Techniques of Mathematical Modelling

SPECIMEN FHS QUESTIONS.

Warning: these are rather longer than actual fhs questions would be. In parts they are also somewhat harder.

1. Explain what is meant by a *conservation law* and a *constitutive law* in a mathematical model. A certain substance has a density ϕ of a certain quantity, which moves with a flux $\mathbf{f}$. Write down an integral conservation law for the quantity of the substance in an arbitrary volume V, and hence deduce carefully that ϕ satisfies the partial differential equation

$$\frac{\partial \phi}{\partial t} + \boldsymbol{\nabla}.\mathbf{f} = 0.$$

The density of cars on a certain one lane highway is ρ (with units of cars per unit length), and the speed v of the cars is measured as a function of the density,

$$v = v_0 \left(1 - \frac{\rho}{\rho_0}\right)^2.$$

Give a physical interpretation of the quantities v_0 and ρ_0, and explain why this relation is intuitively sensible.

Write down an equation governing the traffic density, using the above expression for car speed, and by choosing suitable non-dimensional variables, show that the model can be written in the dimensionless form

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0,$$

where $q = \rho(1-\rho)^2$, and x represents dimensionless distance along the road.

A line of cars of initial (dimensionless) density $\rho_0(x)$ moves along a road $-\infty < x < \infty$, and is governed by the preceding equation. Consider the three situations in which $\rho_0(x)$ is monotonic, with $\rho \to \rho_-$ as $x \to -\infty$ and $\rho \to \rho_+$ as $x \to \infty$, where (i) $\rho_- < \frac{1}{3} < \rho_+ < \frac{2}{3}$, (ii) $\frac{2}{3} > \rho_- > \frac{1}{3} > \rho_+$, (iii) $\frac{1}{3} < \rho_- < \frac{2}{3} < \rho_+$.

Explain qualitatively, using diagrams, what happens in each case. In particular, describe in which situation(s) a shock forms, and give an expression for the resulting shock speed(s). [*An exact solution is not required.*]

2. Explain what is meant by a *regular perturbation* and by a *singular perturbation* for an algebraic equation and for an ordinary differential equation. [*It may be useful to give illustrative examples.*]

   The quantity x satisfies the algebraic equation

   $$x^4 - \varepsilon x - 1 = 0.$$

   Suppose that $\varepsilon \ll 1$. Find approximate expressions, correct to terms of $O(\varepsilon)$, for each of the four solutions of the equation.

   Now suppose $\varepsilon \gg 1$. Show that an approximate solution cannot immediately be found, but that by a suitable rescaling of the equation (which you should find), it can be written in the form

   $$X^4 - X - \delta = 0,$$

   where $\delta = \varepsilon^{-4/3} \ll 1$.

   Hence find leading order (non-zero) approximations for all four of the solutions.

   Find a more accurate approximation to the smallest root in this case.

3. A nonlinear damped pendulum satisfies the equation

$$l\ddot{\theta} + k\dot{\theta} + g\sin\theta = 0.$$

Explain the meaning of the terms in this equation, and how it is derived.

Suppose that $\theta = 0$, $\dot{\theta} = \omega_0$ at $t = 0$. By suitably non-dimensionalising the equation, show that the model can be written in the form

$$\ddot{\theta} + \dot{\theta} + \varepsilon\sin\theta = 0,$$

$$\theta(0) = 0, \quad \dot{\theta} = \mu,$$

and give the definitions of ε and μ.

The pendulum is suspended in a bath of liquid (e. g., water). Why might this be consistent with a value of $\varepsilon \ll 1$?

Assume now that $\varepsilon \ll 1$ and that $\mu = O(1)$. Find an approximate solution in this case, and show that $\theta \to \mu$ for large t.

By rescaling $t = \tau/\varepsilon$, find an approximate equation satisfied by θ over this longer time scale, and explain why a suitable initial condition for θ as $\tau \to 0$ is $\theta \approx \mu$.

Hence show that for $\tau \geq O(1)$,

$$\theta \approx 2\tan^{-1}\left[e^{-\tau}\tan\frac{\mu}{2}\right].$$

Do these results accord with your expectation?

