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The uniqueness of steady state flows of glaciers and ice sheets

A. C. Fowler* *School of Mathematics, 39 Trinity College, Dublin 2, Ireland*
D. A. Larson *Mathematics Department, Indiana University, Bloomington, Indiana 47401, USA*

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Summary. We consider the uniqueness of solutions to a model set of equations describing the temperature and flow of cold (or sub-temperate) glaciers. These equations are based on the model developed by Fowler & Larson (1978), and incorporate the essential features of a free surface boundary, free melting boundary, and a fully non-Newtonian flow law. The only real simplification which is made is that of taking small Péclet number (or Graetz number), which implies that advective heat transport is negligible. Parametric estimates indicate that this is unrealistic, but nevertheless it represents a formally self-consistent way of effectively adopting a 'slab' model, while avoiding the inherent disadvantages of such models. Since the possibility of non-uniqueness is associated with the non-linear viscous heating term, we expect that non-zero advection of heat will only affect the results quantitatively, but not qualitatively. We find that for the various kinds of basal boundary condition which can occur, the solutions for the temperature (and hence the flow field) are unique, with the possible exception of regions where the basal ice is temperate and sliding, but the rest of the glacier is cold. In such regions one can have up to three solutions (even with the exponential approximation to the Arrhenius term), and such solutions can exhibit hysteretic instability: however, we then show that such multiple solutions must transgress the condition that the ice be below its freezing point, and hence they are not relevant in the present study. We therefore conclude that the solutions are unique, and so the non-linear heating term is unlikely to cause surge-like instabilities in real glaciers (or ice sheets). This does not preclude the possibility that the unique solution is linearly unstable to infinitesimal perturbations: such an instability might also lead to surging states, but not in the explosive manner that non-uniqueness would suggest.

1 Introduction

The striking phenomena of glacier surges, described by Meier & Post (1969), and of periodic build-up of ice sheets, as occurs during ice ages, have led to speculation that these large-scale

*Present address: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA.

ice motions can be explained on the basis of the internal dynamics of the ice masses, rather than as a result of variations in the external forcing inputs. In particular, the periodicity of glacier surges implies an internal self-regulating mechanism. As a first step in analysing such a mechanism, it is attractive to seek criteria under which steady state solutions of the equations of glacier flow may be unstable, so that the flow might be expected to develop into a steady, time-dependent mode.

One instability mechanism that has been suggested (Robin 1955, 1969) as a means of triggering surges is thermal instability: if the steady state temperature profile of a cold glacier were unstable to small perturbations, then the basal ice might be warmed to the melting point: basal sliding would be initiated, thus leading to the possibility of a surge. The reason such a thermal instability is feasible is that the flow law for cold ice which relates stress and strain rate tensors is dependent on temperature (Glen 1955), and thus the momentum and energy equations are coupled. In principle, a local increase in temperature decreases the viscosity; this increases the viscous heating, which thus tends to increase the temperature further. This positive feedback mechanism will provide for an instability, but such a tendency will be resisted by the ability of conductive and advective terms to transport the heat away. We might then expect (Nye 1971) that a realistic stability criterion would be that the flow is thermally unstable provided the viscous heating is sufficiently 'large'. A further stabilizing feature, which will be seen to be crucial as far as multiplicity of the steady states is concerned, is the fact that the top surface is a free boundary (i.e. is not given as part of the input data) and is determined from the temperature and flow fields. Thus if a local temperature increase leads to an increase in velocity, this is balanced by a decrease in depth and hence in stress, which tends to lower the viscous heating.

The physical discussion above is suggestive but inconclusive: we need an analytical model in order to evaluate a quantitative stability criterion. Recent work has been done in this area by Clarke, Nitsan & Paterson (1977) and Yuen & Schubert (1979). These authors consider slab models of glacier flow in which the depth is constant, and in which the temperature is a function of the vertical depth ξ from the glacier surface. These assumptions ensure that advective heat transport is either zero (Yuen & Schubert 1979) or a function only of ξ (Clarke *et al.* 1977), the latter being the case if the vertical velocity v is independent of the lengthwise coordinate x. Thus the x dependence of the solution which is essentially due to longitudinal advective heat transport is suppressed, and the governing equations are essentially ordinary differential equations. This simplification is not really justified in practical cases, but in fact it seems unlikely that advective heat transport would have a major qualitative effect on the types of solution. Also, it makes the model problem reasonably tractable, both numerically and analytically.

