Fast Thermoviscous Convection

By A. C. Fowler

This paper studies the asymptotic structure of convection in an infinite Prandtl number fluid with strongly temperature-dependent viscosity, in the limit where the dimensionless activation energy $1/\varepsilon$ is large, and the Rayleigh number R, defined (essentially) with the basal viscosity and the prescribed temperature drop, is also large. We find that the Nusselt number N is given by $N \sim C\varepsilon R^{1/5}$, where C depends on the aspect ratio a. The relative error in this result is $O(R^{-1/10}\varepsilon^{-1/4}, \varepsilon^{1/2}, R^{-2/5}\varepsilon^{-2}, R^{-1/20}\varepsilon^{-1/24})$, so that we cannot hope to find accurate confirmation of this result at moderate Rayleigh numbers, though it should serve as a useful indicator of the relative importance of R and ε. For the above result to be valid, we require $R \gtrsim 1/\varepsilon^5 \gg 1$. More important, however, is the asymptotic structure of the flow: there is a cold (hence rigid) lid with sloping base, beneath which a rapid, essentially isoviscous, convection takes place. This convection is driven by plumes at the sides, which generate vorticity due to thermal buoyancy, as in the constant viscosity case (Roberts, 1979). However, the slope of the lid base is sufficient to cause a large shear stress to be generated in the thermal boundary layer which joins the lid to the isoviscous region underneath (though a large velocity is not generated); consequently, the layer does not "see" the shear stress exerted by the interior flow (at leading order), and therefore *the thermal boundary layer structure is totally self-determined*: it even has a similarity structure (as a consequence). This fact makes it easy to analyse the problem, since the boundary layer uncouples from the rest of the flow. In addition, we find an alternative scaling (in which the lid base is "almost" flat), but it seems that the resulting boundary layer equations have no solution, though this is certainly open to debate: the results quoted above are not for this case. When a free slip boundary condition is applied at the top surface, one finds that there exists a thin "skin" at the top of the lid which is a *stress* boundary layer. The shear stress changes rapidly to zero, and there exists a huge longitudinal

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stress (compressive/tensile) in this skin. For earthlike parameters, this stress far exceeds the fracture strength of silicate rocks.

1. Introduction

The convection which occurs in the mantle of the earth, and which is manifested in the processes of plate tectonics and continental drift, is the slow, creeping motion of a nonlinear viscous solid in response to applied heat sources. The secular cooling of the earth's core, evidenced by the gradual solidification of the inner core, releases heat to the base of the mantle, and radioactive decay within the mantle continues to provide an internal heat source: either of these could be dominant in driving convection in the mantle. The flow law of the solid silicates which constitute the mantle of the earth and other terrestial planets is (almost certainly) given in terms of a viscosity which depends strongly on the temperature and pressure as well as on the stress [7]. Of these, the temperature and pressure dependence is most important, as it occurs exponentially. For a basal (core-mantle) temperature of 3000 K and a surface temperature of 300 K, one finds that an activation energy $E^* = 100$ kcal mole^{-1} (a common estimate) yields a viscosity contrast from top to bottom of about 10^{66}; this would be somewhat reduced by a nonzero activation volume.

In studying the convection of planetary mantles, it is thus useful, as a first step, to study the convection in a fluid whose viscosity is strongly thermally activated, and that is the aim of the present paper. In particular, we will assume that E^* is "large," in a sense to be made precise (but basically the viscosity contrast between top and bottom is large), and also that the (appropriately defined) Rayleigh number is large—this will enable us to exploit the known boundary layer structure of convection at high Rayleigh number [13]. For the earth, the Rayleigh number (as defined here) should lie in the range 10^6–10^8. This is sufficient to warrant an asymptotic analysis, although it is not high enough to yield truly accurate results.

There is not a substantial literature on convection in thermally activated fluids. Early work was done by Foster [5] and Torrance and Turcotte [14], and the importance of a thermoviscous rheology was emphasized by Tozer [15]. Experiments have been carried out by Booker [2], Booker and Stengel [3], and more recently Nataf and Richter [10] and Richter et al. [12] (we have omitted reference to various other studies concerned with planforms, onset of convection, etc.). Lately, Christensen [4] has addressed the problem numerically, and some of his results are discussed in Section 5. Morris and Canright [9] have done a theoretical analysis of the same problem as that addressed here. However, although the basic physical natures of the solution obtained by Morris and Canright and the one given here are essentially the same, the detailed analysis is very different. It remains to be seen if these two analyses are reconcilable.

2. Basic equations and scaling

We consider steady convection of a fluid in two dimensions, $0 < x < ad$, $0 < y < d$, so a is the aspect ratio of a cell. With a density given by $\rho_0[1-\alpha(T-T^*)]$, and

pressure $\rho_0 g(d-y)+p$, where T^* is some ambient temperature (to be defined), and α is the thermal expansion coefficient, the Boussinesq equations of motion ($\alpha\Delta T \ll 1$, ΔT defined below) are

$$\begin{aligned} \operatorname{div}\mathbf{u} &= 0, \\ \rho_0\frac{du}{dt} &= -p_x + \tau_{1x} + \tau_{2y} \\ \rho_0\frac{dv}{dt} &= -p_y + \tau_{2x} - \tau_{1y} + \alpha\rho_0(T-T^*)g, \\ \tau_1 &= 2\eta u_x, \\ \tau_2 &= \eta(u_y+v_x), \\ \frac{dT}{dt} &= \kappa\nabla^2 T, \end{aligned} \tag{2.1}$$

where (u, v) are the velocity components, d/dt is the material derivative, τ_1 and τ_2 are the longitudinal and tangential components of the stress deviator tensor ($\tau_1 = \tau_{11} = -\tau_{22}$, $\tau_2 = \tau_{21} = \tau_{12}$), g is gravity, T is temperature, κ is the thermal diffusivity, and η is the viscosity, which we will take to depend exponentially on the temperature, in the form

$$\eta = \eta_0 \exp\left[-\frac{T-T^*}{\Delta T_{\text{rheol}}}\right]. \tag{2.2}$$

We take as boundary conditions those of prescribed temperature at top and bottom, no heat flux through the sides, and free slip (no stress) at all four boundaries, as would be appropriate in the terrestrial context:

$$\begin{aligned} x = 0, ad&: \quad T_x = 0, \quad \tau_2 = 0, \quad u = 0; \\ y = 0&: \quad T = T_b, \quad \tau_2 = 0, \quad v = 0; \\ y = d&: \quad T = T_0, \quad \tau_2 = 0, \quad v = 0. \end{aligned} \tag{2.3}$$

Our eventual aim is to solve (2.1)–(2.3) to find the Nusselt number, N (here k is thermal conductivity):

$$N = \frac{\dfrac{1}{ad}\displaystyle\int_0^{ad}\left[-k\frac{\partial T}{\partial y}\right]\bigg|_{y=d}dx}{\dfrac{k(T_b-T_0)}{d}}. \tag{2.4}$$

We first make the problem dimensionless. To do so, we choose scales

$$\mathbf{u} \sim \frac{\kappa}{d}, \quad \mathbf{x} \sim d, \quad t \sim \frac{d^2}{\kappa},$$

$$T = T^* + (\Delta T)\theta, \quad p, \tau_1, \tau_2 \sim \frac{\eta_0 \kappa}{d^2}, \quad \eta \sim \eta_0. \tag{2.5}$$

