Introduction to Mathematical Modelling

PROBLEM SHEET 1.

1. Consider the discrete population model

$$u_{n+1} = ru_n\left(1 - \frac{u_n}{K}\right) - hu_n.$$

By writing $u_n = Lw_n$ for some suitable choice of L, show that the model takes the form

$$w_{n+1} = \lambda w_n(1 - w_n),$$

and determine λ. What is the effect of increasing h on the behaviour of the population? What happens if $h > r - 1$?

2. Consider the continuous population model

$$\frac{du}{dt} = \rho u\left(1 - \frac{u}{C}\right) - \mu u.$$

By writing $u = Mw$ for some suitable choice of M, show that the model takes the form

$$\frac{dw}{dt} = \alpha w(1 - w),$$

and determine α. What is the effective of increasing μ on the population? What happens if $\rho < \mu$?

3. Starting from an integral conservation law, derive the heat equation and the mass conservation equation in the forms given in chapter 1 of the notes.

4. Suppose

$$Pe\left[\frac{\partial T}{\partial t} + \mathbf{u}.\boldsymbol{\nabla} T\right] = \nabla^2 T + 1 \quad \text{in} \quad D,$$

with

$$\begin{aligned} T &= 0 \quad \text{on} \quad \partial D, \\ T &= \theta_0 \Theta(\mathbf{x}) \quad \text{in} \quad D \quad \text{at} \quad t = 0, \end{aligned}$$

and $\Theta = O(1)$, $\theta_0 \gg 1$, $Pe \ll 1$. Discuss appropriate scales for the various phases of the solution.

5. A population of size N is subject to immigration at rate I, and mutual pair destruction at a rate kN^2, so that $\dot{N} = I - kN^2$. By appropriate scaling of the variables, show that the model can be written in the form $\dot{x} = 1 - x^2$.

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PROBLEM SHEET 2.

1. Explain why the iterative method

$$x_{n+1} = (1 + \varepsilon x_n)^{1/3}$$

will converge to the real solution of $x^3 - \varepsilon x - 1 = 0$. Does this depend on the value of ε?

2. (i) Find approximations to the solution of

$$\varepsilon x^3 - x - 1 = 0, \quad \varepsilon \ll 1,$$

which is close to $x = -1$. Compare with the numerical solution when $\varepsilon = 0.1$; $\varepsilon = 0.01$.

(ii) Use perturbation methods to find approximate roots to the equation

$$xe^{-x} = \varepsilon, \quad 0 < \varepsilon \ll 1.$$

(Use graphical methods to find the location of the roots. For the larger root, take logs and note that if $x \gg 1$, then $x \gg \ln x$.)

3. Each of the equations

$$z^5 - \varepsilon z - 1 = 0,$$
$$\varepsilon z^5 - z - 1 = 0,$$

has five (possibly complex) roots. Find approximations to these if $\varepsilon \ll 1$. Can you refine the approximations?

4. (a) Sketch the function $y = x^3 e^{-x}$, $x > 0$.

   (b) Sketch $y = x \ln x$, $x > 0$. (Note $\ln x \ll \frac{1}{x}$ when $x \ll 1$; why?)

   (c) If $y = x^x$, $x > 0$, and $y(0) = 1$, sketch $y(x)$.

   (d) Sketch y defined by $y = x + (x + y)^3$.

   (e) Sketch y defined by $\tan x = \tanh y$.

   (f) Sketch $T(\mu)$ defined by $T = \mu \exp\left(\frac{T}{1 + \varepsilon T}\right)$, where $\varepsilon > 0$, for $\mu > 0$.

5. The differential equation

$$\dot{x} = a - xe^{-x}, \quad x > 0, \quad a > 0,$$

may have $0, 1$ or 2 steady states. Determine how these depend on a, and describe how solutions behave for $a > e^{-1}$ and $a < e^{-1}$, depending on the value of $x(0)$.

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PROBLEM SHEET 3.