With a neglect of advection, the energy equation is a second order ordinary differential equation with a non-linear source term, of the form

$$\theta_{\xi\xi} + f(\xi, \theta) = 0, \tag{1}$$

where θ is the (dimensionless) temperature. For glacier flow, a relevant form of f is

$$f = \alpha\xi^{n+1} \exp(\theta), \tag{2}$$

where n is the power law exponent in Glen's flow law (Glen 1955), and α is a dimensionless measure of the importance of viscous heating. Equations of the type (1) are well known in other applications, for example in chemical reactor theory (Gavalas 1968; Luss & Amundson 1967), polymer flows (Pearson 1978) and the study of the convection of the Earth's mantle (Melosh 1976; Yuen & Schubert 1977). In particular, the last two papers cited, the paper by

Clarke *et al.* (1977) and other work, e.g. Joseph (1964, 1965), Gavis & Laurence (1968) all show that with the exponential dependence on temperature (2), the equation (1) has multiple (in fact two) steady state solutions for a geometry with *fixed* boundaries, if α is small enough. Parter (1974) and Clarke *et al.* (1977) and Yuen & Schubert (1977) showed that with a proper Arrhenius-type term

$$f \propto \exp\left[\frac{\theta}{1+\delta\theta}\right],$$

there are in fact three branches of solutions (subcritical, supercritical and hot). For the exponential case (which is sufficient for discussion of ice flow), there are no solutions at all beyond some critical value of α. It is this fact that has led Clarke *et al.* and Yuen & Schubert (1979) to consider the possibility that glacier or ice sheet surges might be triggered by an increase of α beyond its critical value. This process is known as thermal runaway, and the associated growth rate is catastrophic, the temperature reaching infinity in a finite time (Gruntfest 1963). In glaciers we would expect large areas of basal ice to become temperate, and hence to slide.

As far as they go, these results are interesting and relevant, but it is worth enquiring whether more physically realistic models will retain the same characteristics. As pointed out by Clarke *et al.*, it is not likely that heat advection will seriously affect the *possibility* of non-uniqueness in the solutions. However, Fowler (1980a) has observed that determination of the free boundary surface as part of the solution may fundamentally alter this aspect. In fact, if the ice flux s is used as a control variable in Yuen & Schubert's (1979) results (rather than the depth), then it appears by inspection of their figures that the control space of output u_0 (surface velocity) against flux s ($\approx hu_0$, h = depth) will be single-valued, which would negate the possibility of the thermal runaway type of instability occurring.

In order to examine more critically the possibility of multiple steady states, we give in this paper an analysis of a realistic model, based on that of Fowler & Larson (1978), which incorporates such features as a free top surface and a free melting surface, and also the various kinds of basal boundary conditions which occur. The analysis is not limited to completely cold glaciers, although we consider only the uniqueness of the temperature profile of the cold ice. The only unwarranted assumption we make is that the Péclet number is small, that is, heat advection is negligible. As already stated, we surmise that this assumption, while not realistic, will not affect the results qualitatively. In particular, we then find that all solutions are in fact unique, and since it is the viscous source term that might generate non-uniqueness, we expect this conclusion to be also valid when advection is present. The neglect of the Péclet number is in effect equivalent to a slab model assumption, as regards having ordinary differential equations to solve. However, x dependence is still present through the boundary conditions, and thus the essential finiteness of the geometry is retained. This is likely to be of importance in future linear stability studies (since we would then exclude perturbations of infinitely long wavelength, for example).

2 Model equations and boundary conditions

The model we use is the 'reduced' one derived by Fowler & Larson (1978). It is an asymptotic reduction of the full equations, based on the assumptions of slow, shallow flow, and incorporating Glen's flow law in the form of equation (2). We take Cartesian axes (x, ξ), with x along the line of mean bedrock slope, ξ measured transversely downwards from the top surface (which is thus $\xi = 0$). We make the additional (inessential) assumption that $\mu = 0$, that is we neglect the variation of surface slope from the bedrock slope: we emphasize that this assumption is made *purely* for convenience. In this case, the steady state equations of energy and momentum for regions of cold ice may be written for the temperature T and a modified stream function Ψ in the form

$$\Psi_{\xi\xi} = \xi^n \exp(\kappa T), \tag{3}$$

$$\Psi_x T_\xi - \Psi_\xi T_x = \beta_2 T_{\xi\xi} + \beta_1 \xi^{n+1} \exp(\kappa T). \tag{4}$$

Similar equations are considered by Pearson (1977) for the channel flow of non-Newtonian polymers or plastics.

