Techniques of Applied Mathematics

PROBLEM SHEET 2, WEEK 2 LECTURES. [2006]

1. The absolute temperature T of an igniting solid is modelled by the equation
$$c\frac{dT}{dt} = -k(T - T_0) + A\exp\left(-\frac{E}{RT}\right).$$
Find a non-dimensionalisation of this equation so that it can be written in the form
$$\dot{\theta} = \theta_0 - g(\theta),$$
where $\theta_0 = RT_0/E$ and $g = \theta - \alpha e^{-1/\theta}$. Give the definition of α. Hence show that the steady state θ is a multiple-valued function of θ_0 if $\alpha > \frac{1}{4}e^2$.

   Now suppose that $\alpha > \frac{1}{4}e^2$, and θ_0 varies slowly according to
$$\dot{\theta}_0 = \nu(\theta^* - \theta),$$
where $\nu \ll 1$. Show that there are three possible outcomes, depending on the value of θ^*, and describe them.

2. *Phase plane revision.* u and v satisfy the ordinary differential equations
$$\begin{aligned} \dot{u} &= k_1 - k_2 u + k_3 u^2 v, \\ \dot{v} &= k_4 - k_3 u^2 v, \end{aligned}$$
where $k_i > 0$. By suitably scaling the equations, show that these can be written in the dimensionless form
$$\begin{aligned} \dot{u} &= a - u + u^2 v, \\ \dot{v} &= b - u^2 v, \end{aligned}$$
where a and b should be defined. Show that if u, v are initially positive, they remain so. Draw the nullclines in the positive quadrant, show that there is a unique steady state and examine its stability.

   Show that if $k_1 \ll k_4$, then $a \ll b$, and show that an oscillatory (Hopf) instability occurs in this case if $b \lesssim 1$ (i. e., b is approximately less than one). Assuming a stable periodic orbit occurs in this case, sketch its form in the (u, v) phase plane.

3. H and Q satisfy the ordinary differential equations
$$\dot{H} = a - Q,$$
$$\varepsilon\dot{Q} = H - f(Q),$$
where a and ε are positive constants, $\varepsilon \ll 1$, and
$$f(Q) = 4 + (Q - 1)(Q - 2)^2.$$
Show that if a is in a certain range, which you should find, then relaxation oscillations will occur, and describe them in the (H, Q) phase plane.

Techniques of Applied Mathematics

PROBLEM SHEET 3, WEEK 3 LECTURES. [2006]

1. The Van der Pol oscillator is represented by the equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0,$$

and we assume that $0 < \varepsilon \ll 1$.

Use the Poincaré-Lindstedt method to find the amplitude and frequency correction of the periodic solution of the equation.

2. Duffing's equation is

$$\ddot{x} + x + \varepsilon x^3 = 0.$$

By first finding a first integral, show that all solutions are periodic if $\varepsilon > 0$. If $\varepsilon \ll 1$, use the Poincaré-Lindstedt method to construct approximate solutions, and find the frequency correction of a periodic solution in terms of ε and its amplitude A.

3. The heat equation for the temperature $T(r,t)$ of a fluid flowing axially in a pipe $0 < r < a$ satisfies

$$\rho c_p \frac{\partial T}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T}{\partial r}\right] + \frac{\tau^2}{\mu},$$

where ρ is density, c_p is specific heat, k is thermal conductivity, τ is shear stress and μ is viscosity. The boundary conditions are that $T = T_0$ at $r = a$, and $\dfrac{\partial T}{\partial r} = 0$ at $r = 0$.

The shear stress τ satisfies the slow flow momentum equation

$$\frac{1}{r}\frac{\partial(r\tau)}{\partial r} = p',$$

where p' is the axial pressure gradient, taken to be constant, and $\tau = 0$ at $r = 0$.

The stress is related to the velocity gradient by

$$\tau = \mu\frac{\partial u}{\partial r},$$

where μ is the viscosity, and for a shear thinning fluid (like tomato ketchup or Guinness), the viscosity decreases with increasing stress. Suppose such a fluid flows in a cylindrical tube of radius a, and has viscosity

$$\mu = \frac{\mu_0}{|\tau|^{n-1}}\exp\left[-\frac{(T - T_0)}{\Delta T}\right],$$

where the power law exponent $n > 1$ ($n = 1$ corresponds to Newtonian flow).