4. The temperature T of a fluid satisfies the equation

$$\frac{\partial T}{\partial t} + \mathbf{u}.\boldsymbol{\nabla} T = \kappa \nabla^2 T + \frac{H}{\rho c_p},$$

and the quantity H is given by

$$H = A \exp\left(-\frac{E}{RT}\right).$$

The fluid is contained in a vessel D of linear size l, and the fluid velocity is of order of magnitude U. The boundary condition for T is

$$T = T_B \quad \text{on} \quad \partial D,$$

where T_B is constant. Show how to non-dimensionalise the equation to obtain the dimensionless form

$$Pe\left[\frac{\partial \theta}{\partial t} + \mathbf{u}.\boldsymbol{\nabla}\theta\right] = \nabla^2\theta + \lambda \exp\left(\frac{\theta}{1+\varepsilon\theta}\right),$$

with

$$\theta = 0 \quad \text{on} \quad \partial D,$$

and show that

$$Pe = \frac{Ul}{\kappa}, \quad \varepsilon = \frac{RT_B}{E}, \quad \lambda = \frac{\mu}{\varepsilon}\exp(-1/\varepsilon),$$

where

$$\mu = \frac{Al^2}{kT_B},$$

and k is the thermal conductivity, $k = \rho c_p \kappa$.

Suppose that D is one-dimensional, and of length $2l$, so that the dimensionless range of the space variable x is $[-1, 1]$. Suppose also that $Pe \ll 1$, $\varepsilon \ll 1$, and $\lambda = O(1)$. Write down an approximate equation and boundary conditions satisfied by θ, and show that the solution in $x > 0$ can be written in the integral form

$$\int_\theta^{\theta_0} \frac{du}{\sqrt{e^{\theta_0} - e^u}} = \sqrt{2\lambda}\, x,$$

where $\theta_0 = \theta|_{x=0}$.

Evaluate the integral to find $\theta(x)$, and deduce that θ_0 satisfies

$$\sqrt{\frac{\lambda}{2}} = z \operatorname{sech} z,$$

where

$$z = \sqrt{\frac{\lambda e^{\theta_0}}{2}}.$$

Draw a sketch graph of θ_0 in terms of λ, explaining the behaviour of θ_0 for small λ.

Show that no solution exists for $\lambda > \lambda_c$, where

$$\lambda_c = 2\zeta^2 \operatorname{sech}^2\zeta, \quad 1 = \zeta \tanh\zeta.$$

[*It may help to consider the graph of* $z \operatorname{sech} z$.]

5. Draw a graph of the function

$$V(x) = \frac{x^4}{4} - \frac{x^2}{2}.$$

The function $x(t)$ satisfies the ordinary differential equation

$$\ddot{x} + \Lambda \dot{x} + V'(x) = 0.$$

Show that the equation has fixed points at $x = -1$, 0 and 1.

(i) Suppose that $\Lambda = 0$. Show that the quantity

$$E = \tfrac{1}{2}\dot{x}^2 + V(x)$$

is constant. Deduce the form of the trajectories in the $(x, \dot{x})$ phase plane.

Hence or otherwise show that the solutions are oscillatory, and sketch the form of the solutions as graphs of x against t, distinguishing clearly between solutions in which $E < 0$ and those in which $E > 0$.

Now suppose that $\Lambda \ll 1$. Show that $\dot{E} = -\Lambda \dot{x}^2$, and deduce that for almost all trajectories, $E \to -\frac{1}{4}$. Hence, or otherwise, show that the fixed points at ± 1 are stable, and that at 0 is unstable. Sketch (roughly) the form of the trajectories in the phase plane.

(ii) Suppose now that $\Lambda \gg 1$. Find a suitable rescaling of t so that the equation for x can be approximated by a first order differential equation. Hence show that if $x(0) \neq 0$, $x \to \pm 1$ as $t \to \infty$. How does the limiting state depend on $x(0)$?

Describe briefly how you could reconcile the prescription of two initial conditions for x with this approximate first order equation.

6. The function $x(t)$ satisfies the equation

$$\ddot{x} + (x^4 - 1)\dot{x} + \omega^2 x = 0.$$

Show that the steady state $x = 0$ is an unstable spiral, or an unstable node, depending on the size of ω.

Suppose that $\omega^2 \ll 1$. Find a suitable rescaling of t so that the equation can be written in the form

$$\varepsilon\ddot{x} + (x^4 - 1)\dot{x} + x = 0, \qquad (*)$$

and give the definition of ε. Hence show that $\varepsilon \ll 1$.