1. u and v satisfy the ordinary differential equations

$$\begin{aligned}\dot{u} &= k_1 - k_2 u + k_3 u^2 v,\\ \dot{v} &= k_4 - k_3 u^2 v,\end{aligned}$$

where $k_i > 0$. By suitably scaling the equations, show that these can be written in the dimensionless form

$$\begin{aligned}\dot{u} &= a - u + u^2 v,\\ \dot{v} &= b - u^2 v,\end{aligned}$$

where a and b should be defined. Show that if u, v are initially positive, they remain so. Draw the nullclines in the positive quadrant, show that there is a unique steady state and examine its stability. Are periodic solutions likely to exist?

2. The relaxational form of the van der Pol oscillator is

$$\varepsilon\ddot{x} + (x^2 - 1)\dot{x} + x = 0, \quad \varepsilon \ll 1.$$

A suitable phase plane is spanned by (x, y), where $y = \varepsilon\dot{x} + \frac{1}{3}x^3 - x$. Describe the motion in this phase plane, and find, approximately, the period of the relaxation oscillation. What happens if $\varepsilon < 0$?

3. The Belousov-Zhabotinskii chemical reaction can be approximately described by the two component pair of ordinary differential equations

$$\varepsilon\dot{X} = X(1 - X) - \left(\frac{X - \delta}{X + \delta}\right) Z,$$

$$\dot{Z} = \gamma X - Z,$$

where X and Z are positive reactant concentrations, ε and δ are very small, and γ is $O(1)$. Show that relaxation oscillations will occur for γ within a certain range (γ_-, γ_+), and give approximations for the values of $\gamma_\pm$.

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PROBLEM SHEET 4.

1. Find a scaling of the combustion equation
$$c\frac{dT}{dt} = -k(T - T_0) + A\exp(-E/RT)$$
so that it can be written in the form
$$\dot{\theta} = \theta_0 - g(\theta),$$
where $\theta_0 = RT_0/E$ and $g = \theta - \alpha e^{-1/\theta}$. Give the definition of α. Hence show that the steady state θ is a multiple-valued function of θ_0 if $\alpha > \frac{1}{4}e^2$.

2. Suppose that θ satisfies $\dot{\theta} = \theta_0 - g(\theta)$, where $g(\theta)$ is as in question 1 and $\alpha > \frac{1}{4}e^2$, and θ_0 varies slowly according to
$$\dot{\theta}_0 = \varepsilon(\theta^* - \theta),$$
where $\varepsilon \ll 1$. Show that there are three possible outcomes, depending on the value of θ^*, and describe them.

3. A forced pendulum is modelled by the (dimensional) equation
$$l\ddot{\theta} + k\dot{\theta} + g\sin\theta = \alpha\sin\lambda t.$$
By scaling the equation, show how to obtain the dimensionless model
$$\ddot{u} + \beta\dot{u} + \Omega_0^2\sin u = \varepsilon\sin\omega t,$$
and show that the time scale can be chosen so that $\Omega_0 = 1$. In this case, identify the parameters ε, β and ω.

4. It is asserted in the notes that the oscillator frequency $\Omega(A)$ is a decreasing function of A for $0 < A < \pi$, or equivalently, that the function
$$p(A) = \frac{1}{\sqrt{2}}\int_0^A \frac{du}{[\cos u - \cos A]^{1/2}}$$
is increasing. Show that this is true by writing p in the form
$$p = \int_0^1 \left(\frac{\theta}{\sin\theta}\right)^{1/2}\left(\frac{\phi}{\sin\phi}\right)^{1/2}\frac{dw}{(1-w^2)^{1/2}}$$
for some functions $\theta(w, A)$ and $\phi(w, A)$, and using the fact that $\theta/\sin\theta$ is an increasing function of θ in $(0, \pi)$.

[Hint: $\cos u - \cos A = 2\sin\left(\frac{A-u}{2}\right)\sin\left(\frac{A+u}{2}\right)$.]