Show that a suitable dimensionless formulation of the temperature profile across the tube satisfies the dimensionless equation

$$\frac{\partial \theta}{\partial t} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left[\xi \frac{\partial \theta}{\partial \xi}\right] + \beta \xi^{n+1} e^{\theta},$$

with the boundary conditions

$$\theta = 0 \quad \text{on} \quad \xi = 1,$$

and

$$\theta_\xi = 0 \quad \text{at} \quad \xi = 0.$$

Give the definition of the parameter β.

Find a steady solution of this problem,[1] and find a Sturm-Liouville eigenvalue problem, whose eigenvalues determine the linear stability of the steady solution.

[1] The transformations $z = -\ln \xi$, $\theta = \psi + (2\gamma)^{1/2} z$, $e^{\psi} = \frac{\gamma}{\beta} \operatorname{sech}^2 u$ for some appropriate γ, may help in this.

Techniques of Applied Mathematics

PROBLEM SHEET 4, WEEK 4 LECTURES. [2006]

1. Show that the second order differential equation

$$w'' + a(x)w' + [b(x) + \lambda c(x)]w = 0$$

can be transformed to the Sturm-Liouville form

$$(pw')' + (\lambda r - q)w = 0,$$

by suitable definitions of the functions p, q and r. Show further that by suitable definition of ξ, the equation can be written in the form

$$\frac{d^2w}{d\xi^2} + (\lambda R - Q)w = 0,$$

and give the definitions of the functions Q and R.

Hence, or otherwise, find the eigenvalues λ and eigenfunctions w of the equation

$$2xw'' + w' + \lambda w = 0,$$

subject to the boundary conditions

$$w(0) = w(1) = 0.$$

2. Bessel's equation of order ν is

$$w'' + \frac{1}{z}w' + \left(1 - \frac{\nu^2}{z^2}\right) w = 0,$$

and the solutions are called Bessel functions of order ν. Show that the solutions of the Sturm-Liouville equation

$$(rR')' + \left(k^2 r - \frac{m^2}{r}\right) R = 0 \quad (*)$$

can be written as Bessel functions of order m.

By seeking solutions near $z = 0$ of the form $w \sim z^\mu$, show that if $\nu \neq 0$, then independent solutions $w_\pm$ are possible with $w_\pm \sim z^{\pm\nu}$. Deduce that if we require $R(0) = 0$ when m is a positive integer in $(*)$, then we must have $R \propto w_+(kr)$.

(w_+ is normally written J_ν, and is the Bessel function of the first kind; the Bessel function of the second kind is singular at $z = 0$ and is denoted Y_ν.)

Show that if $\nu = 0$, then independent solutions are possible in which

$$J_0 \sim 1 - \tfrac{1}{4}z^2, \quad Y_0 \sim \ln z.$$

[*When $\nu = 0$, the first solution can be found by seeking a Taylor series approximation for small z, i. e., a perturbation series in the small parameter z; the second can be found by writing the equation in Sturm-Liouville form, and then supposing that $zw \to 0$ as $z \to 0$.*]

3. A function $u(x,t)$ satisfies the nonlinear reaction-diffusion equation

$$u_t = u_{xx} + f(u) \quad \text{on} \quad [0,1],$$

with boundary conditions

$$u = 0 \quad \text{on} \quad x = 0,\ 1.$$

Suppose that $u_0(x)$ is a steady solution. By writing $u = u_0 + U$, derive a linearised equation for small perturbations U. Show that it has separable solutions of the form $U = v(x)e^{\sigma t}$, and deduce that v satisfies the Sturm-Liouville equation

$$v'' + [s(x) - \sigma]v = 0, \quad v(0) = v(1) = 0,$$

where

$$s(x) = f'[u_0(x)].$$

Deduce that

$$-\sigma = \frac{\displaystyle\int_0^1 (v'^2 - s(x)v^2)\,dx}{\displaystyle\int_0^1 v^2\,dx},$$

and hence show that if $s(x) < 0$, then all eigenvalues σ are negative, and hence the steady state is stable.