Draw a graph of the function $f(x) = \frac{1}{5}x^5 - x$. Show that the equation $x^5 - 5x - 4 = 0$ has real roots of -1, -1 and x_0 (and no others), where x_0 is the unique positive root of $x^3 - 2x^2 + 3x - 4 = 0$. (You should explain why this root is indeed unique.)

Show that, by defining y suitably, the differential equation $(*)$ for x can be written as the pair

$$\begin{aligned} \varepsilon\dot{x} &= y - f(x), \\ \dot{y} &= -x. \end{aligned}$$

Deduce that for small ω, a relaxation oscillation occurs, and indicate its form in the (x, y) phase plane.

Show that the period of oscillation P is approximately given by

$$P = 2\int_1^{x_0} \frac{f'(x)\,dx}{x},$$

and hence find an approximation for P in terms of x_0.

7. Describe what is meant by a *boundary layer approximation* to the solution of a differential equation.

   Describe, giving brief reasons, where you expect the boundary layers to be located for the solutions of the boundary value problem

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \quad y(0) = 0, \quad y(1) = 1,$$

   if $\varepsilon \ll 1$ and (i) $a(x) > 0$; (ii) $a(x) < 0$.

   Find a leading order approximation (including boundary layers, where necessary) to the solution of the boundary value problem

$$\varepsilon y'' + (1 + x)y' + xy = 0, \quad y(0) = 0, \quad y(1) = 2.$$

   Hence show that, approximately,

$$y'(0) = \frac{e}{\varepsilon}.$$

8. Suppose that

$$\dot{X} = A(t)X, \qquad (*)$$

where $X \in \mathbf{R}^n$ and A is a 2π–perodic $n \times n$ matrix, whose elements are continuous.

Define what is meant by the fundamental matrix $\Phi(t)$, and use it to show that the solution of $(*)$ satisfying $X = X_0$ at $t = 0$ is given by

$$X = \Phi(t)X_0.$$

Suppose that $W = \det \Phi$; show that

$$\dot{W} = (\mathrm{tr}\, A)W,$$

and deduce that W is never zero.

Deduce that the monodromy matrix $M = \Phi(2\pi)$ has no zero eigenvalues.

Suppose the (distinct) eigenvalues of M are $\exp(2\pi\lambda_s)$, $s = 1, 2 \ldots n$, and that Λ is the diagonal matrix diag (λ_s).

If Ψ is the matrix $\exp(t\Lambda)$, show that

$$\Psi(2\pi) = \mathrm{diag}\, [\exp(2\pi\lambda_s)]\,.$$

Suppose now that the constant matrix P satisfies $P^{-1}MP = \Psi(2\pi)$. Show that $P\Psi(2\pi) = \Phi(2\pi)P$, and deduce that the matrix $B = \Phi P \Psi^{-1} P^{-1}$ is 2π–periodic.

Show how this can be used to prove Floquet's theorem, that there is a 2π–periodic matrix $B(t)$ such that $X = BY$, and

$$\dot{Y} = CY,$$

where C is a constant matrix, which you should find.

Explain how the eigenvalues of C are related to those of M.

9. (i) The function $u(x,t)$ satisfies the heat equation

$$u_t = u_{xx}$$

on $0 < x < \infty$, together with the initial condition

$$u = 0 \quad \text{at} \quad t = 0,$$

and the boundary conditions

$$u = 1 \quad \text{at} \quad x = 0,$$

$$u \to 0 \quad \text{as} \quad x \to \infty.$$

Find a similarity solution for u.

(ii) Suppose now that $u(x,t)$ satisfies

$$u_t = (uu_x)_x \ \text{ in } \ 0 < x < \infty,$$

with the same initial and boundary conditions as above.

Show that a similarity solution in the form $u = f(\eta)$, $\eta = \dfrac{x}{ct^\alpha}$ can be found if α is chosen appropriately, and show that if also c takes a certain value, then f satisfies the equation

$$(ff')' + 2\eta f' = 0.$$

Write down the boundary conditions for f.

Suppose now that $f > 0$ for all finite η, and assume that $\eta f \to 0$ as $\eta \to \infty$. By integrating the equation for f, show that f satisfies

$$f' = -2\eta - \frac{2\int_\eta^\infty f(\xi)\,d\xi}{f},$$

and deduce that

$$f < 1 - \eta^2$$

in this case.