Show also that, if $\Omega = \dfrac{\pi\Omega_0}{2p}$, then
$$\Omega \approx \Omega_0\left[1 - \frac{A^2}{16}\dots\right]$$
for small A.

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PROBLEM SHEET 5.

1. The function $u(x,t)$ satisfies

$$u_t + uu_x = \alpha(1-u^2)$$

for $-\infty < x < \infty$, where $\alpha > 0$, and with $u = u_0(x)$ at $t = 0$, and $0 < u_0 < 1$ everywhere.

Write down the characteristic differential equations satisfied by x and u in terms of the variable t along the characteristics, and write down the correponding initial conditions at $t = 0$ in parametric form.

Show that the solution for u is

$$u = \frac{(1+u_0)e^{2\alpha t} - (1-u_0)}{(1+u_0)e^{2\alpha t} + (1-u_0)},$$

and deduce that

$$u = \frac{u_0(s) + \tanh \alpha t}{1 + u_0(s)\tanh \alpha t}. \qquad (1)$$

Hence show that

$$\exp[\alpha(x-s)] = \cosh \alpha t + u_0(s)\sinh \alpha t, \qquad (2)$$

and, by using (1) to eliminate u_0, deduce that

$$\exp[\alpha(x-s)] = \frac{\operatorname{sech} \alpha t}{1 - u\tanh \alpha t}. \qquad (3)$$

Sketch the form of the characteristics for an initial function such as $u_0(s) = a/(1+s^2)$. [*Don't use (2) unless you have to; simply use the characteristic differential equations to infer the graphical behaviour of u with t on a characteristic, and therefore also that of x.*]

Use (1) to show that (note $s = s(x,t)$)

$$u_x = \frac{\operatorname{sech}^2 \alpha t\, u_0' s_x}{(1 + u_0 \tanh \alpha t)^2},$$

and use (2) to show that

$$s_x\left[u_0' \tanh \alpha t + \alpha(1 + u_0\tanh \alpha t)\right] = \alpha\left[1 + u_0 \tanh \alpha t\right].$$

Deduce that, in terms of s and t, u_x is given by

$$u_x = \frac{[\alpha \operatorname{sech}^2 \alpha t] u_0'(s)}{[1 + u_0(s)\tanh \alpha t][\alpha + \{u_0'(s) + \alpha u_0(s)\}\tanh \alpha t]},$$

and deduce that a shock will form if $u_0' + \alpha(1+u_0)$ becomes negative for some s.

Now suppose that $u_0 = a/(1+s^2)$ and that a is small.

Show that

$$u_0' + \alpha(1+u_0) = \frac{\alpha(1+s^2)^2 + \alpha a(1+s^2) - 2as}{(1+s^2)^2},$$

and deduce that a shock will form if

$$p(s,\alpha) = \alpha(1+s^2)^2 + \alpha a(1+s^2) - 2as$$

is negative for some value of s.

Observe that p is a continuous function of s and α.

Show that $p(s,\alpha)$ increases with α.

Show that $p < 0$ for $s > 0$ when $\alpha = 0$.

Show that $p > a(1-s)^2 \geq 0$ when $\alpha > 1$, and that in fact $p > 0$ for $\alpha = O(1)$ when a is small. Deduce that there is a positive small value $\alpha = \alpha_c$ such that $\min p = 0$, and that a shock will form if $\alpha < \alpha_c$.

Show that for small α and a,

$$p \approx \alpha(1+s^2)^2 - 2as,$$

and that at $\alpha = \alpha_c$, the value of s where $p = p' = 0$ satisfies

$$\begin{aligned} \alpha((1+s^2)^2 &\approx 2as, \\ 4\alpha s((1+s^2) &\approx 2a, \end{aligned}$$

and thus

$$s \approx \left(\frac{a}{8\alpha}\right)^{1/3}.$$

Deduce that α_c is approximately given by

$$\alpha_c \approx \frac{3a\sqrt{3}}{8}.$$

2. The function $u(x,t)$ satisfies the first order equation

$$u_t + u^\alpha u_x = \varepsilon[u^\beta u_x]_x$$

for $-\infty < x < \infty$, with $u = u_0(s) > 0$ on the initial curve $x = s$, $t = 0$. The parameters α and β are positive, and $0 \leq \varepsilon \ll 1$. Assume that $u_0 > 0$, and that $u_0(s) \to 0$ as $s \to \pm\infty$.