The relevant boundary conditions to be satisfied by T and Ψ are these:

$$\text{on } \xi = 0: \Psi = s(x), \quad T = T_A(x); \tag{5}$$

$$\begin{aligned} \text{on } \xi = H(x)&: \Psi = 0, \quad \Psi_\xi = -u_b(H, T), \\ &\beta_1 H \Psi_\xi + \beta_2 T_\xi = \lambda\Phi(T), \quad x \in B_C, \\ &T = 0, \quad x \in B_M; \end{aligned} \tag{6}$$

$$\text{on } \xi = \xi_M(x): T = T_\xi = 0\,(x \in B_T). \tag{7}$$

These conditions require some explanation. The surface conditions (5) are that the surface temperature (T_A) and accumulation rate (ds/dx) are prescribed. In equation (6), $H(x)$ is the unknown depth, and the first two conditions are the kinematic conditions of no flow through the base, and that tangential velocity there is equal to the sliding velocity u_b. Here u_b is a temperature dependent sliding law, introduced as such by Fowler & Larson (1978) to conform with the simple physical requirement that an ice–bedrock interfacial water film accumulates gradually as the melting point is approached at the base. The first condition in equation $(6)_3$ is a heat flux balance, where the function $\Phi(T)$, which decreases from one to zero as the melting temperature is approached, reflects the fact that some of the geothermal heat flux λ is used in producing the above mentioned water film when the ice is slightly cold; the regions B_C, B_M and B_T simply refer to basal regions where the ice is cold, at melting point, or where there exists a temperate layer, respectively. The extent of these regions is not known *a priori*, but is determined with the solution. Thus if $x \in B_C$, we proceed solving the equations with the relevant boundary conditions until $T = 0$ at $x = x_Z$, say, when we shift to B_M. We can do this sequentially because of the time-like role of x in equation (4). Note that the treatment is relevant to polythermal glaciers, where regions of temperate ice may exist, due to the possibility of basal regions B_T occurring.

We now wish to consider the asymptotic limit of small Péclet number, or equivalently $\beta_2 \to \infty$ with β_1/β_2 constant. We define

$$\begin{aligned} \theta &= \kappa T, \\ u_b(H, T) &= F(H, \theta), \\ \Phi(T) &= \phi(\theta), \\ \alpha &= \kappa\beta_1/\beta_2, \\ \lambda^* &= \lambda\kappa/\beta_2, \end{aligned} \tag{8}$$

and consider the limit

$$\beta_2 \to \infty; \quad \alpha, \lambda^* = O(1). \tag{9}$$

The equations (3) and (4) become

$$\Psi_{\xi\xi} = \xi^n \exp(\theta), \tag{10}$$

$$\theta_{\xi\xi} + \alpha\xi^{n+1} \exp(\theta) = 0. \tag{11}$$

We suppose also that ϕ and F are as given by Fowler & Larson (1978), that is

$$F = 0, \quad \phi = 1 \text{ for } \theta < \theta_Q, \tag{12}$$

where we expect $|\theta_Q| \sim 10^{-2}$. We then define subsets of B_C to be B_Q, in which $\theta < \theta_Q$ and B_Z, in which $\theta_Q < \theta < 0$. The boundary conditions (5), (6) and (7) then become:

$$\text{on } \xi = 0: \Psi = s(x), \quad \theta = \theta_A(x) = \kappa T_A(x); \tag{13}$$

$$\begin{aligned}
&\text{on } \xi = H: \Psi = 0,\\
&\text{(cold) } \Psi_\xi = 0, \quad \alpha H \Psi_\xi + \theta_\xi = \lambda^*, \quad x \in B_Q,\\
&\text{(subtemperate) } \Psi_\xi = -F[H, \theta], \quad \alpha H \Psi_\xi + \theta_\xi = \lambda^* \phi(\theta), \quad x \in B_Z,\\
&\text{(temperate) } \Psi_\xi = -F[H, 0], \quad \theta = 0, \quad x \in B_M,\\
&\qquad \theta = \theta_\xi = 0, \quad x \in B_T.
\end{aligned} \tag{14}$$

For convenience, we have relabelled ξ_M as H in B_T, since no confusion will accrue. From the definition of λ given by Fowler & Larson (1978), we see that typically $\lambda^* \approx 10^{-1}$ (for glaciers), and it may be useful to neglect it: this is considered further below.