Hence show that in fact f must reach zero at some finite value η_0. Can this same argument be used to show that $\eta_0 < 1$?

10. Jensen's inequality states that

$$f\left[\int_\Omega g(\mathbf{x})\,d\omega\right] \le \int_\Omega f[g(\mathbf{x})]\,d\omega,$$

if f is convex, i.e., $f'' > 0$, and the spatial domain Ω has unit measure, i.e., $\displaystyle\int_\Omega d\omega = 1$.

The function u satisfies

$$u_t = \nabla^2 u + \lambda f(u) \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{for} \quad \mathbf{x} \in \partial\Omega, \qquad u = 0 \quad \text{at} \quad t = 0,$$

where Ω is a bounded domain, $\lambda > 0$ and f is a positive convex function for $u \ge 0$. Assume that $u \ge 0$ for $t \ge 0$.

Assume that the principal eigenfunction ϕ and the corresponding eigenvalue μ of Helmholtz's equation

$$\nabla^2\phi + \mu\phi = 0$$

satisfy $\phi > 0$ and

$$\mu = \min_\phi \frac{\displaystyle\int_\Omega |\boldsymbol{\nabla}\psi|^2\,dV}{\displaystyle\int_\Omega |\psi|^2\,dV}.$$

Using the definitions

$$d\omega = \frac{\phi\,dV}{\displaystyle\int_\Omega \phi\,dV}$$

and

$$v = \int_\Omega u\,d\omega,$$

use Jensen's inequality to show that

$$\dot{v} \ge -\mu v + \lambda f(v),$$

where $\dot{v} = dv/dt$.

Suppose now that $w(t)$ satisfies the equation

$$\dot{w} = -\mu w + \lambda f(w), \quad w(0) = 0,$$

and that $f(w) > 0$, $f'(w) > 0$ for all $w \ge 0$. By writing the solution as a quadrature in the form $t = I(w)$, show that if λ is sufficiently large that $\lambda f(w) - \mu w$ is positive for all $w \ge 0$, then $w \to \infty$ as $t \to t_c$, where

$$t_c = I(\infty) = \int_0^\infty \frac{dw}{\lambda f(w) - \mu w},$$

assuming this integral exists.

Show that $I(v)$ is a monotone increasing function of v, and that v satisfies the inequality $I(v) \ge t$. Deduce that $v \ge w$, and therefore that u must blow up in finite time.

11. A forced pendulum satisfies the equation

$$l\ddot{\theta} + k\dot{\theta} + g\sin\theta = a\sin\alpha t.$$

By scaling the equation suitably, show that this can be written in the dimensionless form

$$\ddot{\theta} + \beta\dot{\theta} + \sin\theta = \varepsilon\sin\omega t,$$

and give the definitions of β, ε and ω.

Suppose that $\varepsilon \ll 1$ and $\theta \ll 1$. Find an approximate linear equation for θ and obtain its (particular) solution if $\beta = 0$. Plot the amplitude $A = \max|\theta|$ as a function of the driving frequency ω.

Now suppose $\beta \neq 0$ (but is still small). Find the solution (ignoring the transient) and show that its amplitude A is given by

$$A = \frac{\varepsilon}{|1-\omega^2+i\beta\omega|}.$$

Next consider the oscillator when $\beta = 0$ and $\varepsilon = 0$. Find a first integral of the motion, plot the trajectories in the $(\theta, \dot{\theta})$ phase plane, and hence deduce that the period of oscillation P is given by

$$P = 4\int_0^A \frac{du}{\sqrt{2}[E-(1-\cos u)]^{1/2}},$$

where A is the amplitude of the oscillation, and

$$E(A) = 1-\cos A.$$

Hence show that the frequency of the undamped, unforced pendulum is given by

$$\Omega(A) = \frac{\pi}{\sqrt{2}\displaystyle\int_0^A \frac{du}{[\cos u - \cos A]^{1/2}}}.$$

Show that $\Omega \to 1$ as $A \to 0$.

Suppose that when $\beta \neq 0$ and $\varepsilon \neq 0$, the amplitude of oscillation is related to the forcing frequency by

$$A = R[\Omega(A)],$$

where

$$R(\Omega) = \frac{\varepsilon}{|\Omega^2-\omega^2+i\beta\omega|}.$$

Assume that $\Omega(A) = 1-\frac{1}{8}A^2$; write down an expression for the inverse function $A = I(\Omega)$.