If $s(x) > 0$, this argument does not work. Show in this case that by writing $q = s_{\max} - s$ and $\lambda = s_{\max} - \sigma$, that λ is an eigenvalue of a Sturm-Liouville problem, and by writing λ as the minimum of a suitable functional, show that instability can occur ($\sigma > 0$) if a function v satisfying $v(0) = v(1) = 0$ can be found for which

$$\min_v \frac{\displaystyle\int_0^1 (v'^2 - s(x)v^2)\,dx}{\displaystyle\int_0^1 v^2\,dx} < 0.$$

(This shows that the constraint that $q > 0$ in Sturm-Liouville theory is not necessary.)

By consideration of the minimum value of $\dfrac{\displaystyle\int_0^1 v'^2\,dx}{\displaystyle\int_0^1 v^2\,dx}$ (with $v(0) = v(1) = 0$), show that instability occurs if $s > \pi^2$ for all $x \in (0,1)$.

Techniques of Applied Mathematics

PROBLEM SHEET 5, WEEK 5 LECTURES. [2006]

1. Show that the Wronskian $W = y_1 y_2' - y_1' y_2$ of two solutions y_1, y_2 of the differential equation
$$[p(x)y']' - q(x)y = 0, \quad p(x) > 0,$$
satisfies $pW = C$, a constant. Show that the solutions are dependent (one a multiple of the other) if $C = 0$, and independent if $C \neq 0$.

   Use the method of variation of parameters to find the solution in integral form of
$$y'' + \omega^2 y = -f(x), \quad y(0) = y'(0) = 0.$$
By constructing the Green's function, find the solution of
$$y'' + \omega^2 y = -f(x), \quad y(0) = y(1) = 0,$$
in the form
$$y = \int_0^1 G(x,\xi) f(\xi)\, d\xi.$$

2. (i) Use the Neumann series iterative method to solve the integral equation
$$\phi(x) = 1 + \lambda \int_0^1 \phi(y)\, dy.$$
Show that the successive approximations suggest an exact solution, and check that it is indeed a solution. Could you have found this solution directly?

   (ii) Use the Neumann method to find successive approximations to the nonlinear integral equation
$$\phi(x) = 1 + \lambda \int_0^1 \{\phi(y)\}^2\, dy.$$
Show that the approximations suggest a series solution
$$\phi = 1 + a_1\lambda + a_2\lambda^2 + \ldots,$$
and by direct substitution, find the values of a_1 and a_2. Are these values consistent with the iterative method?

   Solve the integral equation directly, and show that there are two solutions if $0 < \lambda < \frac{1}{4}$ or $\lambda < 0$, and one if $\lambda = \frac{1}{4}$ or $\lambda = 0$. Show that one of the solutions is represented by the series solution. Why is the other not?

   (iii) Find the solution of the equation
$$\phi(x) = 1 + \lambda \int_0^1 xy\phi(y)\, dy$$
if $\lambda \neq 3$. Show that there is no solution if $\lambda = 3$.

Show that the homogeneous equation

$$\phi(x) = \lambda \int_0^1 xy\phi(y)\, dy$$

has an infinite number of solutions if $\lambda = 3$, but only the trivial solution $\phi = 0$ if $\lambda \neq 3$.

(iiii) Solve the integral equation

$$\phi(x) = 1 + \lambda \int_0^\pi \cos(x-y)\phi(y)\, dy,$$

and find the value of λ for which no solution exists.

3. (i) The function $y(x)$ satisfies the Fredholm integral equation

$$y(x) = \lambda \int_a^b G_n(x,\xi) r(\xi) y(\xi)\, d\xi + g(x), \quad (*)$$

where $r > 0$, and the real kernel G_n is *degenerate*, that is,

$$G_n(x,\xi) = \sum_1^n \phi_i(x)\psi_i(\xi).$$

(Neither ϕ_i nor ψ_i are assumed to be eigenfunctions for the homogeneous equation.)

Show that the solution takes the form

$$y(x) = g(x) + \lambda \sum_1^n c_i \phi_i(x),$$

where the vector $\mathbf{c} = (c_1, \ldots, c_n)^T$ satisfies the vector equation

$$(I - \lambda A)\mathbf{c} = \mathbf{s},$$

and the vector $\mathbf{s}$ and real-valued matrix A should be determined.