In equation (14) H is not determined, and we see that to uncouple completely θ from Ψ, we need to be able to write two boundary conditions on H involving θ only. This is the case for $x \in B_T$ (which is why we need not consider the basal temperate ice zone there): for the other regions we argue as follows. The equations (10) and (11) can be amalgamated to form

$$\theta_{\xi\xi} + \alpha\xi\Psi_{\xi\xi} = 0. \tag{15}$$

Integrating this between 0 and H yields

$$[\theta_\xi]_0^H = \alpha[\Psi - \xi\Psi_\xi]_0^H = \alpha[-s(x) + HF(H, \theta_b)] \tag{16}$$

where $\theta_b = \theta(H)$, and we suppress the explicit x dependence of θ. Using equation (16) we see that we may obtain two temperature conditions from equation (14) as follows:

$$\begin{aligned}
&x \in B_Q: \theta_\xi = \lambda^*, \quad \theta_\xi(0) - \theta_\xi(H) = \alpha s(x);\\
&x \in B_Z: \theta(H) = \theta_b, \quad \theta_\xi(H) = \lambda^* \phi(\theta_b) + \alpha HF(H, \theta_b),\\
&\qquad \theta_\xi(0) - \theta_\xi(H) = \alpha s(x) - \alpha HF(H, \theta_b);\\
&x \in B_M: \theta(H) = 0, \quad \theta_\xi(0) - \theta_\xi(H) = \alpha[s(x) - HF(H, 0)];\\
&x \in B_T: \theta(H) = 0, \quad \theta_\xi(H) = 0.
\end{aligned} \tag{17}$$

With these conditions, we can solve for θ and H, and then determine Ψ uniquely from equations (10), (13) and (14). The question of interest is thus whether equations (11) and (17), together with $\theta(0) = \theta_A$, have multiple solutions. Since x only appears parametically, this is an ordinary differential equation, and we can examine each of the regions B_Q, B_Z, B_M and B_T in turn.

3 Steady state solutions

We now examine the solution of the equation

$$\theta_{\xi\xi} + \alpha\xi^{n+1} \exp(\theta) = 0 \tag{18}$$

with the various basal boundary conditions given in equation (17).

(i) $x \in B_Q$ (cold, non-sliding base).

Here the boundary conditions are

$$\theta(0) = \theta_A, \quad \theta_\xi(H) = \lambda^*, \quad \theta_\xi(0) = \alpha s(x) + \lambda^*. \tag{19}$$

We note that $\theta_A < 0$, and θ_ξ decreases monotonically from $\theta_\xi(0) > 0$. Using the first and third conditions in equation (19), we solve equation (18) as an initial value problem, and determine H as the unique value of ξ for which the second condition holds. The solution is therefore unique, and the maximum temperature is at the base.

(ii) $x \in B_Z$ (sub-temperate, sliding base).

When the basal temperature is in the range $\theta_Q < \theta < 0$, the relevant boundary conditions, from equation (17), are

$$\theta(0) = \theta_A, \quad \theta_\xi(0) = \alpha s(x) + \lambda^* \phi[\theta(H)], \quad \theta_\xi(H) = \alpha HF[H, \theta(H)] + \lambda^* \phi[\theta(H)]. \tag{20}$$

The ranges of ϕ and F vary by $0(1)$ for $\theta \in (\theta_Q, 0)$, and no clear simplification is possible. We might use the fact that $\theta = O(|\theta_Q|)$ as an approximate boundary condition, but no simplification then ensues. Alternatively, we utilize the estimate that $\lambda^* \sim 10^{-1}$, and so may be considered small, at least for glaciers (this will not be realistic for ice sheets). If we let $\lambda^* \to 0$, then equation (20) becomes

$$\theta(0) = \theta_A, \quad \theta_\xi(0) = \alpha s(x), \quad \theta_\xi(H) = \alpha HF[H, \theta(H)]. \tag{21}$$

It is now clear that we can solve equation (21) as an initial value problem, giving a concave profile $\theta(\xi)$. In principle, equation $(21)_3$ then determines H, but it is more profitable to use the fact that $|\theta_Q| \sim 10^{-2} \ll 1$, so that to $O(|\theta_Q|)$, H is determined by the intersection of $\theta(\xi)$ with the ξ-axis. There are either two roots, one or none of this equation. By assumption that $x \in B_Z$, there is at least one root. If there are two, then we take the smaller one since the other corresponds to a solution in which there is a temperate region, i.e. $x \in B_T$. Thus H is again determined uniquely (up to $O(|\theta_Q|)$).*