Then the nondimensional equations and boundary conditions (2.1)–(2.3) can be written, in terms of the dimensionless variables corresponding to (2.5) (i.e., we write $\mathbf{x} = d\mathbf{x}^*$, etc., and then omit the asterisks), as follows:

$$\begin{aligned}
\operatorname{div}\mathbf{u} &= 0,\\
\frac{1}{\sigma}\frac{du}{dt} &= -p_x + \tau_{1x} + \tau_{2y},\\
\frac{1}{\sigma}\frac{dv}{dt} &= -p_y + \tau_{2x} - \tau_{1y} + R\theta,\\
\frac{d\theta}{dt} &= \nabla^2\theta,\\
\tau_1 &= 2\eta u_x,\\
\tau_2 &= \eta(u_y + v_x),\\
\eta &= \exp[-\theta/\varepsilon],
\end{aligned} \tag{2.6}$$

where

$$\sigma = \frac{\eta}{\rho_0\kappa}, \quad R = \frac{\alpha\rho_0 g d^3 \Delta T}{\eta_0\kappa}, \quad \varepsilon = \frac{\Delta T_{\text{rheol}}}{\Delta T}, \tag{2.7}$$

and

$$\begin{aligned}
&\text{on } x = 0, a: && u = \tau_2 = \theta_x = 0;\\
&\text{on } y = 0: && \theta = \frac{T_b - T^*}{\Delta T}, \quad v = \tau_2 = 0;\\
&\text{on } y = 1: && \theta = \frac{T_0 - T^*}{\Delta T}, \quad v = \tau_2 = 0.
\end{aligned} \tag{2.8}$$

The temperatures ΔT and T^* are yet to be chosen (but roughly $\Delta T \sim T_b - T_0$, $T^* \sim T_b$); the parameters σ and R are the Prandtl number and Rayleigh number relevant for the problem, and ε is an inverse measure of the activation energy. We will assume $\sigma \gg 1$, $R \gg 1$, $\varepsilon \ll 1$, so that inertia is irrelevant.

We now seek to *rescale* (2.6)–(2.8) in keeping with the following discussion of the expected (or intuitive) physical nature of the solutions. It is by now well known that when $\varepsilon \ll 1$, convection (of essentially isoviscous type) occurs beneath a cold (hence rigid) lid, which is essentially stagnant. This is because the tendency of the lid to sink is vastly counteracted by its much greater viscosity. There is thus a large temperature increase across the lid, and a much smaller one beneath, where the viscosity changes only by $O(1)$. Within this convecting zone, the structure is similar to that given by Roberts [13] for isoviscous convection: thermal buoyancy at the side walls balances vorticity and produces a vorticity (or shear stress in the present context) which drives an internal biharmonic flow. This balance of terms tells us how to scale the equations in order to do asymptotic analysis. There is one complication, however, associated with the fact that the lid "base" is to be determined as part of the solution. Since it is sloping, one obtains a much larger buoyancy-produced stress than for a flat base. In fact, a related analysis [6] shows that the thermal layer which connects the cold lid to the (essentially isothermal) interior flow develops a much larger stress than exists in the interior. This enables this layer to be treated separately. Further discussion of the motivation for the following rescaling can be found in [6].

Let us suppose the lid "base" is at $y = 1 - \gamma s(x)$, $\gamma \lesssim 1$, $s \sim 1$, and that the thermal boundary layer beneath the lid is of thickness $\delta \ll 1$. We anticipate that $\theta = O(\varepsilon)$ in the convecting region (and thermal layer). Equality of heat flux then implies

$$\gamma = \delta/\varepsilon. \tag{2.9}$$

Let δ_p be the width of the thermal plumes at the side walls, and suppose $\theta \sim \varepsilon\phi_p$ there. Now ϕ_p may be less than 1, since the base and sides of the convecting zone are free, not rigid, suggesting $\delta_p \ll \delta$, and therefore indeed $\phi_p \ll 1$, since (analogously to [13]) heat is advected round the corners. In fact, advection round the corner (with $\theta \sim \varepsilon$ in the top thermal layer, below the lid) directly implies

$$\phi_p = \delta_p/\delta. \tag{2.10}$$

Balance of advection and conduction in the top layer (beneath the lid) implies $\psi \sim 1/\delta$ there, where ψ is a stream function for the flow, so that $\psi \sim 1/\delta_p$ in the plumes (from conservation of $\int\theta\, d\psi$ going round corners; see Section 3 or [13]). By matching this plume value of ψ to the interior flow, we see that $\psi \sim 1/\delta_p^2$ in the interior. Hence $p, \tau_1, \tau_2 \sim 1/\delta_p^2$ in the interior. Now a balance of shear stress with buoyancy in the plumes gives

$$\delta = R\varepsilon\delta_p^4. \tag{2.11}$$

Lastly, if $\gamma s' \sim \lambda$ is the lid base slope, then a balance of shear stress with buoyancy in the top thermal layer yields

$$\delta^4 R\varepsilon\lambda = 1. \tag{2.12}$$

As already mentioned, Fowler [6] finds that such a balance is the only possible self-consistent one. The alternative possibilities are that $\tau_2 \sim 1/\delta_p^2$ in this layer, or that it is a shear layer with velocity comparable to the interior flow, which implies $\psi \sim \delta/\delta_p^2$ there. Neither works. The first is ruled out because, if $\tau_2 \sim 1/\delta_p^2$, then $\psi \sim \delta^2/\delta_p^2$ through the constitutive law, but also $\psi \sim 1/\delta$ in the thermal layer, so $\delta_p = \delta^{3/2}$. Then

$$\tau_{2_{yy}}/R\theta_x \sim \frac{1}{\delta_p^2}\cdot\frac{1}{\delta^2}\Big/\frac{R\varepsilon\lambda}{\delta};$$

thus $\tau_{2_{yy}}/R\theta_x \sim \delta_p^2/\lambda\delta^2$ [from (2.11)] $= \delta/\lambda$. This scaling can only be appropriate if $\lambda = \delta$ (see Section 4), since we can certainly take $\lambda \gtrsim \delta$: if $\lambda \gg \delta$, the momentum equation scales wrongly. The second possibility is ruled out because, if $\psi \sim \delta/\delta_p^2 \sim 1/\delta$, then $\delta_p = \delta$. But then $\tau_2 \sim 1/\delta^3$ (from the constitutive law) $\gg 1/\delta^2$, its value in the interior. Consequently, we must have $\tau_{2_{yy}} \sim R\theta_x$ in the layer if τ_2 is to become smaller in the interior. But now $\tau_{2_{yy}} \sim 1/\delta^5$, $R\theta_x \sim R\varepsilon\lambda/\delta \sim \lambda/\delta^4$ from (2.11), so $\lambda = 1/\delta$, patently absurd.

The relations (2.9), (2.11), and (2.12) give three relations for the four unknowns δ, λ, δ_p, and γ. We can restrict $\delta \lesssim \lambda \lesssim \gamma$, but no further dynamic restriction seems possible. We are faced with the possibility of a multiplicity of solutions. In Section 3 we consider the case of $\lambda = \gamma$ (shear layer). In Section 4 we consider the almost flat roof $\lambda = \delta$ (or what amounts to the same thing, $\lambda = 0$). An argument of the lid-stripping type, (cf. [10]) can be used to eliminate $\delta \ll \lambda \ll \gamma$, so that we have only two possible solutions. We will present some arguments in Section 4 that the "flat roof" ($\lambda = \delta$) problem, although consistently scaled, nevertheless has no solution. This is a matter of some delicacy, however, and the final answer is as yet unclear.