From linear algebra we (should) know the Fredholm alternative for solutions of the (real-valued) matrix equation $T\mathbf{v} = \mathbf{c}$, which takes the following form: if $\det T \neq 0$, then T is invertible, and $\mathbf{v}$ is uniquely determined as $\mathbf{v} = T^{-1}\mathbf{c}$; or, $\det T = 0$, in which case the null space $\mathcal{N}(T^T)$ of T^T (as well as that of T) is non-empty, and then the equation has a solution iff $\boldsymbol{\eta}^T\mathbf{c} = 0$ for all $\boldsymbol{\eta} \in \mathcal{N}(T^T)$ (and then are infinitely many solutions, as any element of $\mathcal{N}(T)$ can be added to a solution).[2]

[2]The proof of the necessity of this orthogonality condition is simple. The proof of the sufficiency uses the fundamental theorem of linear algebra (see the book by Gil Strang, Introduction to applied mathematics): if T is an $n \times n$ matrix on $\mathbf{R}^n$, then $\mathbf{R}^n = \mathcal{R}(T) \oplus \mathcal{N}(T^T)$, where $\mathcal{R}(T)$ is the range of T. This means the direct sum, i. e., there is no intersection between range and adjoint null space. That there is no intersection is clear since (easily) $\mathcal{N}(T^T) \perp \mathcal{R}(T)$; the fact that $\mathbf{R}^n = \mathcal{R}(T) + \mathcal{N}(T^T)$ then follows from the dimension of each. If the dimension of $\mathcal{R}(T)$ is r, then so is that of $\mathcal{R}(T^T)$ (because row rank equals column rank), and thus that of $\mathcal{N}(T^T)$ is $n - r$. Hence $\mathbf{c}$ is in the range of T if and only if it is orthogonal to the null space of T^T.

Use this to deduce the Fredholm alternative in the following form: for each λ, either there is a unique solution of $(*)$, or else there is no solution unless a set of subsidiary conditions are satisfied, in which case there are an infinite number of solutions. Explain what the conditions are, and the form of the consequent solutions, in terms of the eigenvectors of A and A^T.

(ii) Now suppose that $G_n \to G$ uniformly as $n \to \infty$. Show that if the solution to $(*)$ is denoted as y_n, and the solution when G_n is replaced by G is denoted y, assuming these exist, then $e_n = \sup_{[a,b]} |y - y_n|$ satisfies

$$e_n \leq \lambda \left[\varepsilon + M_n e_n \right],$$

for any $\varepsilon > 0$ and sufficiently large n, if M_n is chosen appropriately. Hence show that

$$e_n \leq \frac{\lambda \varepsilon}{1 - \lambda M_n},$$

and deduce that $y_n \to y$ as $n \to \infty$ if $|\lambda| < \dfrac{1}{M}$, where

$$M = (b - a) \max_{[a,b]} (r|G|).$$

Techniques of Applied Mathematics

PROBLEM SHEET 6, WEEK 6 LECTURES. [2005]

1. The integral operator $\mathcal{G}$ is defined on the space of continuous functions on $[a, b]$, $C[a, b]$, by
$$\mathcal{G}u = \int_a^b G(x, \xi) r(\xi) u(\xi)\, d\xi,$$
where G is real, symmetric and continuous on $[a, b] \times [a, b]$, and r is positive and continuous on $[a, b]$.

   Suppose that the eigenfunctions ϕ_n with corresponding eigenvalues λ_n of
$$(I - \lambda\mathcal{G})\phi = 0$$
form a complete, orthonormal sequence, and that a continuous function f has a Fourier expansion
$$f = \sum_i c_i \phi_i.$$
By writing $(I - \lambda\mathcal{G})^{-1} = \sum_n \lambda^n \mathcal{G}^n$, show that a formal solution of the inhomogeneous Fredholm integral equation
$$(I - \lambda\mathcal{G})y = f$$
can be found in the form
$$y = \sum_i \frac{c_i \lambda_i \phi_i}{\lambda_i - \lambda}. \quad (*)$$
Show that the derivation is only valid for $|\lambda| < \dfrac{1}{||\mathcal{G}||}$, but that the series in $(*)$ converges for all λ except the eigenvalues (where there are simple poles).

