduce the differential equation to the form

$$\dot{y} = Ay + O(r+1),$$

providing p_r satisfies the homological equation

$$L_A p_r \equiv P'_{r-1} Ay - Ap_r = g_r,$$

where P'_{r-1} is the Jacobian of p_r.

Assuming $\{e_i\}$ is a basis set of eigenvectors of A, with eigenvalues λ_i, show that

$$L_A x^m e_s = [(m, \lambda) - \lambda_s] x^m e_s$$

where $x^m = x_1^{m_1} \dots x_n^{m_n}$, $\lambda = (\lambda_1 \dots \lambda_n)$, $m = (m_1, \dots m_n)$, and deduce that a nonlinear system $\dot{x} = Ax + g_2(x) + g_3(x) + \dots$ may be formally reduced to the linear system $\dot{y} = Ay$, providing $\lambda_s \neq (m, \lambda)$ for all choices of s and m.

I 4. Explain in outline the procedure for the reduction of the differential equation

$$\dot{x} = Ax + g(x), \quad x \in \mathbf{R}^n,$$

to normal form. (You may assume A is diagonalisable.)

Find the normal form for the pair of differential equations

$$\dot{x} = x - 2y + \dots,$$

$$\dot{y} = 3x - y + \dots.$$

Hence find the resonant quadratic term(s) in the normal form reduction of

$$\dot{x} = x - 2y,$$

$$\dot{y} = 3x - y - x^2.$$

I 5. Describe what is meant by a Hopf bifurcation of an ordinary differential equation

$$\dot{x} = f(x, \mu), \quad x + \mathbf{R}^n.$$

Show that the real two-dimensional system

$$\dot{x}_1 = \mu x_1 - \Omega x_2 + f(x_1, x_2),$$

$$\dot{x}_2 = \Omega x_1 + \mu x_2 + g(x_1, x_2),$$

has a Hopf bifurcation at $\mu = 0$. By writing $z = x_1 + ix_2$ and embedding the system with the auxiliary equation

$$\dot{\mu} = 0,$$

show that the normal form can be written in the form

$$\dot{z} = (\mu + i\Omega)z + \sum_{m \geq 1} c_m(\mu) \left|z\right|^{2m} z.$$

I 6. Use the Poincaré-Lindstedt method to give solutions correct up to and including terms of $O(\epsilon^2)$ of the equations

$$\begin{array}{ll} \text{(i)} & \ddot{x} - \epsilon x\dot{x} + x = 0; \\ \text{(ii)} & (1 + \epsilon\dot{x})\ddot{x} + x = 0, \end{array}$$

where $\epsilon \ll 1$.

I 7. Describe two different kinds of bifurcation associated with a zero eigenvalue of the Jacobian Df of the differential equation

$$\dot{x} = f(x, \mu), \quad x \in \mathbf{R}^n,$$

where $f(0, \mu) = 0$ for all μ.

A feedback control system is modelled by

$$\ddot{x} + \delta \dot{x} + x(x^2 - 1) = -z,$$

$$\dot{z} + \alpha z = \alpha \gamma x,$$

where x is the amplitude of a nonlinear oscillator, and z is the controller. Evaluate the fixed points and their stability, and show that bifurcations can occur on three distinct surfaces in (α, δ, γ) space. Show that these surfaces meet at $\gamma = 1$, $\delta = 1/\alpha$.

II

1. Write down Hamilton's equations for the generalised coordinates p_i, q_i of the Hamiltonian $H(p_i, q_i, t)$. If H is autonomous show that it is a conserved quantity. Describe the use of a generating function $S(q_i, P_i)$ to integrate an autonomous hamiltonian system by solving the Hamilton-Jacobi equation.

A simple harmonic oscillator has Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2 q^2).$$

By solving the Hamilton-Jacobi equation for $S(q, I)$, find the action angle variables I, θ and hence integrate the system.

II 2. Describe what is meant by a canonical transformation of a Hamiltonian system.

For a one degree of freedom Hamiltonian $H(p, q, t)$ determine which of the following are canonical transformations:

(i) $Q = \frac{1}{2}q^2,\ P = p/q$;
(ii) $Q = \tan q,\ P = (p - k)\cos^2 q$;
(iii) $Q = \sin q,\ P = (p - k)/\cos q$;
(iv) $Q = q^{1/2} e^t \cos p,\ P = q^{1/2} e^{-t} \sin p$.

II 3. Write down the Hamilton-Jacobi equation for the perturbed one degree of freedom Hamiltonian

$$H = H_0(I) + \epsilon H_1(I,\theta) + \epsilon^2 H_2(I,\theta)\ldots,$$

where $H_1, H_2 \ldots$ are 2π-periodic in θ. By expressing the generating function $S(\theta, J)$ as a power series,

$$S = J\theta + \epsilon S_1 + \epsilon^2 S_2 + \ldots,$$

show how S_1, S_2 may be chosen to express the Hamiltonian in the form $K(J) = K_0(J) + \epsilon K_1(J) + \epsilon^2 K_2(J)\ldots$, providing $\omega_0(J) = H_0'(J) \neq 0$.