Now let us choose the temperatures T^* and ΔT in such a way that on $y = 1$, $\theta = -1$ and on $y = 0$, $\theta = \varepsilon\beta$. Then

$$\Delta T = \frac{T_b - T_0}{1 + \varepsilon\beta},$$

$$T^* = T_b - \varepsilon\beta\Delta T. \tag{2.13}$$

The parameter β will be chosen in the solution in such a way that the interior (isothermal) temperature is zero: this convenience is the reason for the convoluted choice of temperature scales. Apparently, we can write

$$\beta = \delta_p\bar{\beta}/\delta, \qquad \bar{\beta} = O(1). \tag{2.14}$$

We now *rescale* the equations (2.6) and boundary conditions (2.8) in accord with the above motivations. Thus

$$\theta = \varepsilon\phi, \quad \psi \sim \frac{1}{\delta_p^2}, \quad p, \tau_1, \tau_2 \sim \frac{1}{\delta_p^2}, \quad t \sim \delta_p^2, \tag{2.15}$$

where we have introduced the stream function

$$u = -\psi_y, \qquad v = \psi_x \tag{2.16}$$

into (2.6). That is, we now rewrite $\psi = \psi^*/\delta_p^2$, etc., and, omitting asterisks, obtain

$$\frac{1}{\sigma\delta_p^2}\frac{du}{dt} = -p_x + \tau_{1_x} + \tau_{2_y},$$

$$\frac{1}{\sigma\delta_p^2}\frac{dv}{dt} = -p_y + \tau_{2_x} - \tau_{1_y} + \delta_p^2 R\varepsilon\phi,$$

$$\frac{d\phi}{dt} = \psi_y\phi_x - \psi_x\phi_y = \delta_p^2\nabla^2\phi,$$

$$\tau_1 = -2\eta\psi_{xy},$$

$$\tau_2 = \eta(\psi_{xx} - \psi_{yy}),$$

$$\eta = e^{-\phi}, \tag{2.17}$$

with boundary conditions

$$\text{on } x = 0, a: \qquad \phi_x = \tau_2 = \psi = 0,$$

$$\text{on } y = 0: \qquad \phi = \beta, \qquad \tau_2 = \psi = 0,$$

$$\text{on } y = 1: \qquad \phi = -1/\varepsilon, \qquad \tau_2 = \psi = 0. \tag{2.18}$$

We will seek an asymptotic solution of (2.17) and (2.18) with (2.9), (2.11), and (2.12) for the cases $\lambda = \gamma$ (Section 3) and $\lambda = \delta$ (Section 4).

To summarize, we have written successively a dimensional set of equations, a nondimensionalized set, and finally a scaled set, suitable for analysis. Rather than clutter up the symbology with different notations, we have used essentially the same notation in each set of equations, the point being that each set supersedes the previous one. In the analysis to follow, there are so many different scalings involved that we will adhere to the symbols in (2.17) and (2.18) as defining the basic usage of these quantities. Then, in each subsection where a new local scaling is necessary, we will indicate the scales in terms of the basic scaled variables, and use capitals for the rescaled variables. Thus, the definition of the capitalized boundary layer variables will be specific to the subsection in which they occur. In a few cases, subsidiary notation will be introduced where required.

3. Shear layer: $\lambda = \gamma$

This means that the slope of the lid base is comparable to its thickness. From (2.9), (2.11), and (2.12), we have

$$\delta = R^{-1/5}, \qquad \gamma = \frac{R^{-1/5}}{\varepsilon}, \qquad \delta_p = R^{-3/10}\varepsilon^{-1/4}, \tag{3.1}$$

and from (2.4) and (2.13),

$$N = a^{-1}(1+\varepsilon\beta)^{-1}\int_0^a \left[-\theta_y|_{y=1}\right] dx. \tag{3.2}$$

Since the lid is stagnant, hence conductive, we can immediately deduce

$$N = C\varepsilon R^{1/5}; \tag{3.3}$$

we wish to compute C (and in so doing, verify the physically motivated scaling of the equations). Notice that we require $\gamma \lesssim 1$ to proceed, that is,

$$R \gtrsim \varepsilon^{-5}; \tag{3.4}$$

otherwise the lid fills most of the cell.

A. *Convecting core*

Beneath the lid, and away from boundary layers, all the variables are (assumed) of $O(1)$. In particular, ϕ satisfies

$$\frac{d\phi}{dt} = O(\delta_p^2); \tag{3.5}$$

note that $\delta_p^2 = R^{-3/10}\varepsilon^{-1/4} \lesssim \varepsilon^{5/4} \ll 1$, from (3.4). Thus $\phi \sim \phi(\psi)$ at leading order. It is well known, both computationally and experimentally, that in fact ϕ satisfies

$$\phi \sim \text{constant} \tag{3.6}$$

to all orders, since if $\phi(\psi) \neq$ constant, the analogue of the Prandtl-Batchelor theorem on vorticity inside closed streamlines [1] implies that $\phi =$ constant in a steady state (ϕ leaks out conductively through the boundary layers). We now choose β in such a way that

$$\phi = 0 \tag{3.7}$$

to all orders in the core; hence $\eta = 1$, and ψ satisfies the biharmonic equation

$$\nabla^4\psi = 0 \tag{3.8}$$

provided $\sigma\delta_p^2 \gg 1$, that is,

$$\sigma \gg R^{3/5}\varepsilon^{1/2}. \tag{3.9}$$

If $\sigma \ll R^{3/5}\varepsilon^{1/2}$, then the core flow is inertial and will have viscous boundary layers. It is not clear whether the analysis needs adjusting in that case, and it has not been examined. We will assume σ is as large as necessary for us to neglect it throughout (this is certainly true for solid-state convection).

B. *Plumes*

We suppose that convection is clockwise in the cell. We analyse the plume at $x = 0$. Then

$$x \sim \delta_p, \quad \psi \sim \delta_p, \quad \tau_2 \sim 1, \quad \phi \sim \delta_p/\delta, \tag{3.10}$$

and at leading order we find (using capitals to denote boundary layer variables, where these are rescaled)

$$\Psi_{XX} \sim 0 \quad \Rightarrow \quad \Psi \sim Xv_p(y), \tag{3.11}$$

$$\tau_{2X} \sim -\Phi \quad \Rightarrow \quad \tau_2 \sim -\int_0^X \Phi\, dX, \tag{3.12}$$

and

$$\Psi_X\Phi_y - \Psi_y\Phi_X \sim \Phi_{XX}; \tag{3.13}$$

a von Mises transformation to independent coordinates (y, Ψ) yields

$$\Phi_y = v_p(y)\Phi_{\Psi\Psi}, \tag{3.14}$$

or

$$\Phi_\eta = \Phi_{\Psi\Psi}, \qquad \eta = \int_0^y v_p(y)\, dy, \tag{3.15}$$

with

$$\Phi_\Psi = 0, \quad \Psi = 0; \qquad \Phi \to 0, \quad \Psi \to \infty; \tag{3.16}$$

consequently

$$\int_0^\infty \Phi\, d\Psi = C_l, \text{ constant.} \tag{3.17}$$

We notice that matching of the core flow to the plume requires

$$\psi = 0 \quad \text{on} \quad x = 0,$$

$$\psi_{xx} = \tau_2|_{X\to\infty} = -\int_0^\infty \Phi\, dX \quad \text{on} \quad x = 0. \tag{3.18}$$

Since also $\psi_x = v_p(y)$ on $x = 0$ defines $v_p(y)$, then $\psi_x\psi_{xx} = -\int_0^\infty \Phi v_p(y)\, dX = -\int_0^\infty \Phi\, d\Psi = -C_l$, so

$$\psi = 0, \quad \psi_x\psi_{xx} = -C_l \quad \text{on} \quad x = 0 \tag{3.19}$$

are the core flow boundary conditions at the left wall. Equally we have

$$\psi = 0, \quad \psi_x\psi_{xx} = C_r \quad \text{on} \quad x = a, \tag{3.20}$$

where

$$C_r = -\int_0^\infty \Phi\, d\Psi|_{\text{right hand plume}} \tag{3.21}$$

(the scaling being the same).