By consideration of the intersection of the graphs of $I(\Omega)$ and $R(\Omega)$, show that, for sufficiently small ε and β, there can be one, three, or exceptionally two values of Ω (and thus A) for given ω. Draw a graph of the amplitude A in terms of ω.

12. The size of courgette plants in my garden is measured by the leaf area L. The rate of growth of the plants is proportional to leaf area, and also to received sunlight, which is itself proportional to leaf area. Additionally, the plants grow to a maximum size. Explain why a growth rate for L of $gL^2\left(1-\dfrac{L}{L_0}\right)$ represents these assumptions.

    Slugs consume courgette leaves at a rate rS, where S is slug density, and I plant out seedlings at a rate p (leaf area per unit time). Explain why a model equation for leaf area can be assumed to be

    $$\dot{L} = p - rS + gL^2\left(1-\frac{L}{L_0}\right),$$

    where $\dot{L} = dL/dt$.

    Assume to begin with that the slug density S is constant. Non-dimensionalise the equation to obtain the dimensionless model

    $$\dot{l} = 1 - \rho + \gamma l^2(1-l),$$

    and define the parameters ρ and γ.

    Show that if $\rho < 1$, the plants grow to a healthy size.

    Show that if $\rho > 1 + \dfrac{4\gamma}{27}$, I cannot grow courgettes.

    Show that if $1 < \rho < 1 + \dfrac{4\gamma}{27}$, there is a threshold value l_c such that if $l > l_c$, plants will thrive, but if $l < l_c$, plants will die out.

    Draw a graph of the steady state l versus ρ, and show that this response diagram indicates hysteresis as ρ varies. Indicate where $\dot{l} > 0$ and $\dot{l} < 0$ on the diagram, and thus determine the stability of the steady state(s).

    In reality, the slug consumption rate depends on leaf area. Suppose now that

    $$r = \frac{r_0 L}{L + L_c}.$$

    Write down the corresponding form of the dimensionless model in this case. Suppose that $l_0 = L_c/L_0$ is small. Draw a response diagram of steady state l versus $\rho_0 = r_0 S/p$ in this case, indicating carefully how this diagram differs from the previous one.

    Show that the diagram again displays hysteresis, and that there are two values ρ_- and ρ_+ such that for $\rho_- < \rho_0 < \rho_+$, there are three possible steady state values of l, only two of which are stable. Show that $\rho_- \to 1$ and $\rho_+ \to 1 + \dfrac{4\gamma}{27}$ as $l_0 \to 0$, and that the corresponding values $l_- \to 0$ and $l_+ \to 2/3$.

    Slugs are attracted towards foliage at a rate proportional to leaf area, and I kill them at a rate proportional to their number. Explain why a suitable model equation for slug density is

    $$\dot{S} = aL - bS,$$

and show that by suitable choice of the slug density scale, the dimensionless slug model can be written in the form

$$\dot{s} = l - \mu s,$$

and give the definition of μ.

Hence show that for small killing rate or high planting rate, a stable state occurs in which slugs win, but for high killing rate or low planting rate, the plants win. Show also that for intermediary rates, oscillations are possible.

13. The variables x and y satisfy the ordinary differential equations

$$\dot{x} = f(x, y),$$

$$\dot{y} = g(x, y),$$

where $\dot{x} = dx/dt$, $\dot{y} = dy/dt$, and t denotes time. Suppose (x_0, y_0) is a fixed point of these equations. Describe how the stability of the fixed point is determined by the trace T and determinant D of the community matrix

$$M = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix},$$

and indicate on a diagram in which regions of (T, D) space the fixed point is a saddle, a node, or a spiral. Give a necessary and sufficient condition on T and D for (x_0, y_0) to be stable.

The functions $G(x)$ and $H(x)$ are defined for positive x by

$$G(x) = x^2 e^{-x}, \quad H(x) = \beta e^{-2x},$$

where $\beta > 0$. Show that the equation $G(x) = H(x)$ has a unique positive root. If this root is denoted by $x_0(\beta)$, show (for example, graphically) that x_0 increases with β, and that $x_0 \to 0$ as $\beta \to 0$ and $x_0 \to \infty$ as $\beta \to \infty$.