2. The function $G(x, \xi)$ is defined in $[0, 1]$ by
$$G(x, \xi) = \begin{cases} x(1 - \xi), & x < \xi, \\ \xi(1 - x), & x > \xi. \end{cases}$$
Show by direct calculation that the functions $u = \sin n\pi x$ are eigenfunctions of
$$u = \lambda \int_0^1 G(x, \xi) u(\xi)\, d\xi,$$
with eigenvalues $\lambda = n^2\pi^2$.

   Hence show (assuming this set of eigenfunctions is complete) that the solution of the inhomogeneous integral equation
$$u = \lambda \int_0^1 G(x, \xi) u(\xi)\, d\xi + 1$$

can be written in the form

$$u = \sum_{n \text{ odd}} \frac{4n\pi}{n^2\pi^2 - \lambda} \sin n\pi x,$$

providing λ is not an eigenvalue.

3. An integral operator is defined by

$$\mathcal{G}u = \int_a^b G(x,\xi) r(\xi) u(\xi)\, d\xi,$$

where G is real, symmetric and continuous, r is positive and continuous, and u is continuous.

Show that

$$||u|| = \left\{ \int_a^b r u^2 \, d\xi \right\}^{1/2}$$

defines a norm for u, and that

$$||\mathcal{G}|| = \sup \frac{||\mathcal{G}u||}{||u||}$$

defines a norm for $\mathcal{G}$.

Write down the corresponding inner product (u, v), and write down the Cauchy-Schwarz inequality for it. Hence, by writing $\mathcal{G}u = (G, u)$, or otherwise, show that

$$||\mathcal{G}|| \leq \left\{ \int_a^b \int_a^b r(\xi) r(\eta) G^2(\xi,\eta)\, d\xi\, d\eta \right\}^{1/2}.$$

Bessel's equation of order zero is

$$-(xu')' = \lambda x u.$$

Suppose we require u bounded at $x = 0$ and $u = 0$ at $x = 1$. Show that the Green's function of the left hand side of this equation with these boundary conditions is

$$G = \begin{cases} -\ln x, & x > \xi, \\ -\ln \xi, & x < \xi, \end{cases}$$

and deduce that the corresponding Bessel function solves the integral equation

$$u = \lambda \mathcal{G}u \equiv \lambda \int_0^1 \xi G(x,\xi) u(\xi)\, d\xi.$$

Hence show that $\mathcal{G}$ is bounded.

Techniques of Applied Mathematics

PROBLEM SHEET 7, WEEK 7 LECTURES. [2005]

1. The function $u(x,t)$ satisfies the nonlinear advection-diffusion equation

$$u_t + u^\alpha u_x = \varepsilon[u^\beta u_x]_x$$

for $-\infty < x < \infty$, with $u = u_0(s)$ on the initial curve $x = s$, $t = 0$. The parameters α and β are positive, and $0 \le \varepsilon \ll 1$. Assume that $u_0 > 0$, and that $u_0(s) \to 0$ as $s \to \pm\infty$.

Write down the solution in the case $\varepsilon = 0$, and show that a shock will form at

$$t = t_c = \min_{s:u_0'<0} -\frac{1}{(u_0^\alpha)'}.$$

For $t > t_c$ show that the shock at $x = x_s(t)$ will travel at a speed

$$\dot{x}_s = \frac{[u^{\alpha+1}]_-^+}{(\alpha+1)[u]_-^+}, \qquad (1)$$

defining what you mean by $u_\pm$. Sketch the form of the solution.

Now suppose that $0 < \varepsilon \ll 1$. By writing $x = x_s + \varepsilon X$, derive an approximate equation describing the *shock structure* of the solution for u within the shock. Write down suitable boundary conditions for this equation. By integrating the equation, show that

$$u_X = \frac{\dfrac{u^{\alpha+1}}{\alpha+1} + K - cu}{u^\beta}, \qquad (2)$$

where $c = \dot{x}_s$ is the (constant) shock speed and K is a constant, and by applying the boundary conditions, confirm that the shock speed c is indeed given by (1).