C. *Basal layer*

This is much the same, except τ_2 is now small:

$$y \sim \delta_p, \quad \psi \sim \delta_p, \quad \tau_2 \sim \delta_p, \quad \phi \sim \delta_p/\delta. \tag{3.22}$$

At leading order, we have (we use T as capital of τ)

$$0 = -p_x + \tau_{1x} + \mathrm{T}_{2Y},$$

$$0 \sim -p_x - \tau_{1Y} + \Phi,$$

$$-\Psi_Y\Phi_x + \Psi_x\Phi_Y \sim \Phi_{YY},$$

$$\tau_1 = -2\eta\Psi_{xY},$$

$$\Psi_{YY} \sim 0,$$

$$\eta \sim 1, \tag{3.23}$$

and thus

$$\Psi \sim -u_b(x)Y, \tag{3.24}$$

where u_b is the basal velocity (<0). The von Mises transform (to coordinates x, Ψ) yields

$$\Phi_x = u_b(x)\Phi_{\Psi\Psi}, \tag{3.25}$$

or

$$\Phi_\zeta = \Phi_{\Psi\Psi}, \qquad \zeta = \int_x^a [-u_b]\, dx. \tag{3.26}$$

together with

$$\Phi = \bar\beta \quad \text{on} \quad \Psi = 0, \qquad \Phi \to 0 \quad \text{as} \quad \Psi \to \infty. \tag{3.27}$$

As already stated, we choose $\bar\beta$ so that the core temperature is zero. If we assume that $\int_0^\infty \Phi\, d\Psi$ is advected round the corners [13], as can be checked in due course, then since $[\int_0^\infty \phi\, d\Psi]_{x=a}^{x=0} = C_l + C_r$, we therefore choose $\bar\beta$ so that the solution of (3.26) and (3.27) satisfies this relation; from (3.25), this is

$$\int_0^a (-\Phi_Y|_{Y=0})\, dx = C = C_l + C_r, \tag{3.28}$$

where C is the same coefficient as in (3.3). That is, we choose $\bar\beta$ so that the heat flux in at the base equals that out the top (which is the same as that transferred by the plumes).

D. *Lid temperature*

In the lid, $\phi = O(1/\varepsilon)$, $\eta = \exp[-O(1/\varepsilon)]$, and hence ψ is exponentially small. Consequently, ϕ (or θ) satisfies Laplace's equation. For simplicity, we will assume that $\gamma \ll 1$, that is, $R \gg 1/\varepsilon^5$. This assumption then renders the heat equation trivially solvable, since then $\partial/\partial y \sim 1/\gamma \gg \partial/\partial x \sim 1$. Thus we have to solve

$$\nabla^2\theta = 0 \tag{3.29}$$

with the boundary conditions

$$\theta = \begin{cases} -1 & \text{on} \quad y = 1, \\ 0 & \text{on} \quad y = 1 - \gamma s(x). \end{cases} \tag{3.30}$$

The second of these is really a matching condition to the thermal (shear) layer, but clearly it is appropriate. The unknown lid base is determined by an extra matching condition on θ_y there. For $\gamma \ll 1$, the solution of (3.30) is

$$\theta = -1 + z/s + O(\gamma^2), \tag{3.31}$$

where we define $y = 1 - \gamma z$.

The no heat flux condition is not satisfied at $x = 0$ or $x = a$, so that a region $x \sim \gamma$ involves the full Laplace's equation. We return to a consideration of slab stresses later.

E. Shear (thermal) layer

This is located near $y = 1 - \gamma s$. We put

$$y = 1 - \gamma s - \delta Y,$$

$$p \sim \delta\delta_p^2 R\varepsilon, \quad \tau_2 \sim \delta_p^2 R\delta^2, \quad \tau_1 \sim \frac{\delta_p^2 R\delta^3}{\varepsilon}, \quad \psi \sim \delta_p^2 R\delta^4 \tag{3.32}$$

(note that $\psi \sim \delta_p^2/\delta$, as required). We find these scales are

$$\begin{aligned} p &\sim R^{1/5}\varepsilon^{1/2} \geq \varepsilon^{-1/2} \gg 1, \\ \tau_2 &\sim \varepsilon^{-1/2} \gg 1, \\ \psi &\sim R^{-2/5}\varepsilon^{-1/2} \leq \varepsilon^{3/2} \ll 1, \\ \psi_y &\sim R^{-1/5}\varepsilon^{-1/2} \leq \varepsilon^{1/2} \ll 1; \end{aligned} \tag{3.33}$$

thus the stress scale is very large, but the horizontal velocity scale is very small. Consequently, matching of the thermal layer to the core ($Y \to \infty$) will require $P, T_2 \to 0$ at leading order. Since the stress condition is only correct to $O(\varepsilon^{1/2})$, the solutions can only be correct to $O(\varepsilon^{1/2})$ (in particular, there will be an error of relative order $\varepsilon^{1/2}$ in the Nusselt number).

With the choice (3.32), substitution into (2.17) yields, at leading order,

$$\begin{aligned} s'P_Y - T_{2Y} &= O(\gamma^2), \\ \phi + P_Y &= O(\gamma^2), \\ \Psi_Y\phi_x - \Psi_x\phi_Y &= \phi_{YY} + O(\gamma^2), \\ T_1 &= -2\eta s'\Psi_{YY}, \\ T_2 &= -\eta\Psi_{YY} + O(\gamma^2), \\ \eta &= e^{-\phi}. \end{aligned} \tag{3.34}$$

The boundary conditions on (3.34) are the following:

$$\text{as} \quad Y \to \infty, \qquad P, T_2, \phi \to 0; \tag{3.35}$$

as $Y \to -\infty$, ϕ must match to the solution θ of (3.29) and (3.30). Writing this as

$$\theta_{zz} + \gamma^2\theta_{xx} = 0, \qquad \theta = \begin{cases} -1 & \text{on} \quad z = 0 \\ 0 & \text{on} \quad z = s \end{cases}. \tag{3.36}$$

then

$$\theta = \varepsilon\phi = (z - s)\theta_z(s, x) + O\left[(z - s)^2\right] \tag{3.37}$$

as $z \to s$; since $z = (1 - y)/\gamma = s + \varepsilon Y$, it follows that the lid temperature is $\phi = \theta_z(s, x)Y + O(\varepsilon)$. For $\gamma \ll 1$, $\theta_z = 1/s + O(\gamma^2)$, so the appropriate condition on ϕ in (3.4) is

$$\phi \sim Y/s \qquad \text{as} \quad Y \to -\infty \tag{3.38}$$

(i.e., $\phi_Y \to 1/s$). From (3.34) and (3.35), we find

$$s'P = T_2, \tag{3.39}$$

and

$$\begin{aligned} s'\phi + T_{2Y} &= 0, \\ \Psi_{YY} &= -T_2 e^{\phi}, \\ \Psi_Y\phi_x - \Psi_x\phi_Y &= \phi_{YY}, \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} T_2 \to 0, \quad \phi \to 0 \qquad &\text{as} \quad Y \to \infty, \\ \phi \sim Y/s, \quad \Psi \to 0 \qquad &\text{as} \quad Y \to -\infty, \end{aligned} \tag{3.41}$$

where $\Psi \to 0$ to match the stagnant lid. These equations admit a similarity solution, which it is plausible to assume is consistent with the initial conditions at $x = 0$ (though strictly a separate analysis is necessary to verify this).