Now suppose that in the differential equations for x and y,

$$f(x, y) = y - G(x), \quad g(x, y) = H(x) - y,$$

and suppose that x and y are initially non-negative. Show that x and y remain positive for $t > 0$.

Show that there is a unique fixed point P at $(x_0, H(x_0))$ in the positive quadrant. Draw the nullclines in the phase plane, indicating on your diagram the direction of trajectories in the four regions of the positive quadrant delineated by the nullclines.

Show that the trace T and the determinant D of the community matrix at the fixed point P are given by

$$T = -G'(x_0) - 1, \quad D = G' - H',$$

and deduce that $D > 0$ for all positive values of β.

Derive an expression for $-G'(x)$, and show that it has a maximum at $x = x_M = 2 + \sqrt{2}$. *Assuming* that $-G'(x_M) < 1$, deduce (explaining why) that P is stable for all positive values of β.

14. The Fisher equation is given by

$$u_t = ru\left[1 - \frac{u}{K}\right] + \{Du_x\}_x.$$

If r, K and D are constant, show how to non-dimensionalise the equation to the form

$$u_t = u\,[1 - u] + u_{xx}.$$

If, initially, $u = 0$ except on a finite interval where it is small and positive, explain using diagrams how you would expect the solution to evolve. Verify your description by seeking travelling wave solutions of the form $u = f(\xi)$, $\xi = x - ct$, and write down the resulting equation and boundary conditions which f must satisfy, assuming that $c > 0$. How should the boundary conditions be modified if $c < 0$?

By examining the solutions in a suitable phase plane, show that a travelling wave in which $u > 0$ is positive is possible if $c \geq 2$. What happens if $c < 2$?

Sketch the form of the wave as a function of ξ. If, instead, $D = D_0 u$, what form would you expect a travelling wave to take, assuming such a wave exists?

15. The function $u(x,t)$ satisfies the equation

$$u_t + uu_x = -\beta u,$$

with initial conditions

$$u = u_0(x) \quad \text{on} \quad -\infty < x < \infty,$$

and $u_0 > 0$, $u \to 0$ as $x \to \pm\infty$.

Use the method of characteristics to derive the solution in the parametric form

$$u = u_0(s)e^{-\beta t},$$

$$x = s + \frac{u_0(s)\left(1 - e^{-\beta t}\right)}{\beta}.$$

Assume that $\beta > 0$ and that $t \geq 0$. By first writing $u = F(x,t,u)$, or otherwise, find an expression for u_x, and hence show that a shock will develop if $\max |u_0'| > \beta$.

Suppose that $u_0(s) = \alpha(1 - |s|)$ for $s \in (-1,1)$, and $u_0 = 0$ otherwise. If $0 < \alpha < \beta$, show that characteristics do not intersect, and deduce that $u = 0$ for $|x| > 1$. By considering the characteristics from $0 < s < 1$, show also that

$$u = \frac{\alpha\beta(1-x)e^{-\beta t}}{\beta - \alpha(1 - e^{-\beta t})}$$

for $\dfrac{\alpha((1 - e^{-\beta t})}{\beta} < x < 1$. Write down a corresponding expression for u in $0 < x < \dfrac{\alpha((1 - e^{-\beta t})}{\beta}$. Draw the characteristic diagram for the solution.

16. The function $u(x,t)$ satisfies the equation

$$u_t + u^2 u_x = \varepsilon u_{xx}$$

on $(-\infty, \infty)$, where $\varepsilon \ll 1$. At $t = 0$, $u = u_0(x)$, where u_0 is positive and $u_0(\pm\infty) = 0$. Show that $\int_{-\infty}^{\infty} u\, dx$ is conserved in time.

If the diffusion term is neglected, use the method of characteristics to solve the equation, and hence derive an expression for u_x as a function of x, t and u. Use this to show that a shock will form for $t > 0$ if $u_0' < 0$, and this occurs at $t = \min\limits_{u_0'<0} \dfrac{1}{2u_0|u_0'|}$.

Suppose $u_0(x) = V(1 - |x|)$ for $|x| < 1$, $u_0 = 0$ otherwise, where $V > 0$. Show that a shock first forms when $t = \dfrac{1}{2V^2}$, at the point $x = \frac{1}{2}$. Draw the shape of u as a function of x at this time.