Apply the method to the perturbed Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega_0^2 q^2) + \epsilon q^3,$$

by first writing H_0 in action angle variables. Hence show that the perturbed frequency is given by

$$\omega(J) = \omega_0 - \epsilon^2 \left(\frac{15J}{4\omega_0^4}\right) + \ldots.$$

II 4. Show that the equation of motion describing a simple pendulum,

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0,$$

describes a Hamiltonian system with Hamiltonian

$$H^*(p^*, q^*) = \frac{1}{2}p^{*2} - \omega_0^2 \cos q^*.$$

Show that, for small amplitude oscillations, the Hamiltonian can be written in the perturbed form

$$H(p,q) = H_0 + \epsilon H_1 + \epsilon^2 H_2 \ldots, \quad \epsilon \ll 1,$$

where $H^* = \epsilon H$, $p^* = \sqrt{\epsilon}p,$, $q^* = \sqrt{\epsilon}q$. By transforming to action angle variables and applying canonical perturbation theory, show that the perturbed frequency is

$$\omega = \omega_0 - \frac{\epsilon}{8}J + \ldots,$$

where J is the (perturbed) action variable.

II 5. Describe the use of canonical perturbation theory in solving the perturbed n degree of freedom Hamiltonian system

$$H = H_0(I) + \epsilon H_1(I, \theta) + \ldots$$

where I, θ are n-dimensional vectors. Show that a perturbed generating function

$$S(J, \theta) = \theta.J + \epsilon S_1 + \epsilon^2 S_2 + \ldots$$

can be defined, and find an expression for S_1 if

$$H_1(I, \theta) = \bar{H}_1(I) + \sum_{m \neq 0} H_{1m} e^{im.\theta}.$$

Show that perturbation theory fails if the unperturbed frequencies are *resonant*, i.e. $m.\omega_0 = 0$, where $\omega_0 = \boldsymbol{\nabla} H_0$.

Give a *statement* of the problem of small divisors, and a *statement* of how the KAM theorem surmounts this.

II 6. Prove the *Poincaré-Birkhoff fixed point theorem* for the area-preserving twist map T_ϵ, given by

$$T_\epsilon \begin{pmatrix} I \\ \theta \end{pmatrix} = \begin{pmatrix} I + \epsilon g(I,\theta) \\ \theta + 2\pi\omega(I) + \epsilon f(I,\theta) \end{pmatrix},$$

where f and g are smooth 2π-periodic functions; that is, if $\omega(I_0) = r/s \in \mathbf{Q}$, $\omega'(I_0) \neq 0$, then for sufficiently small ϵ, there are (in general) $2ks$ orbits of period s near $I = I_0$, which are alternatively saddles and sinks.

Explain why, if the stable and unstable manifolds of neighbouring saddles intersect transversely, stochastic behaviour can be expected.

III

1. A twist map is defined by $\theta \to T_\epsilon(\theta)$, where

$$T_\epsilon(\theta) = \theta + 2\pi\Omega(\epsilon) + \epsilon \sin s\theta,$$

where $\epsilon \ll 1$ and Ω is a smooth function of ϵ. Show that if $\Omega(0) = r/s \in \mathbf{Q}$, then for sufficiently small ϵ, there are $2s$ fixed points of T_ϵ^s providing $|\Omega'(0)| < 1/2\pi$, and determine their stability. Deduce that T_ϵ has two period s cycles, one stable and the other unstable (you may assume r and s have no common factor).

If $\Omega'(0)$ varies independently of ϵ, what happens as $|\Omega'(0)|$ increases through $1/2\pi$?

III 2. Define a *Cantor set*. Describe the construction of the Cantor middle-thirds set, and show that it is a Cantor set. Show that, at the n-th stage of construction, there are 2^n intervals, each of length 3^n. Deduce that the Cantor set has measure zero.

III 3. Define what is meant by a Cantor set. Show that the Cantor middle-fifths set, constructed as follows, is a Cantor set. Delete the middle fifth (0.4, 0.6) from the unit interval [0,1]. Then delete the middle fifth from each of the two remaining intervals, and continue in this fashion. Find the number of intervals at the n-th stage, and compute their total length. Hence deduce that the middle-fifths set has zero measure.

III 4. Use symbolic dynamics to show that the *baker map*

$$\phi_{n+1} = 2\phi_n \bmod 1$$

has

(i) a countably infinite number of periodic orbits;
(ii) an uncountable number of aperiodic orbits;
(iii) a countable number of homoclinic points to the origin;
(iv) sensitive dependence on initial conditions;
(v) a dense orbit,

taking care to define the meaning of the terms in (iii), (iv) and (v).

Deduce that the logistic map $x_{n+1} = \mu x_n(1 - x_n)$ is chaotic when $\mu = 4$.

III 5. What is meant by *chaos* in a one-dimensional map?