We put

$$\xi = \frac{Y}{s(x)}, \quad \phi = g(\xi), \quad T_2 = ss'h(\xi), \quad \Psi = s^3 s' f(\xi); \tag{3.42}$$

then

$$\begin{aligned} h' + g &= 0, \\ f'' &= -he^g, \\ g'' + Afg' &= 0, \end{aligned} \tag{3.43}$$

where

$$g(\infty) = h(\infty) = 0,$$
$$f(-\infty) = 0, \qquad g'(-\infty) = 1, \tag{3.44}$$

and

$$(s^3 s')' s = A. \tag{3.45}$$

It follows from (3.45) that

$$s = kx^{2/5}, \qquad A = 0.24k^5. \tag{3.46}$$

Notice that $f(-\infty) = 0$ is two conditions [$f'(-\infty) = 0$ as well], but A must be chosen. This is similar to a situation encountered by Ockendon and Ockendon [11, p. 184]. The extra condition arises because the origin is arbitrary. That is, we solve (with given $\tilde{A}$)

$$\tilde{h}' + \tilde{g} = 0,$$
$$\tilde{f}'' + \tilde{h}e^{\tilde{g}} = 0,$$
$$\tilde{g}'' + \tilde{A}\tilde{f}\tilde{g}' = 0, \tag{3.47}$$

requiring that in addition to (3.44),

$$\tilde{g} - c\xi \to 0 \quad \text{as} \quad \xi \to -\infty. \tag{3.48}$$

This is easily done by shooting with c and h_0 from $\xi = -M$, where $\tilde{f} = 0$, $\tilde{f}' = 0$, $\tilde{g}' = c$, $g = -cM$, $h = h_0$, towards infinity for $\tilde{g}(\infty) = 0 = \tilde{h}(\infty)$, for large M. Then

$$g = \tilde{g}(b\xi), \qquad h = \frac{1}{b}\tilde{h}(b\xi), \qquad f = \frac{1}{b^3}\tilde{f}(b\xi), \tag{3.49}$$

together with

$$A = b^4\tilde{A}, \tag{3.50}$$

satisfies the equations (3.43) and boundary conditions (3.44), with

$$b = 1/c. \tag{3.51}$$

Finally, (3.2) and (3.3) show that

$$N = C\varepsilon R^{1/5}, \tag{3.52}$$

where

$$C = a^{-1}(1+\varepsilon\beta)^{-1}\int_0^a \frac{dx}{s} \sim \bar{C}a^{-2/5}, \tag{3.53}$$

and

$$\bar{C} = \frac{5}{3k}. \tag{3.54}$$

Computation of (3.43) and (3.44) yields $g'(-\infty) = 1.843$, $\tilde{h} \sim -1.325 - 0.5\xi^2$ as $\xi \to -\infty$, and

$$\bar{C} = 2.043. \tag{3.55}$$

F. Slab stress

It is now necessary to compute the state of stress in the rigid lid. This does not enter the calculation of the Nusselt number; however, it is of interest because the expansion is rather curious, and also because we can estimate (numerically) just how small ε must be for advection to be truly negligible. We will treat separately the two cases of a no-slip condition and a no-stress condition at the top surface. We begin with the no-slip condition.

As $Y \to -\infty$ in (3.40),

$$\phi \sim \frac{Y}{s}, \qquad \mathrm{T}_2 \sim \frac{s'}{2s}Y^2, \qquad \Psi \sim \tfrac{1}{2}ss'Y^2e^{Y/s}; \tag{3.56}$$

for the lid, an appropriate vertical coordinate is z, where

$$y = 1 - \gamma z. \tag{3.57}$$

Thus, $Y = (z - s)/\varepsilon$, so (3.56) and (3.33) (with $\delta = R^{-1/5}$) give

$$\tau_2 \sim \varepsilon^{-5/2}\left[-\frac{s'}{2s}(z-s)^2\right],$$
$$\psi \sim \delta^2\varepsilon^{-5/2}\left[\tfrac{1}{2}ss'(z-s)^2\exp\left\{\frac{z-s}{\varepsilon s}\right\}\right]; \tag{3.58}$$

from (3.32), (3.33), (3.34), and (3.39), we additionally have

$$p \sim \delta^{-1}\varepsilon^{-3/2}\left[-\frac{1}{2s}(z-s)^2\right],$$
$$\tau_1 \sim \delta\varepsilon^{-7/2}\left[-\frac{s'^2}{s}(z-s)^2\right]. \tag{3.59}$$

(3.58) and (3.59) give the appropriate stress scales and matching conditions for the slab. Specifically, we put

$$\tau_2 = \varepsilon^{-5/2}\mathrm{T}_2,$$
$$\psi = \delta^2\varepsilon^{-5/2}e^{\theta/\varepsilon}\Psi,$$
$$\tau_1 = \delta\varepsilon^{-7/2}\mathrm{T}_1,$$
$$p = \delta^{-1}\varepsilon^{-3/2}P \tag{3.60}$$

in the slab, where

$$\theta = \varepsilon\phi. \tag{3.61}$$

Together with (3.57) (and $\gamma = \delta/\varepsilon$), we find (at leading order)

$$\mathrm{T}_1 \sim 2\Psi\theta_x\theta_z,$$
$$\mathrm{T}_2 \sim -\Psi\theta_z^2; \tag{3.62}$$

the heat equation is (at leading order)

$$\theta_{zz} \sim \varepsilon^{-3}e^{\theta/\varepsilon}[\Psi_z\theta_x - \Psi_x\theta_z]. \tag{3.63}$$

Asymptotically, as $\varepsilon \to 0$ advection is negligible to all orders. *Numerically*, heat advection is of relative order $\varepsilon^{-3}\exp[-O(1/\varepsilon)]$. For example, half way through the lid, this is $\varepsilon^{-3}\exp[-1/2\varepsilon]$: for $\varepsilon = \frac{1}{10}$ (viscosity ratio 22,000), it is actually larger than one (6.7); for $\varepsilon = \frac{1}{20}$ (viscosity ratio $\sim 5 \times 10^8$) it is about 0.36. Only for $\varepsilon = \frac{1}{30}$ is it as small as 0.008. The actual magnitude of the surface velocity relative to the interior is given by $-\psi_y = \delta\varepsilon^{-3/2}e^{\theta/\varepsilon}\Psi_z \sim R^{-1/5}\varepsilon^{-3/2}e^{-1/\varepsilon}$. For $\varepsilon = \frac{1}{10}$ ($R \geq \varepsilon^{-5}$) this is already $\leq \varepsilon^{-1/2}e^{-1/\varepsilon} \sim 1.4 \times 10^{-4}$.