Write down the solution in the case $\varepsilon = 0$, and show that a shock will form at

$$t = t_c = \min_{s:u_0'<0} -\frac{1}{(u_0^\alpha)'}.$$

For $t > t_c$ show that the shock at $x = x_s(t)$ will travel at a speed

$$\dot{x}_s = \frac{[u^{\alpha+1}]_-^+}{(\alpha+1)[u]_-^+}, \qquad (1)$$

defining what you mean by $u_\pm$.

Now suppose that $0 < \varepsilon \ll 1$. By writing $x = x_s + \varepsilon X$, derive an approximate equation describing the *shock structure* of the solution for u within the shock. Write down suitable boundary conditions for this equation. By integrating the equation, show that

$$u_X = \frac{\dfrac{u^{\alpha+1}}{\alpha+1} + K - cu}{u^\beta}, \qquad (2)$$

where $c = \dot{x}_s$ is the (constant) shock speed and K is a constant, and by applying the boundary conditions, confirm that the shock speed c is indeed given by (1).

By using the definitions of K and c needed to satisfy the boundary conditions, show that the numerator $N(u)$ of the fraction in (2) can be written as

$$N(u) = \frac{(u_+ - u)(u_-^{\alpha+1} - u_+^{\alpha+1})}{(\alpha+1)(u_- - u_+)} + \frac{u^{\alpha+1} - u_+^{\alpha+1}}{\alpha+1},$$

and deduce that the derivative of N is

$$N'(u) = u^\alpha - \left\{ \frac{u_-^{\alpha+1} - u_+^{\alpha+1}}{(\alpha+1)(u_- - u_+)} \right\},$$

and that $N(u_+) = N(u_-) = 0$.

Use the mean value theorem for derivatives to show that

$$N'(u) = u^\alpha - u_*^\alpha$$

for some u_* between u_- and u_+.

Deduce that $N < 0$ for u between u_- and u_+, and hence show that a solution of (2) exists only if $u_- > u_+$.

3. Show that the equation

$$u_t + uu_x = \varepsilon u u_{xx}$$

admits a shock structure joining u_- to a lower value u_+, in which the wave speed is

$$c = \frac{[u]_-^+}{[\ln u]_-^+}.$$

Naïvely, one might have expected the wave speed to be $c = \left[\frac{1}{2}u^2\right]_-^+ / [u]_-^+$. Why? And why is it not?

Introduction to Mathematical Modelling

PROBLEM SHEET 6.

1. Suppose that we wish to find positive travelling wave solutions (if they exist) to the equation
$$u_t = u^p(1-u^q) + [u^r u_x]_x,$$
where $p, q, r \geq 0$, with boundary conditions
$$u \to 1 \quad \text{as} \quad x \to -\infty \quad \text{and} \quad u \to 0 \quad \text{as} \quad x \to \infty.$$

   Show that travelling wave solutions exist of the form $u = f(\xi)$, $\xi = x - ct$, if f satisfies the equation
$$(f^r f')' + f^p((1-f^q) + cf' = 0,$$
with $f(-\infty) = 1$, $f(\infty) = 0$. Deduce that the quantity
$$E = \tfrac{1}{2} f^{2r} f'^2 + \frac{f^{p+r+1}}{p+r+1} - \frac{f^{p+q+r+1}}{p+q+r+1}$$
decreases as ξ increases along a trajectory, if $f > 0$ and $c > 0$.