We now demonstrate that for this case ($\lambda^* = 0$), the extent of the sub-temperate sliding zone B_Z does not vanish as $\theta_Q \to 0$. Thus the discontinuous sliding laws proposed by other authors (e.g. Grigoryan, Krass & Shumskiy 1976) are not consistent with the limiting behaviour as $\theta_Q \to 0$ of a continuous sliding law, and there may exist a finite region of the bedrock surface on which the ice is (almost) temperate but where the sliding velocity is below that predicted for temperate ice. This fundamental physical fact has not to our knowledge been considered before: since it is essentially a consequence of the continuity of the sliding law as θ approaches zero, we do not expect that non-zero λ^* or finite β_2 will alter the result, and thus such sub-temperate basal regions may be expected to occur generally under ice sheets and glaciers. We consider this result to be of major significance to both theoretical and practical glaciological studies. To prove this we simply observe that if $B_Z = (x_Q, x_Z)$, say, where $\theta = \theta_Q$ at x_Q and $\theta = 0$ at x_Z, and if $|x_Q - x_Z| \to 0$ as $\theta_Q \to 0$, then in this limit $\theta_A = \theta_A^* + o(1)$, $s = s^* + o(1)$, where θ_A^* and s^* are constants, and we have

* *Note added in proof.* Using Figs 1 and 4 below, we can easily extend this result to $\lambda^* \neq 0$, if we make the reasonable assumptions $d\phi/d\theta < 0$, $\partial F/\partial\theta > 0$.

assumed continuity in x of s and θ_A. Then the initial value problem for θ in B_Z satisfies $\theta(0) = \theta_A^* + o(1)$, $\theta_\xi(0) = \alpha s^* + o(1)$. If the unique solution of equation (18) with $\theta(0) = \theta_A^*$, $\theta_\xi(0) = \alpha s^*$, has its first zero at $\xi = H^*$, then since the solution is continuously dependent on the boundary conditions, we have that $H = H^* + o(1)$ as $\theta_Q \to 0$. It follows that $\theta_\xi(H) = \theta_\xi(H^*) + o(1)$; however, in B_Z, $\theta_\xi(H)$ must range from zero to $\alpha H^* F(H^*, 0)$, as θ increases from θ_Q to zero. Thus the last condition in equation (20) cannot be valid as $\theta_Q \to 0$, and we obtain a contradiction: our original supposition that $|x_Q - x_Z| \to 0$ is invalid, and the sub-temperate basal zone does not vanish.

Since in fact a prescription of F and ϕ would pose a major difficulty in obtaining solutions (analytically *and* numerically), it is satisfying to note that in the limit as $\theta_Q \to 0$ with $\lambda^* = 0$, we can replace equation (20) by the *effective* boundary conditions

$$\theta(0) = \theta_A, \quad \theta_\xi(0) = \alpha s(x), \quad \theta(H) = 0. \tag{22}$$

These conditions are valid while $\theta_\xi(H) < \alpha HF(H, 0)$, and become invalid when this inequality no longer holds. We see that the solution in B_Z is *effectively* unique.

(iii) $x \in B_M$: (temperate sliding base).

The relevant boundary conditions for this case are

$$\theta(0) = \theta_A, \quad \theta(H) = 0, \quad \theta_\xi(0) - \theta_\xi(H) = \alpha[s(x) - HF(H, 0)]. \tag{23}$$

The problem is no longer an initial value problem, and some subtlety is needed to evaluate the multiplicity of solutions. For *fixed H*, the problem (18) with the first two boundary conditions in equation (23) is known to have two solutions for sufficiently small H (this follows from work by Joseph 1966). In a control space, we can take an 'input' coordinate H and any convenient 'output' we desire. We choose

$$p = \theta_\xi(0) - \theta_\xi(H); \tag{24}$$

then a control map of solutions in the (H, p) plane is shown in Fig. 1. Note that if (for fixed H)

$$z = \xi/H, \quad f(z) = \theta(\xi), \tag{25}$$

then

$$f'' + \alpha H^{n+3} z^{n+1} \exp(f) = 0, \quad f(0) = \theta_A, \quad f(1) = 0, \tag{26}$$