Use symbolic dynamics to show that the maps

$$\begin{array}{lll} \text{(i)} & x \to 4x(1-x), & x \in [0,1]; \\ \text{(ii)} & x \to 4x^3 - 3x, & x \in [-1,1], \end{array}$$

are chaotic. How could you generalise these to obtain other smooth chaotic maps?

III 6. Describe the dynamics of the Smale horseshoe map, and show that the invariant set is the cartesian product of two Cantor sets. Show that the dynamics on the invariant set are topologically conjugate to the action of the shift map on the space Σ_2 of sequences of two symbols. Hence show that the invariant set contains a countably infinite number of periodic orbits, and an uncountably infinite number of aperiodic orbits.

IV

1. Define what is meant by a *dissipative* differential equation.

Show that the Lorenz equations

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y,\\ \dot{y} &= (r - z)x - y,\\ \dot{z} &= xy - bz,\end{aligned}$$

are dissipative, assuming $\sigma, b > 0$.

Show that there is a pitchfork bifurcation from the origin at $r = 1$, and a Hopf bifurcation at $r = \sigma(\sigma+b+3)/(\sigma-b-1)$. By writing $\epsilon = r^{-1/2}$, $x = \epsilon^{-1}\xi$, $y = \epsilon^{-2}\sigma^{-1}\eta$, $z = \epsilon^{-2}(\sigma^{-1}\zeta + 1)$, $t = \epsilon\tau$, show that for $\epsilon = 0$, the solution is periodic.

Give a brief description of the fate of both types of periodic solution as r varies.

IV 2. Let

$$\dot{x}_1 = \lambda_1 x_1 + P(x_1, x_2, x_3, \mu),$$

$$\dot{x}_2 = \lambda_2 x_2 + Q(x_1, x_2, x_3, \mu),$$

$$\dot{x}_3 = \lambda_3 x_3 + R(x_1, x_2, x_3, \mu),$$

where $P, Q, R = O(|x|^2)$ at $x = 0$, possess a homoclinic orbit Γ to the origin at $\mu = 0$. Suppose also that $-\lambda_2 < \lambda_1 > -\lambda_3 > 0$. Show how to construct an approximate Poincaré map on the surface of section $x_3 = c$, if c is small. (Assume that Γ is tangent to the positive x_1 axis and the positive x_3 axis at the origin.)

By appropriately scaling x and μ if necessary, show that this map may be approximated by the one-dimensional map

$$\xi \to \alpha\xi^\delta - \mu, \quad (\xi > 0)$$

where $x_1 \propto \xi$, and $\delta = |\lambda_3| / \lambda_1$. If $\alpha > 0$, show that this map has two fixed points if $\mu < [(1 - \delta)/\delta](\alpha\delta)^{1/(1-\delta)}$.

IV 3. The Lorenz equations have a homoclinic bifurcation at $r = r_h \approx 13.926\ldots$. For $\mu = r - r_h$ close to zero, the dynamics are smoothly approximated by the map

$$\xi \to \operatorname{sgn}(\xi)[d_1 |\xi|^{\delta_3} + \mu b_1] + a_1 \eta |\xi|^{\delta_2},$$

$$\eta \to \operatorname{sgn}(\xi)[\eta^* + d_2 |\xi|^{\delta_3} + \mu b_2] + a_2 \eta |\xi|^{\delta_2},$$

where $\xi, \eta \ll 1$, and $\delta_3 < 1 < \delta_2$, $d_1, b_1 > 0$. Describe *briefly* how this map can be derived, and show how symbolic dynamics can be used to show that there is a *strange invariant set* for $0 < \mu \ll 1$.

***IV 4.** Shil'nikov bifurcations occur for systems of the form

$$\begin{aligned}\dot{x} &= -\lambda_2 x - \omega y + P(x, y, z, \mu),\\ \dot{y} &= \omega x - \lambda_2 y + Q(x, y, z, \mu),\\ \dot{z} &= \lambda_1 z + R(x, y, z, \mu),\end{aligned}$$

where P, Q, R are $O(x^2 + y^2 + z^2)$. Suppose there is a homoclinic orbit for $\mu = 0$, tangent to the positive z axis. Construct an approximate Poincaré map on the surface of section $S : r = (x^2 + y^2)^{1/2} = c$, where c is small. By writing $z = c\zeta$, show that this map may be written in the form

$$\zeta \to d\zeta^\delta \cos[\theta + \frac{\omega}{\lambda_1} \ln(1/\zeta) + \Phi_1] + \mu b_1,$$

$$\theta \to a\zeta^\delta \cos[\theta + \frac{\omega}{\lambda_1} \ln(1/\zeta) + \Phi_2] + \mu b_2,$$

where $\delta = \lambda_2/\lambda_1$. Assuming $\zeta, \theta, \mu \ll 1$, show that the map is approximated (after rescaling ζ and μ) by the one-dimensional map

$$\zeta \to a\zeta^\delta \cos\left[\frac{\omega}{\lambda_1} \ln(1/\zeta)\right] + \mu.$$

Deduce that for $\delta < 1$, an infinite number of periodic orbits bifurcate at $\mu = 0$, but for $\delta > 1$, only one does.