   Show that such a travelling wave solution can be found providing a connecting trajectory from $(1,0)$ to $(0,0)$ with $f > 0$ exists of the system
$$\begin{aligned} f^r f' &= -g, \\ g' &= f^p(1-f^q) - \frac{cg}{f^r}. \end{aligned} \qquad (1)$$

   Now consider the specific case $p = 1$, $q = 2$, $r = 0$, and assume that $c > 0$. Find and draw the nullclines of (1). Show that $(1,0)$ is a saddle, and that $(0,0)$ is a stable spiral or node.

   Show in this case that
$$E = \tfrac{1}{2} g^2 + \tfrac{1}{2} f^2 - \tfrac{1}{4} f^4,$$
and thus that E decreases along trajectories even when $f < 0$, and deduce that the trajectory which leaves $(1,0)$ in the direction of increasing g and decreasing f *always* terminates at $(0,0)$. Show that this approach is oscillatory if $c < 2$, and deduce that a travelling wave in which f is positive can only exist if $c > 2$.

2. In a model of snow melting, it is assumed that the permeability is $k = k_0 S^\alpha$, and the capillary suction is $p_c(S) = p_0(S^{-\beta} - S)$, where $\alpha, \beta > 0$, and S is the saturation. The saturation for one-dimensional flow is described by
$$\phi \frac{\partial S}{\partial t} + \frac{\partial K}{\partial z} = \frac{\partial}{\partial z}\left[D \frac{\partial S}{\partial z}\right],$$

where ϕ is porosity, $K = k\rho g/\mu$ is the hydraulic conductivity, and $D = -kp_c'(S)/\mu$ is the hydraulic diffusivity; z is the vertical coordinate pointing downwards from the surface $z = 0$.

If a suitable depth scale is h, show how to non-dimensionalise the equation to obtain the form

$$\frac{\partial S}{\partial t} + \frac{\partial S^\alpha}{\partial z} = \kappa \frac{\partial}{\partial z}\left[S^\alpha \left(1 + \frac{\beta}{S^{\beta+1}} \right) \frac{\partial S}{\partial z} \right],$$

where

$$\kappa = \frac{p_0}{\rho g h}.$$

Suppose that an initially dry snowpack ($S = 0$ at $t = 0$ and $S \to 0$ as $z \to \infty$) is inundated at the surface (i. e., $S = 1$ at $z = 0$ for $t > 0$). Assume also that $\kappa = 0$. Write down the characteristic equations for the model, and show, by drawing the characteristic diagram, that if $\alpha > 1$, then a shock must form, but that if $\alpha < 1$, a solution can be found with an expansion fan emanating from $z = t = 0$ (note that S is indeterminate at this point on the initial boundary curve, and can take any value between 0 and 1). For this latter case, show that the solution is

$$S = 1, \quad z < \alpha t,$$
$$S = \left(\frac{\alpha t}{z} \right)^{1/(1-\alpha)}, \quad z > \alpha t.$$

For the case $\alpha > 1$, write down a suitable jump condition across a shock, and hence show that the wetting front (i. e., a shock) $z = z_w(t)$ moves downwards at a speed $\dot{z}_w = 1$.

Now suppose that $0 < \kappa \ll 1$ and $\alpha > 1$. Write $z = z_w + \kappa\zeta$, $\dot{z}_w = c$, and show that to leading order S satisfies the equation

$$-cS' + (S^\alpha)' = \left[S^\alpha \left(1 + \frac{\beta}{S^{\beta+1}} \right) S' \right]',$$

with appropriate boundary conditions being $S \to 1$ as $\zeta \to -\infty$, $S \to 0$ as $\zeta \to \infty$. By integrating this equation, show that $c = 1$ and that

$$S' = -\frac{(S^{1-\alpha} - 1)}{\left(1 + \frac{\beta}{S^{\beta+1}} \right)}.$$

Deduce that a shock structure connecting $S = 1$ to $S = 0$ only exists if $\alpha > 1$.