of which the subcritical solution $f_- \sim |\theta_A|(z - 1) + O(H^{n+3})$ as $H \to 0$, whence $p \to 0$ as $H \to 0$ along the subcritical branch; along the supercritical branch f develops a large maximum of $O[\log(1/\alpha H^{n+3})]$, and it follows that $p \to \infty$ as $H \to 0$ along this branch. This suffices to give the schematic diagram in Fig. 1. Since in equation (23), H is *a priori* unknown, we obtain solutions of equations (18) and (23) by plotting intersections of the graph of $p(H)$ given in Fig. 1 with the curve

$$p = \alpha[s(x) - HF(H, 0)]. \tag{27}$$

For fixed α and x, this is a concavely decreasing function if we assume that HF increases convexly with H. For basal flow by deformation only, $HF \propto H^{n+1}$ (e.g. Fowler 1980b); Weertman (1957, 1964, 1971) proposed an empirical law incorporating regelation effects in which $HF \propto H^{(n+3)/2}$. For Newtonian flow, Nye (1969, 1970) showed that $HF \propto H^2$. Thus it is sensible to suppose that p defined by equation (27) decreases concavely. However,

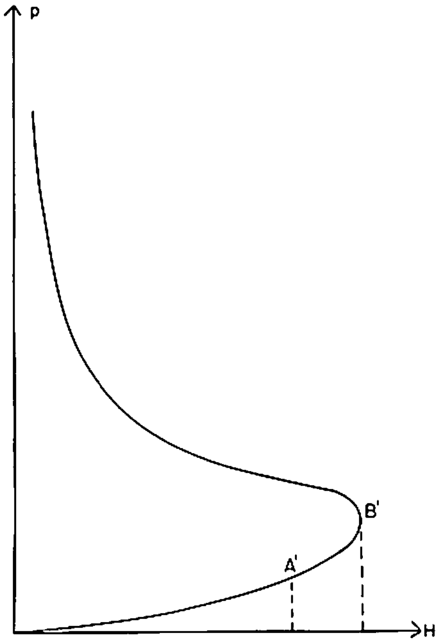

Figure 1. Control space of p (given by equation 24) versus H for the solution of $\theta_{\xi\xi} + \alpha\xi^{n+1}\exp(\theta) = 0$, $\theta(0) = \theta_A$, $\theta(H) = 0$.

Lliboutry (1968) and Fowler (1979) warn that the effects of cavitation may make F multi-valued, and in this case HF may not be convex, or even single-valued. We suppose that cavitation is not important in this sense, and that equation (27) is indeed concave. Possible intersections of equations (27) and (24) are shown in Fig. 2, for fixed α. As we increase s

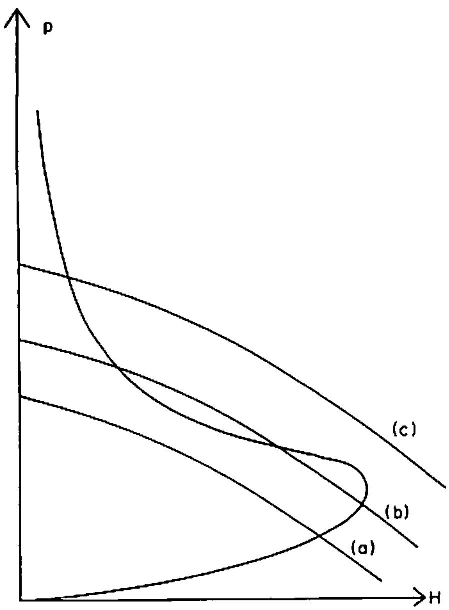

Figure 2. Possible intersections of p defined by equation (27) with p defined by equation (24), same equation and boundary conditions as Fig. 1; (a), (b) and (c) correspond to increasing values of s.

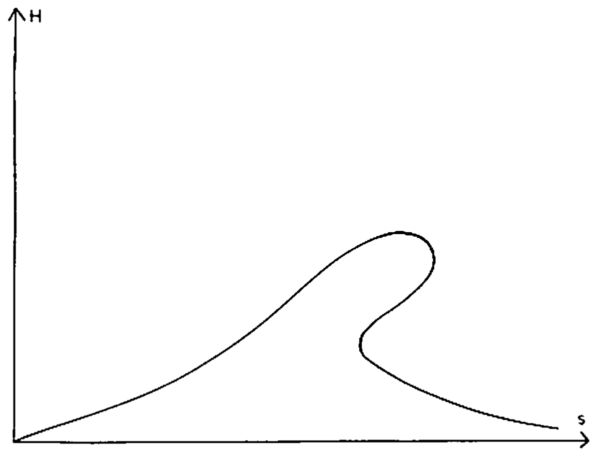

Figure 3. Solution space of H versus s for equations (18) and (23). The flux s is the *input* (given data), and H is the corresponding *output*.