The momentum equations, at leading order, yield

$$P_z + \theta = O(\gamma^2),$$
$$P_x + \mathrm{T}_{2z} = O(\gamma^2). \tag{3.64}$$

With $\theta \sim -1 + z/s$, there follows

$$\mathrm{T}_{2zz} \sim \theta_x \sim -s'z/s^2, \tag{3.65}$$

with matching conditions, from (3.58),

$$\mathrm{T}_2, \mathrm{T}_{2z} = 0 \quad \text{on} \quad z = s; \tag{3.66}$$

thus

$$\mathrm{T}_2 \sim -\frac{s'}{2s^2}\left[\tfrac{1}{3}z^3 - s^2z + \tfrac{2}{3}s^3\right]. \tag{3.67}$$

There is a residual shear stress $\mathrm{T}_0 = -ss'/3$ at the surface, and we are not able to satisfy the zero stress condition there. In retrospect, this is not too surprising, since if $\lambda \sim \gamma$ (and $\delta \ll \gamma$), the slab baroclinicity must generate a larger stress than in the thermal layer, and there is seemingly no way to balance this stress. Thus, although the stresses in the slabs uncouple from the determination of the convective flow beneath, it seems that their determination yields a first-order problem of consistency. The resolution of this problem is presented later. For the moment, we apply the no slip condition at the surface. The preceding analysis then goes through, as follows. The full expression for T_2 [of which $(3.62)_1$ is the leading order term] is

$$\mathrm{T}_2 = -\left[\Psi\theta_z^2 + \varepsilon(2\Psi_z\theta_z + \Psi\theta_{zz}) + \varepsilon^2\Psi_{zz}\right]$$
$$+ \gamma^2\left[\Psi\theta_x^2 + \varepsilon(2\theta_x\Psi_x + \Psi\theta_{xx}) + \varepsilon^2\Psi_{xx}\right]. \tag{3.68}$$

The stress expression (3.67) is uniformly valid to first order, but no slip at the surface requires

$$\Psi = \Psi_z = 0 \quad \text{at} \quad z = 0. \tag{3.69}$$

With

$$\mathrm{T}_2^0 = \mathrm{T}_2|_{z=0} = -ss'/3 \tag{3.70}$$

and

$$z = \varepsilon\zeta, \tag{3.71}$$

(3.68) gives, at leading order (and with $\theta_z = 1/s$),

$$\mathrm{T}_2^0 \sim \Psi_{\zeta\zeta} + \frac{2\Psi_\zeta}{s} + \frac{\Psi}{s^2}. \tag{3.72}$$

This is the boundary layer equation for Ψ. Curiously, it has *two* exponentially decaying solutions, which enables us to satisfy *both* of (3.69): the solution is

$$\Psi = -s^2\mathrm{T}_2^0\left[1 - \left(1 + \frac{\zeta}{s}\right)e^{-\zeta/s}\right]. \tag{3.73}$$

Let us turn to the case of prescribed zero traction at the surface. The initial discussion of this subsection remains valid [up to (3.59)], but in $(3.60)_2$ we make the following crucial distinction:

$$\psi = \frac{\lambda}{\eta_s} z u(x) + \delta^2 \varepsilon^{-5/2} e^{\theta/\varepsilon} \Psi, \tag{3.74}$$

where

$$\eta_s = e^{1/\varepsilon} \tag{3.75}$$

is the surface viscosity. Here, $u = O(1)$, and λ is an (as yet unknown) scaling factor. This allows explicitly for the fact that a small surface velocity is possible. It should be noticed that (3.74) is asymptotically consistent with $(3.58)_2$, provided λ is algebraic in ε [for then $\lambda z u/\eta_s$ is transcendentally small compared to $(3.58)_2$]. The other definitions in (3.60) being the same, (3.62) becomes

$$\mathrm{T}_2 \sim -\Psi \theta_z^2,$$
$$\mathrm{T}_1 \sim 2\left[\Psi \theta_x \theta_z + \frac{\lambda \varepsilon^{9/2}}{\delta^2} \frac{\eta}{\eta_s} u'\right], \tag{3.76}$$

where we retain the term in λ with the expectation (or anticipation) that it will become important near $z = 0$, where $\eta/\eta_s = O(1)$.

The complete momentum equations [cf. (3.64)] are

$$-P_x + \gamma^2 \mathrm{T}_{1x} - \mathrm{T}_{2z} = 0,$$
$$P_z + \gamma^2 \mathrm{T}_{2x} + \gamma^2 \mathrm{T}_{1z} + \theta = 0. \tag{3.77}$$

As before, we find (to leading order with $\gamma \ll 1$) (3.65)–(3.67) for $z \sim 1$: also the pressure satisfies

$$P \sim z - \frac{z^2}{2s} - \tfrac{1}{2}s, \tag{3.78}$$

by integrating once and using $P \to 0$ as $z \to s$ [from (3.59) and (3.60)]. As $z \to 0$,

$$P \sim P^0 = -\tfrac{1}{2}s,$$
$$\mathrm{T}_2 \sim \mathrm{T}_2^0 = -ss'/3. \tag{3.79}$$

As before, there exists a boundary layer in which we put

$$z = \varepsilon \xi, \tag{3.80}$$

but this time, it is the stresses which change. This is effected by ensuring that T_1 is large when $\xi \sim 1$. From (3.76), we evidently have

$$\mathrm{T}_1 \sim \frac{\lambda \varepsilon^{9/2}}{\delta^2} \gg 1 \tag{3.81}$$

since $\eta/\eta_s \sim 1$ when $\xi \sim 1$. In fact

$$\frac{\eta}{\eta_s} \sim \exp\left[-\frac{\xi}{s}\right]. \tag{3.82}$$

It turns out that we can effect the appropriate balance by choosing

$$\lambda = 1/\varepsilon^{7/2} \tag{3.83}$$

(algebraic as required), and defining

$$P = \frac{\tilde{P}}{\varepsilon}, \qquad \mathrm{T}_1 = \frac{\varepsilon}{\delta^2} \tilde{\mathrm{T}}_1. \tag{3.84}$$

The momentum equations (3.77) are then

$$-\tilde{P}_x + \tilde{\mathrm{T}}_{1x} - \mathrm{T}_{2\xi} = 0,$$
$$\tilde{P}_\xi + \tilde{\mathrm{T}}_{1\xi} = -\delta^2 \mathrm{T}_{2x} - \varepsilon^2 \theta, \tag{3.85}$$

where

$$\tilde{\mathrm{T}}_1 = 2u'(x) e^{-\xi/s}. \tag{3.86}$$

On $\xi = 0$, we have $\mathrm{T}_2 = 0$. The matching conditions at infinity are [from (3.84) and (3.79)]

$$\mathrm{T}_2 \to \mathrm{T}_2^0, \quad \tilde{P} \to 0 \quad \text{as} \quad \xi \to \infty. \tag{3.87}$$

It follows from (3.85) that

$$\tilde{P} \sim -\tilde{\mathrm{T}}_1, \tag{3.88}$$
$$\mathrm{T}_2 = 4\left[(su')' - \{(su')' - s'u'\xi/s\} e^{-\xi/s}\right], \tag{3.89}$$

which latter satisfies the condition for T_2 on $\xi = 0$. The requirement that $(3.87)_1$ be satisfied gives the equation for the unknown u, $4(su')' = \mathrm{T}_2^0$, with boundary conditions $u = 0$ at $x = 0$ and $x = a$. Using the previous expression for s, (3.46) with (3.54) and (3.55), we find

$$u \approx 0.097 a^{4/5} x^{3/5} \left[1 - (x/a)^{4/5}\right]. \tag{3.90}$$

This completes the solution.