By using the definitions of K and c needed to satisfy the boundary conditions, show that the numerator $N(u)$ of the fraction in (2) can be written as

$$N(u) = \frac{(u_+ - u)(u_-^{\alpha+1} - u_+^{\alpha+1})}{(\alpha+1)(u_- - u_+)} + \frac{u^{\alpha+1} - u_+^{\alpha+1}}{\alpha+1},$$

and deduce that the derivative of N is

$$N'(u) = u^\alpha - \left\{\frac{u_-^{\alpha+1} - u_+^{\alpha+1}}{(\alpha+1)(u_- - u_+)}\right\},$$

and that $N(u_+) = N(u_-) = 0$.

Use the mean value theorem for derivatives to show that

$$N'(u) = u^\alpha - u_*^\alpha$$

for some u_* between u_- and u_+.

Deduce that $N < 0$ for u between u_- and u_+, and hence show that a solution of (2) exists only if $u_- > u_+$.

2. Show that the equation

$$u_t + uu_x = \varepsilon u u_{xx},$$

with u positive, admits a shock structure joining u_- to a lower value u_+, in which the wave speed is

$$c = \frac{[u]_-^+}{[\ln u]_-^+}.$$

Naïvely, one might have expected the wave speed to be $c = \left[\frac{1}{2}u^2\right]_-^+ / [u]_-^+$. Why? And why is it not?

3. In a model of snow melting, it is assumed that the permeability is $k = k_0 S^\alpha$, and the capillary suction is $p_c(S) = p_0(S^{-\beta} - S)$, where $\alpha, \beta > 0$, and S is the saturation. The saturation for one-dimensional flow is described by

$$\phi \frac{\partial S}{\partial t} + \frac{\partial K}{\partial z} = \frac{\partial}{\partial z}\left[D \frac{\partial S}{\partial z}\right],$$

where ϕ is porosity, $K = k\rho g/\mu$ is the hydraulic conductivity, and $D = -kp_c'(S)/\mu$ is the hydraulic diffusivity; z is the vertical coordinate pointing downwards from the surface $z = 0$.

If a suitable depth scale is h, show how to non-dimensionalise the equation to obtain the form

$$\frac{\partial S}{\partial t} + \frac{\partial S^\alpha}{\partial z} = \kappa \frac{\partial}{\partial z}\left[S^\alpha \left(1 + \frac{\beta}{S^{\beta+1}}\right) \frac{\partial S}{\partial z}\right],$$

where

$$\kappa = \frac{p_0}{\rho g h}.$$

Suppose that an initially dry snowpack ($S = 0$ at $t = 0$ and $S \to 0$ as $z \to \infty$) is inundated at the surface (i. e., $S = 1$ at $z = 0$ for $t > 0$). Assume also that $\kappa = 0$. Write down the characteristic equations for the model, and show, by drawing the characteristic diagram, that if $\alpha > 1$, then a shock must form, but that if $\alpha < 1$, a solution can be found with an expansion fan emanating from $z = t = 0$ (note that S is indeterminate at this point on the initial boundary curve, and can take any value between 0 and 1). For this latter case, show that the solution is

$$S = 1, \quad z < \alpha t,$$

$$S = \left(\frac{\alpha t}{z}\right)^{1/(1-\alpha)}, \quad z > \alpha t.$$

For the case $\alpha > 1$, write down a suitable jump condition across a shock, and hence show that the wetting front (i. e., a shock) $z = z_w(t)$ moves downwards at a speed $\dot{z}_w = 1$.

Techniques of Applied Mathematics

VACATION PROBLEM SHEET, WEEK 8 LECTURES. [2005]

1. The function $u(x,t)$ satisfies the heat equation

$$u_t = u_{xx}$$

on $0 < x < \infty$, together with the initial condition

$$u = 0 \quad \text{at} \quad t = 0,$$

and the boundary condition

$$u \to 0 \quad \text{as} \quad x \to \infty.$$

Find the form of possible similarity solutions for u when the boundary condition at $x = 0$, $t > 0$ has each of the following forms:

(i) $u = 1$;

(ii) $u_x = -1$;

(iii) $uu_x = -1$;

(iiii) $u_x = -u^\beta$.

If you can, solve the resulting boundary value problems (but don't expect that this will always be possible).