For time $t > \dfrac{1}{2V^2}$, a shock propagates forwards at a position $x = X(t)$. If the value of u behind the front is $U(t)$, show that, for large enough t,

$$tU^2 + \frac{U}{V} - 1 - X = 0,$$

and that (also for large enough t)

$$\dot{X} = \tfrac{1}{3}U^2.$$

Deduce that for large t and X,

$$U \approx \left(\frac{X}{t}\right)^{1/2},$$

and hence that

$$X \approx at^{1/3},$$

and use the value of the conserved integral $\int_{-\infty}^{\infty} u\, dx$ to show that

$$a = \left(\frac{3V}{2}\right)^{2/3}.$$

If ε is small but non-zero, show that a shock structure consistent with this description is possible, by writing $x = X + \varepsilon\xi$, and solving the resulting approximate equation for u with the boundary conditions $u \to U$ as $\xi \to -\infty$, $u \to 0$ as $\xi \to \infty$.

17. A model of snow melt run off from a snow pack is described by the equations

$$\phi\frac{\partial S}{\partial t}+\frac{\partial u}{\partial z}=0,$$

$$u=-\frac{k(S)}{\mu}\left[\frac{\partial p}{\partial z}-\rho g\right],$$

$$p_a-p=p_c(S),$$

where z measures distance downwards from the surface of the snowpack, and $S\in(0,1)$ is the relative water saturation ($S=1$ represents total saturation, $S=0$ represents complete dryness).

Show that the equations can be reduced to a single equation for S of the form

$$\phi\frac{\partial S}{\partial t}=\frac{\partial}{\partial z}\left[D(S)\frac{\partial S}{\partial z}-K(S)\right],$$

and give the definitions of the hydraulic diffusivity $D(S)$ and hydraulic conductivity $K(S)$.

If $k(S)=k_0k_r(S)$ and $p_c(S)=p_0f(S)$, where k_r and f are dimensionless and of $O(1)$, show that a dimensionless model can be written in the form

$$\frac{\partial S}{\partial t}=-\frac{\partial u}{\partial z}=\frac{\partial}{\partial z}\left[D(S)\frac{\partial S}{\partial z}-\varepsilon k_r(S)\right],$$

where now

$$D(S)=-k_r(S)f'(S),$$

u is the dimensionless flux, and

$$\varepsilon=\frac{\rho g d}{p_0},$$

d being a suitable length scale.

Assume now that $k_r(S)=S^2$ and $f(S)=\ln(1/S)$. Find the form of the equation for S in this case.

An initially dry snowpack (i. e., with $S=0$) of depth d begins to melt, so that the dimensional surface flux for $t>0$ is $u|_{z=0}=q$. Show that the corresponding dimensionless surface flux is $q^*=\dfrac{\mu d q}{k_0p_0}$.

Supposing that $\varepsilon\ll 1$, show that a similarity solution in the form $S=t^\alpha g(\eta)$, $\eta=z/t^\beta$, which describes the advance of the wetting front into the snowpack can be found, if

$$(gg')'=\tfrac{1}{3}g-\tfrac{2}{3}\eta g',$$

and

$$gg'=-q^*\quad\text{on}\quad\eta=0.$$

Explain briefly why for this model there is a finite wetting front, i. e., $g = 0$ at $\eta = \eta_0$.

Show that $\displaystyle\int_0^{\eta_0} g\, d\eta = q^*$.

Show that

$$g' = -\tfrac{2}{3}\eta - \frac{\displaystyle\int_\eta^{\eta_0} g\, d\eta}{g},$$

and deduce that $g' < -\frac{2}{3}\eta$.

If $g = g_0$ at $\eta = 0$, show that $g < g_0 - \frac{1}{3}\eta^2$, and deduce that $g_0 > \frac{1}{3}\eta_0^2$. Show also that the integral constraint on g implies $g_0 > \frac{1}{9}\eta_0^2 + \dfrac{q^*}{\eta_0}$.

By graphical means, show that these two inequalities together imply that $g_0 > (3q^{*2}/4)^{1/3}$.