In the skin, the original dimensionless stresses are of the following orders:

$$\tau_2 \sim \varepsilon^{-5/2}, \qquad p, \tau_1 \sim \varepsilon^{-5/2} \delta^{-1}. \tag{3.91}$$

These values can be *extremely* high (see the conclusions).

It is also of interest to compute the topographic uplift. If the *actual* (dimensionless) top surface is given by $y = 1 + \Delta$, then, since the surface longitudinal stress $-\sigma_{22}$ must be atmospheric (i.e. effectively zero), we find, using the original definition $\rho_0 g(d - y) + p$ of the pressure [Section 2, before (2.1)] and the various subsequent rescalings, that this means that

$$-\rho_0 g d\Delta + \frac{\eta_0\kappa}{d^2}\cdot\frac{1}{\delta_p^2}\frac{\varepsilon^{-5/2}}{\delta}\Sigma\Big|_{\xi=0} = 0, \tag{3.92}$$

where in the thin skin ($\xi \sim 1$), we define

$$\tilde{P} + \tilde{\mathrm{T}}_1 = \Sigma. \tag{3.93}$$

From (3.85),

$$\Sigma_\xi = -\delta^2 \mathrm{T}_{2x} - \varepsilon^2\theta \tag{3.94}$$

$= O(\varepsilon^2)$, since $\delta \ll \varepsilon$ ($\gamma \ll 1$). The boundary condition on Σ comes from matching as $\xi \to \infty$. Now as $z \to 0$ in the slab, we have $P \sim P^0 = -\frac{1}{2}s$, and using (3.62), $\mathrm{T}_1 \sim 2\varepsilon s' \mathrm{T}_2^0 \xi/s$. From $P = \tilde{P}/\varepsilon$, (3.84), and $\mathrm{T}_1 = \varepsilon\tilde{\mathrm{T}}_1/\delta^2$ [from (3.83) and (3.81)], it follows that as $\xi \to \infty$,

$$\tilde{\mathrm{T}}_1 \sim 2\delta^2 s' \mathrm{T}_2^0 \xi/s, \qquad \tilde{P} \sim -\tfrac{1}{2}\varepsilon s. \tag{3.95}$$

Since $\delta^2 \ll \delta \ll \varepsilon$, this gives, at leading order,

$$\Sigma \sim -\tfrac{1}{2}\varepsilon s, \tag{3.96}$$

both as the matching condition *and* as the leading order solution to (3.94) [$\Sigma_\xi = O(\varepsilon^2)$]. Therefore (3.92) gives, using the definitions of R, δ_p in (3.1),

$$\Delta = \{\alpha\Delta\mathrm{T}\}\,\frac{\delta}{\varepsilon^2}\Sigma\Big|_{\xi=0} \sim -\tfrac{1}{2}\{\alpha\Delta\mathrm{T}\}\gamma k x^{2/5}, \tag{3.97}$$

k being given by (3.54) and (3.55), $k \approx 0.82$. Notice that the uplift becomes comparable to the skin thickness if $\alpha\Delta\mathrm{T} \geq \varepsilon$, in which case the preceding analysis requires a slight modification (for the skin only). This would probably be the case in the context of terrestrial mantles.

4. Flat roof: $\lambda = \delta$

In this section we will examine the possibility that the slope of the roof of the convection zone is comparable to the top thermal layer thickness. That is, we put $\lambda = \delta$ in (2.12). Then (2.9), (2.11), and (2.12) yield

$$\delta = R^{-1/5}\varepsilon^{-1/5}, \qquad \delta_p = \delta^{3/2}, \qquad \gamma = R^{-1/5}\varepsilon^{-6/5}, \tag{4.1}$$

and we need

$$R \geq 1/\varepsilon^6 \tag{4.2}$$

for $\gamma \leq 1$. The values (4.1) lead to the following restatement of (2.17):

$$\begin{aligned} -p_x + \tau_{1x} + \tau_{2y} &= 0, \\ -p_y + \tau_{2x} - \tau_{1y} + \phi/\delta^2 &= 0, \\ \psi_x\phi_y - \psi_y\phi_x &= \delta^3\nabla^2\phi, \\ \tau_1 &= -2\eta\psi_{xy}, \\ \tau_2 &= \eta(\psi_{xx} - \psi_{yy}), \\ \eta &= e^{-\phi}. \end{aligned} \tag{4.3}$$

The analysis is now very similar to Roberts's [13] case of rigid horizontal boundaries, although couched in somewhat different language.

A. Slab temperature

Since the lid base is flat, the temperature is (to leading order) a linear function of depth: thus

$$\phi \sim -\frac{1}{\varepsilon} + \frac{B(1-y)}{\delta}, \tag{4.4}$$

where B is a constant, to be determined.

B. Core

As before,

$$\phi \sim 0 \tag{4.5}$$

to all orders of $\delta, \eta \sim 1$, and so

$$\nabla^4\psi = 0 \tag{4.6}$$

together with the usual boundary conditions

$$\begin{aligned} \psi &= 0 \quad \text{on all walls}, \\ \psi_y &= 0 \quad \text{on lid base and bottom}, \\ \psi_x\psi_{xx} &= -C_l \quad \text{at } x = 0, \\ \psi_x\psi_{xx} &= C_r \quad \text{at } x = a, \end{aligned} \tag{4.7}$$

where C_l and C_r are determined from the plume solutions.

C. Plumes

For example, at $x=0$, we rescale:

$$x \sim \delta^{3/2}, \quad \psi \sim \delta^{3/2}, \quad \phi \sim \delta^{1/2}\ (=\delta_p/\delta). \tag{4.8}$$

Then

$$C_l = \int_0^\infty \Phi\, d\Psi, \tag{4.9}$$

and Φ satisfies

$$\Phi_T = \Phi_{\Psi\Psi}, \quad T = \int^y v_p(y)\, dy, \tag{4.10}$$

v_p being the vertical velocity in the plume. The right hand plume is similar.

D. Basal thermal layer

This is as for the plumes, and as in 3.C, we get

$$\Phi_T = \Phi_{\Psi\Psi}, \quad T = \int_x [-u_b]\, dx, \tag{4.11}$$

u_b being the basal velocity. $\Phi = \bar{\beta}$ on $\Psi = 0$, $\bar{\beta}$ being chosen so that (3.28) is satisfied.

E. Top thermal layer

We put

$$y = 1 - \frac{\gamma}{B} - \delta Y, \quad \psi \sim \delta^2, \quad \tau_1 \sim \delta, \quad p \sim \frac{1}{\delta}; \tag{4.12}$$

at leading order, the boundary layer equations are

$$\begin{aligned} \tau_{2YY} &= \phi_x, \\ \Psi_{YY} &= -\tau_2 e^{\phi}, \\ \Psi_Y\phi_x - \Psi_x\phi_Y &= \phi_{YY}, \end{aligned} \tag{4.13}$$

with boundary conditions

$$\begin{aligned} &\Psi \to 0, \quad \phi \sim BY, \quad Y \to -\infty, \\ &\tau_2 \to \tau_c(x), \quad \phi \to 0, \quad Y \to +\infty. \end{aligned} \tag{4.14}$$

Here τ_c is the stress experienced by the core at the lid base [i.e. $-\psi_{yy}(x, 1-\gamma/B)$]. Thus (4.13) couples to the core flow, and in particular, there is no similarity solution. This is analogous to the constant viscosity case. However, the initial condition for (4.13) is determined by matching to a corner flow, and for this there is (as one expects) a similarity solution. We therefore inspect the corner flow a little more carefully.