from zero, we see that there is initially one solution in which H increases with s (curve a). If equation (27) is sufficiently sharply curved, two further solutions may exist (curve b). For larger s, two of the solutions coalesce and vanish, and the other decreases to zero as $s \to \infty$. This leads to the solution space (H, s) as shown in Fig. 3. Note that this figure gives multiple solutions involving a hysteretic effect, and is a genuine multiplicity of the solutions of equations (18) and (23) in which s is an input and H is an output of the model. It remains to consider whether these mathematical solutions are all glaciologically relevant.

Now it is not necessary to use p as the output variable in Fig. 1: one can alternatively use the initial gradient g, where

$$g = \theta_\xi(0). \tag{28}$$

For the subcritical solution f, we have $df/dz \sim |\theta_A|$ as $H \to 0$, hence $g = \theta_\xi(0) \sim |\theta_A|/H \to \infty$ as $H \to 0$. It is obvious that $g \to \infty$ as $H \to 0$ along the supercritical branch and hence a solution space of g versus H must be as in Fig. 4. This graph is a topological map of Fig. 1: thus for any H, the sub- and supercritical values of g correspond to sub- and supercritical values of p. What we wish to do is to ascertain what values of g (and hence p) correspond to solutions in which $\theta > 0$, and hence are inadmissible. Now consider the solution of equation (18) with $\theta(0) = \theta_A$, $\theta'(0) = g$. With H defined by $\theta(H) = 0$, Fig. 4 shows that for sufficiently large g, there are two solutions. This is obvious, and in particular the one with smaller H will have $\theta'(H) > 0$, and hence $\theta < 0$ throughout $(0, H)$, whereas the larger will have $\theta'(H) < 0$, hence $\theta > 0$ in a subset of $(0, H)$ and is therefore inadmissible. It is easy to see from Fig. 4 that the admissible solutions are precisely those on the subcritical branch of Fig. 4 which lie to the left of A. Now this part of the curve corresponds directly to the part of the subcritical branch of Fig. 1 lying to the left of A′ (having the same associated value of H as A). It follows that all admissible solutions ($\theta < 0$) lie on this section of the subcritical branch, and thus the essential physics dictates that the supercritical branch is entirely irrelevant for the glaciological flows considered here. Thus, if equation (27) is concavely decreasing, and if $dp/dH > 0$ along the subcritical branch in Fig. 1, the solution for $x \in B_Z$ is unique. It is proved in an appendix that the second condition is indeed valid, so that uniqueness is assured.

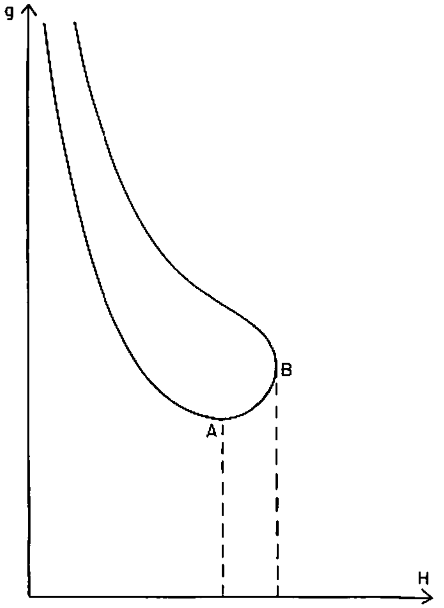

Figure 4. Control space of g (given by equation 28) versus H for the solution of $\theta_{\xi\xi} + \alpha\xi^{n+1}\exp(\theta) = 0, \theta(0) = \theta_A, \theta(H) = 0$.

(iv) $x \in B_T$ (region of temperate ice at base).

Finally, if there exists a region of temperate ice underlying the cold ice, then we have the boundary conditions

$$\theta(0) = \theta_A, \quad \theta(H) = 0, \quad \theta_\xi(H) = 0. \tag{29}$$

In this case, we solve equation (18) with the first two conditions in equation (29) for fixed H; there are thus two solutions, which we can represent in the solution space of g versus H in Fig. 4. Observe that for values of g greater than that at A, the two values of H are such that $\theta_\xi(H) > 0$ for the smaller, and $\theta_\xi(H) < 0$ for the larger (as already observed). As we approach A, the two values of H approach each other, and hence $\theta_\xi(H) = 0$ at A, and only there. Hence the solution in B_T is unique.