Show that the dimensionless breakthrough time (when the wetting front reaches the base of the snow pack) is $t_B = 1/\eta_0^{3/2}$, and the dimensionless ponding time (when the surface saturation reaches $S = 1$) is $t_S = 1/g_0^3$. By means of the above inequalities, show that $(t_B/t_S)^{1/3} > (g_0^3/3)^{1/4}$, and hence deduce that $t_B > t_S$ if $q^* > 2$.

18. The function $u(x,t)$ satisfies the nonlinear diffusion equation

$$u_t = [(D+u)u_x]_x \quad \text{on} \quad 0 < x < \infty,$$

where D is a non-negative constant, together with the boundary conditions $u = 1$ at $x = 0$, $u \to 0$ as $x \to \infty$, and the initial condition $u = 0$ at $t = 0$. Show that a similarity solution in the form $u = f(\eta)$, $\eta = x/ct^{\beta}$, can be found, where for suitable constants c and β, f satisfies

$$[(D+f)f']' + 2\eta f' = 0. \tag{1}$$

Write down the boundary conditions satisfied by f.

Suppose that $f(\eta)$ is *any* analytic function satisfying an equation of the form

$$f'' = \phi_{2,1}(\eta, f, f')f',$$

where $\phi_{2,1}(\eta, f, g)$ has derivatives of all orders with respect to f and g. Show by induction that

$$f^{(n)} = \phi_{n,1}(\eta, f, f')f' + \ldots + \phi_{n,n-1}(\eta, f, f')f^{(n-1)},$$

and deduce that if $f' = 0$ at a point, then $f' \equiv 0$.

Hence show that the solution of equation (1) satisfying the boundary conditions you have prescribed must be monotonically decreasing.

Deduce from this that if $D > 0$, then $f > 0$ for all finite η. Why does this conclusion not apply if $D = 0$?

Show, contrarily, that if $D = 0$, then $ff' < -2\eta f$, and deduce that f must reach 0 at a finite value of $\eta = \eta_0$, and that in fact $\eta_0 < 1$.

19. A small drop of fluid of depth h sits on a horizontal plane. The equation of conservation of mass can be written in the form

$$h_t + \boldsymbol{\nabla}.\mathbf{q} = 0,$$

where $\mathbf{q}$ denotes the horizontal fluid flux (i. e., the integral over the depth of the horizontal velocity). Assume that the horizontal fluid flux, $\mathbf{q}$, can be approximated as

$$\mathbf{q} = -\frac{\rho g h^3}{3\mu}\boldsymbol{\nabla} h,$$

where ρ is density, g is gravity, and μ is the viscosity.

Hence derive the evolution equation

$$h_t = \frac{\rho g}{3\mu}\boldsymbol{\nabla}.[h^3\boldsymbol{\nabla} h],$$

and show that it can be written in the dimensionless form

$$h_t = \boldsymbol{\nabla}.[h^3\boldsymbol{\nabla} h].$$

A lava dome is modelled by a shallow two-dimensional fluid flow of dimensionless depth $h(x,t)$ satisfying the above equation in one space variable x. The eruption rate is modelled by a prescribed dimensionless flux $q = 1$ at $x = 0$. Restricting attention to the dome shape in $x > 0$, show that it may be modelled by the equation

$$h_t = \left[h^3 h_x\right]_x \quad \text{in} \quad x > 0,$$

with

$$h^3 h_x = -1 \quad \text{on} \quad x = 0, \qquad h \to 0 \quad \text{as} \quad x \to \infty.$$

Look for a similarity solution of the form $h = t^\alpha f(\eta)$, $\eta = x/t^\beta$ to the above equations and boundary conditions for a suitable choice of α and β, and show that f satisfies

$$\left(f^3 f'\right)' + \tfrac{4}{5}(\eta f)' = f,$$

with

$$\left(f^3 f'\right)' = -1 \quad \text{at} \quad \eta = 0, \qquad f(\infty) = 0.$$

If f first reaches zero when $\eta = \eta_0$ (which may be infinite) (thus $f > 0$ for $\eta < \eta_0$), show that

$$f^3 f' = -\tfrac{4}{5}\eta f - \int_\eta^{\eta_0} f\, d\eta.$$

Deduce that $f' < 0$, and that

$$f^2 f' < -\tfrac{4}{5}\eta$$

while $\eta < \eta_0$. Hence show that η_0 is in fact finite, and that

$$\eta_0 < \sqrt{\frac{5f(0)^3}{6}}.$$