By considering a suitable approximation to this equation for small S, show that S reaches zero at a finite value of ζ if $\alpha > \beta + 1$. Sketch the consequent solution structure. What happens if $\alpha < \beta + 1$?

Introduction to Mathematical Modelling

PROBLEM SHEET 7.

1. Write down the equation satisfied by a similarity solution of the form $u = t^\beta f(\eta)$, $\eta = x/t^\alpha$, for the equation

$$u_t = (u^m u_x)_x \quad \text{in} \quad 0 < x < \infty,$$

where $m > 0$, with boundary conditions $u^m u_x = -1$ at $x = 0$, $u \to 0$ as $x \to \infty$, and the initial condition $u = 0$ at $t = 0$. Show that an ordinary differential equation for f with time independent boundary conditions is obtained providing

$$\alpha = \frac{m+1}{m+2}, \qquad \beta = \frac{1}{m+2}.$$

Integrate the equation to show that f satisfies

$$f^m f' = -1 - \left(\frac{m+1}{m+2}\right) \eta f + \int_0^\eta f \, d\eta,$$

and deduce that $\int_0^\infty f \, d\eta = 1$. Hence show that

$$f^{m-1} f' < -\left(\frac{m+1}{m+2}\right) \eta,$$

and deduce that f reaches zero at a finite value η_0.

2. u satisfies the equation

$$u_t = [D(u) u_x]_x \quad \text{in} \quad 0 < x < \infty,$$

with $u = 0$ at $x \to \infty$ and $t = 0$, and $u = u_0$ (constant) at $x = 0$, where the function D is non-negative (do not assume it is a power of u). Show that a similarity solution of the form $u = f(\eta)$ exists, where you should determine a suitable form for the function $\eta(x, t)$. Write down the resulting equation and boundary conditions for the similarity function f.

Now suppose that $D = D(u_x)$, and that the boundary and initial conditions are as before, except that $-D u_x = 1$ at $x = 0$. Show that a similarity solution can be found in this case, in the form $u = t^\alpha f(x/ct^\beta)$ for suitable values of α, β and c, and write down the equation and boundary conditions for the similarity function f in this case.

3. A small droplet satisfies the surface-tension controlled equation

$$h_t = -\frac{\gamma}{3\mu} \boldsymbol{\nabla} . [h^3 \boldsymbol{\nabla} \nabla^2 h]$$

in $\mathbf{R}^n$ ($n = 1$ or 2), where γ is the surface tension and μ is the viscosity. A small quantity $\int h\, dS = V$ is released at time zero at the origin. Show that the equation can be written in the dimensionless form

$$h_t = -\boldsymbol{\nabla}.[h^3 \boldsymbol{\nabla} \nabla^2 h],$$

where the scales are chosen so that

$$\int_{\mathbf{R}^n} h\, dS = 1.$$

Show that a similarity solution can be found in the form $h = t^{-\beta} f(r/t^\alpha)$ in both one and two spatial dimensions (i. e., $n = 1$ and $n = 2$). Show that for $n = 1$,

$$f^2 f''' = \tfrac{1}{7}\eta,$$

while for $n = 2$,

$$f^2 \left[f'' + \frac{1}{\eta} f' \right]' = \tfrac{1}{10}\eta.$$

What might suitable boundary conditions for these equations be?

[The dimensionless equation describing radially symmetric solutions takes the form

$$h_t = -\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left[r^{n-1} h^3 \frac{\partial}{\partial r} \nabla^2 h \right],$$

where

$$\nabla^2 h = \frac{\partial^2 h}{\partial r^2} + \frac{n-1}{r} \frac{\partial h}{\partial r}$$

in n dimensions.]

Introduction to Mathematical Modelling

VACATION PROBLEM SHEET.