4 Conclusions

We find that a realistic model of glacier flow, incorporating non-linear heating effects and free surface and melting boundaries but neglecting heat transport by advection, has temperature and flow fields which are uniquely determined by the input data. Since advection is liable to be a stabilizing influence counteracting the source viscous heating term, we speculate that the uniqueness will be unaffected by advective terms. The uniqueness examined here is due essentially to the existence of the free surface boundary. We have specifically examined only the possibility of non-uniqueness due to non-linear heating: other mechanisms, such as a multi-valued sliding law or source heating in temperate ice zones, have been excluded from the discussion. We conclude that thermal runaway types of instability as a triggering mechanism for ice sheet disintegration or glacier surges (Paterson *et al.* 1977; Cary *et al.* 1979) is not a viable prospect. This is not to say that the steady state solution is necessarily stable to small perturbations: a linear stability analysis of the solutions of the present model will be presented in a subsequent paper.

Finally we remark that of course the inclusion of heat transport terms would lend the conclusions of this paper a great deal more authority. Such an inclusion, however, appears to render the problem almost intractable analytically, and numerical methods for such a double free boundary problem are beset with similar difficulties.

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Appendix

Consider the equation

$$\theta_{\xi\xi} + \alpha\xi^{n+1} \exp(\theta) = 0 \tag{A1}$$

with boundary conditions

$$\theta(0) = \theta_A < 0, \quad \theta(H) = 0 \tag{A2}$$

where H is given. We seek to show that the function p defined by

$$p = \theta_\xi(0) - \theta_\xi(H) \tag{A3}$$

is an increasing function of H for subcritical solutions in which $\theta < 0$ for $\xi \in (0, H)$ (i.e. $\theta_\xi(H) > 0$).

Suppose we take two such solutions θ_1, θ_2 in which $H_1 < H_2$. We wish to show $p_1 < p_2$. From Fig. 4 it follows that $\theta_{1\xi}(0) = g_1 > g_2 = \theta_{2\xi}(0)$. Further, integrating equation (A1) implies that

$$p_i = \int_0^{H_i} \alpha\xi^{n+1} \exp(\theta_i)\, d\xi. \tag{A4}$$

Now θ_1 and θ_2 are monotonic in ξ, thus to any value of $\theta^* \in (-|\theta_A|, 0)$, there corresponds a unique ξ_i such that $\theta_i(\xi_i) = \theta^*$. We can thus define a function $Z(\xi)$ by the implicit relation

$$\theta_1(\xi) = \theta_2[Z(\xi)]. \tag{A5}$$

It follows that

$$p_2 - p_1 = \int_0^{H_2} \alpha Z^{n+1} \exp[\theta_2(Z)]\, dZ - \int_0^{H_1} \alpha\xi^{n+1} \exp[\theta_1(\xi)]\, d\xi$$

$$= \int_0^{H_1} \exp[\theta_1(\xi)] \left[Z^{n+1}\frac{dZ}{d\xi} - \xi^{n+1}\right] d\xi \tag{A6}$$

(note that $Z(H_1) = H_2$). If we differentiate equation (A5) twice with respect to ξ, we obtain

$$\theta_{1\xi\xi} = \theta_{2ZZ}Z'^2 + \theta_{2Z}Z'',$$

whence from equation (A1)

$$\frac{d^2Z}{d\xi^2} = \frac{\alpha \exp\,[\theta_1(\xi)]}{\theta_{2Z}}\,[Z'^2 Z^{n+1} - \xi^{n+1}]. \tag{A7}$$

Observe that $\theta_{2Z} > 0$. At $\xi = 0, g_1 > g_2$ so that initially $\theta_1(\xi) > \theta_2(\xi)$. It follows that initially $Z > 1$, and $dZ/d\xi > 1$. But then equation (A7) implies that while these conditions hold (i.e. sufficiently near $\xi = 0$), $d^2Z/d\xi^2 > 0$: hence $dZ/d\xi$ increases, and also Z, and therefore $Z > 1$ and $dZ/d\xi > 1$ for all $\xi \in (0, H)$. It immediately follows from equation (A6) that $p_2 - p_1 > 0$ as required, and hence p increases with H along the subcritical branch of Fig. 1, at least as far as A$'$.