Show in case (iiii) that no similarity solution exists if $\beta = 1$. What is it about this case that prevents a similarity solution existing?

2. Write down the equation satisfied by a similarity solution of the form $u = t^\beta f(\eta)$, $\eta = x/t^\alpha$, for the equation

$$u_t = (u^m u_x)_x \quad \text{in} \quad 0 < x < \infty,$$

where $m > 0$, with boundary conditions $u^m u_x = -1$ at $x = 0$, $u \to 0$ as $x \to \infty$, and the initial condition $u = 0$ at $t = 0$. Show that an ordinary differential equation for f with time independent boundary conditions is obtained providing

$$\alpha = \frac{m+1}{m+2}, \quad \beta = \frac{1}{m+2}.$$

Integrate the equation to show that f satisfies

$$f^m f' = -1 - \left(\frac{m+1}{m+2}\right)\eta f + \int_0^\eta f\, d\eta,$$

and deduce that $\int_0^\infty f\, d\eta = 1$. Hence show that

$$f^{m-1} f' < -\left(\frac{m+1}{m+2}\right)\eta,$$

and deduce that f reaches zero at a finite value η_0.

3. *Blasius equation.* The stream function ψ of a boundary layer flow over a flat plate is described by the equation

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \psi_{yyy},$$

valid in $0 < x > \infty$, $0 < y < \infty$, with boundary conditions

$$\psi = \psi_y = 0 \quad \text{on} \quad y = 0, \quad \psi_y \to 1 \quad \text{as} \quad y \to \infty \quad \text{and} \quad x \to 0.$$

Show that a similarity solution exists in the form

$$\psi = \alpha x^\mu f(\eta), \quad \eta = \frac{y}{\alpha x^\mu}.$$

Find the appropriate value of μ, and show that α can be chosen so that the equation for f takes the form

$$f''' + f f'' = 0.$$

What are the boundary conditions for f?

This nonlinear boundary value problem must be solved numerically. A useful way to do this is to solve the initial value problem

$$F''' + F F'' = 0,$$

with

$$F(0) = F'(0) = 0, \quad F''(0) = 1.$$

Show that if in this solution, it is found that $F'(\infty) = c$, then

$$f(\eta) = \frac{1}{\sqrt{c}} F\left(\frac{\eta}{\sqrt{c}}\right).$$

4. *Stefan problem* An Arctic pond of water of depth d at the freezing point $T = 0°$ C begins to freeze when the air temperature above is suddenly lowered (at $t = 0$) to $-\Delta T°$ C. A layer of ice forms at the surface, of depth $s(t)$. If z is distance downwards from the ice surface, the temperature of the ice is governed by the heat equation

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2},$$

with boundary conditions

$$T = -\Delta T \quad \text{at} \quad z = 0, \quad T = 0 \quad \text{at} \quad z = s(t).$$

The free boundary $s(t)$ is unknown, and a second condition (a Stefan condition) is required to determine it. This follows from energy considerations. The temperature below the ice/water interface is isothermal at $T = 0$ (so there is no heat flux), but there is a heat flux $k\dfrac{\partial T}{\partial z}$ away from the interface towards the

surface, and this removes the latent heat per unit mass L which is released on converting water at 0° C to ice at 0° C. The resultant Stefan condition is

$$\rho L \dot{s} = k \left. \frac{\partial T}{\partial z} \right|_{z=s-} .$$

Non-dimensionalise the model using length scale d, time scale d^2/κ (where κ is the thermal diffusivity $k/\rho c_p$), and temperature scale ΔT, and show that the dimensionless model takes the form

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2},$$

with boundary conditions

$$T = -1 \quad \text{at} \quad z = 0, \quad T = 0 \quad \text{at} \quad z = s(t),$$

$$\sigma \dot{s} = \left. \frac{\partial T}{\partial z} \right|_{z=s-}, \quad s(0) = 0,$$

and define the dimensionless *Stefan number* σ.

Show that a similarity form of solution exists in which $T = f(\eta)$, $\eta = \dfrac{z}{2\sqrt{t}}$, $s = 2\sqrt{\alpha t}$, and solve the resulting boundary value problem. Show that the Stefan condition leads to an equation for α (which must be solved numerically), and show that this equation must have a unique positive solution.