1. By direct integration, show that the solution of

$$u'' + \lambda e^u = 0$$

satisfying $u = 0$ on $x = \pm 1$ is

$$u = 2 \ln \left[A \,\mathrm{sech} \left\{ \sqrt{\frac{\lambda}{2}} Ax \right\} \right],$$

and find a transcendental equation for A. Hence show that no solution exists for $\lambda > \lambda_c$, and derive and solve (numerically) an algebraic equation for λ_c.

If the equation is to be solved in $[0, 1]$, with $u' = 0$ on $x = 0$ and $u' = -1$ on $x = 1$, find the solution, and plot $u(0)$ as a function of λ. Is there a critical value λ_c? If so, find it; if not, why not?

2. (i) Find an exact solution of the Gel'fand equation

$$\nabla^2 \theta + \lambda e^\theta = 0 \quad \text{in} \quad 0 < r < 1,$$

where r is the cylindrical polar radius, and $\theta = 0$ on $r = 1$. [*Assume cylindrical symmetry, and a suitable condition of regularity at* $r = 0$.] Show that there is a critical parameter λ_c such that no solution exists for $\lambda > \lambda_c$, and find its value.

(ii) Write down the ordinary differential equation satisfied by a spherically symmetric solution of the Gel'fand equation in part (i). Suppose that $\theta = 0$ on $r = 1$ and $\theta_r = 0$ on $r = 0$ (why?). By putting

$$p = \lambda r^2 e^\theta, \quad q = 2 + r\theta_r, \quad r = e^{-t},$$

show that $p(t)$ and $q(t)$ satisfy the ordinary differential equations

$$\begin{aligned} \dot{p} &= -pq, \\ \dot{q} &= p + q - 2. \end{aligned}$$

By consideration of trajectories for p and q in the phase plane, show that multiple solutions exist for $\lambda \approx 2$, and infinitely many at $\lambda = 2$. Sketch the corresponding response diagram of $\theta(0)$ versus λ.

3. The complex reactant concentration c in $\mathbf{R}^2$ satisfies the reaction diffusion system

$$c_t = f(|c|)c + D\nabla^2 c,$$

where $f(|c|) = \lambda(|c|) + i\omega(|c|)$, and λ and ω are real-valued. Suppose that $\lambda(|c|)$ is monotonically decreasing, $\lambda(0) > 0$, and $\lambda \to -\infty$ as $|c| \to \infty$.

(i) Show that if $D = 0$, then $\dot{|c|} = |c|\lambda(|c|)$, and deduce that as $t \to \infty$, $|c| \to c^*$, where c^* is the unique positive value of $|c|$ such that $\lambda(c^*) = 0$. Hence show that the eventual solution is the limit cycle $c = c^* \exp[i\omega(c^*)t]$.

(ii) Now suppose $D \neq 0$. Show that travelling waves exist provided $k\sqrt{D} < \{\lambda(0)\}^{1/2}$.

Show also that if $\lambda(0) < 0$, and the boundary condition for c is that $c \to 0$ as $|\mathbf{x}| \to \infty$, then $c \to 0$ as $t \to \infty$. [*Hint: consider the time evolution of* $\int_{\mathbf{R}^2} |c|^2 \, dS$.]

4. *Finals 2001.* An activator–inhibitor reaction–diffusion model takes the non-dimensional form

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{u^2}{v} - bu + \frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial t} &= u^2 - v + d\frac{\partial^2 v}{\partial x^2},\end{aligned}$$

where $u(x,t)$, $v(x,t)$ are the concentrations of the chemicals at spatial coordinate x and time t, and b, d are positive parameters.

Which is the activator and which is the inhibitor? Find the non-zero spatially uniform steady state and, from first principles, determine the conditions for it to be driven unstable by diffusion. Show that the parameter domain for which diffusion-driven instability is given by $0 < b < 1$, $db > 3 + 2\sqrt{2}$, and sketch the instability region in (b, d) parameter space. Show that the critical wave number k_c at the onset of instability is

$$k_c^2 = \frac{1 + \sqrt{2}}{d}.$$