F. Corner flow

The core flow has a singularity in the corners due to the discontinuous nature of the boundary conditions there. Following Roberts [13], if (assuming a rectangular corner) we put

$$x \sim r, \quad 1 - \frac{\gamma}{B} - y \sim r, \tag{4.15}$$

then as $r \to 0$, we have

$$\psi \sim r^{3/2}, \quad \tau_1, \tau_2, p \sim r^{-1/2}; \tag{4.16}$$

since the singularity arises through the contraction of the buoyancy term due to a line source on $x=0$, it is reasonable to suppose that it can be alleviated by a scaling in which buoyancy ~ vorticity in the momentum equation. This gives, with $\phi \sim \delta^{1/2}$,

$$r \sim \delta, \tag{4.17}$$

and using (4.15) and (4.16), the rescaled corner layer equations are

$$\begin{aligned} -P_X + T_{1X} - T_{2Y} &= 0, \\ P_Y + T_{2X} + T_{1Y} + \Phi &= 0, \\ \Psi_Y\Phi_X - \Psi_X\Phi_Y &= \delta^{3/2}\nabla^2\Phi, \\ \eta &= \exp[-\delta^{1/2}\Phi], \\ T_1 &= 2\eta\Psi_{XY}, \\ T_2 &= \eta(\Psi_{XX} - \Psi_{YY}), \end{aligned} \tag{4.18}$$

where $\nabla^2 = \partial^2/\partial X^2 + \partial^2/\partial Y^2$. At leading order

$$\Phi = \Phi(\Psi), \tag{4.19}$$

$$\nabla^4\Psi + \Phi'(\Psi)\Psi_X = 0. \tag{4.20}$$

This shows that (as already assumed) Φ convects round the corner; then (4.20) together with

$$\Psi = 0,\quad \Psi_{XX} = 0 \quad \text{on} \quad X = 0,$$
$$\Psi = 0,\quad \Psi_Y = 0 \quad \text{on} \quad Y = 0 \tag{4.21}$$

and a matching condition at $X^2 + Y^2 \to \infty$ will (we assume) alleviate the core singularity.

Evidently there is a further, thinner corner thermal layer within this layer, which will match the temperature to the lid temperature.

G. *Corner thermal layer*

With the above corner layer variables still in mind, we will need $\phi \sim 1$ to match to the lid (as we need $\eta \sim 1$). To keep the same stress level, we require $Y \sim \delta^{1/2}$, thus also $\Psi \sim \delta$, $T_1 \sim \delta^{1/2}$. We put

$$\Phi = \frac{\phi}{\delta^{1/2}},\quad Y = \delta^{1/2}\tilde{Y},\quad \Psi = \delta\tilde{\Psi},\quad T_1 = \delta^{1/2}\tilde{T}_1, \tag{4.22}$$

so that

$$\begin{aligned}
\eta &= e^{-\phi},\\
\tilde{T}_1 &= 2\eta\tilde{\Psi}_{X\tilde{Y}},\\
T_2 &= -\eta[\tilde{\Psi}_{\tilde{Y}\tilde{Y}} - \delta\tilde{\Psi}_{XX}],\\
\tilde{\Psi}_{\tilde{Y}}\phi_X - \tilde{\Psi}_X\phi_{\tilde{Y}} &= \phi_{\tilde{Y}\tilde{Y}} + \delta\phi_{XX},\\
P_{\tilde{Y}} + \delta^{1/2}[T_{2X} + \tilde{T}_{1\tilde{Y}}] + \phi &= 0,\\
-\delta^{1/2}P_X + \delta\tilde{T}_{1X} - T_{2\tilde{Y}} &= 0.
\end{aligned} \tag{4.23}$$

At leading order,

$$T_2 = -T(X), \tag{4.24}$$

where $T(X)$ can be obtained from matching to the corner layer $[T(X) = -T_2(X,0)]$, and $\tilde{\Psi}$ and ϕ satisfy

$$\begin{aligned}
\tilde{\Psi}_{\tilde{Y}\tilde{Y}} &= T(X)e^{\phi},\\
\tilde{\Psi}_{\tilde{Y}}\phi_X - \tilde{\Psi}_X\phi_{\tilde{Y}} &= \phi_{\tilde{Y}\tilde{Y}}.
\end{aligned} \tag{4.25}$$

The initial conditions for these equations must be found by matching to a *corner subthermal layer* in which $X \sim \delta^{1/2}$, $\tilde{\Psi} \sim \delta^{1/2}$, $T_2 \sim \delta^{1/2}$. We leave it as an exercise to show that the solution for this sublayer [in which $\phi = \phi(\tilde{Y})$] is consistent with the similarity solution of (4.25),

$$\tilde{\Psi} = h(X)f(\xi),\qquad \xi = \frac{\tilde{Y}}{k(x)},$$
$$\phi = g(\xi), \tag{4.26}$$

where

$$k = T^{-1/2}\left\{\int_0^X T^{1/2}dx\right\}^{1/3},\qquad h = \left\{\int_0^X T^{1/2}dx\right\}^{2/3}, \tag{4.27}$$

and

$$f'' = e^g,\qquad g'' + \tfrac{2}{3}fg' = 0,$$
$$f(-\infty) = 0,\qquad g(\infty) = 0. \tag{4.28}$$

These equations are discussed by Ockendon and Ockendon [11], who suggest that there is a solution with $g'(-\infty) = \bar{B} \approx 0.986$, this being determined from the fact that the origin of ξ is arbitrary, which provides an extra condition to supplement the three (f, f', and g) in (4.28). For large X, we know $T \sim X^{-1/2}$ (to match to the core stress singularity); therefore $\xi \sim \tilde{Y}/X^{1/2}$, so $\phi_{\tilde{Y}} \sim 1/X^{1/2}$ as $X \to \infty$. Evidently, this is not consistent with a constant gradient for the thermal boundary layer equations, as, in terms of these boundary layer variables (x and Y) one would have $\phi_Y \sim x^{-1/2}$: it then seems possible that (4.13) may not have a solution at all (at least with constant B).

H. *Acute corner thermal layer*

One way round this is the relaxation of the arbitrary assumption that the corner flow domain is rectangular. Let us suppose that the corner flow equation (4.20) should rather be solved in $Y > Y_s(x)$, to be found. For the corner thermal layer, we put

$$Y = Y_s(x) + \delta^{1/2}\tilde{Y}, \tag{4.29}$$

and other scales as in (4.22). We define

$$T_2 = -(1 - Y_s'^2)\omega; \tag{4.30}$$

then, at leading order, we have

$$\begin{aligned}
\tilde{\Psi}_{\tilde{Y}\tilde{Y}} &= \omega e^{\phi},\\
(1 + Y_s'^2)\omega_{\tilde{Y}} &= \phi,\\
\tilde{\Psi}_{\tilde{Y}}\phi_X - \tilde{\Psi}_X\phi_{\tilde{Y}} &= (1 + Y_s'^2)\phi_{\tilde{Y}\tilde{Y}}.
\end{aligned} \tag{4.31}$$

with

$$\tilde{\Psi} \to 0, \qquad \tilde{Y} \to -\infty$$

$$\phi \to 0, \qquad \tilde{Y} \to +\infty$$

$$\omega \to \frac{T(x)}{1 - Y_s'^2}, \qquad \tilde{Y} \to +\infty. \tag{4.32}$$

The similarity solution is

$$\omega = \frac{T(X)}{1 - Y_s'^2} h(\xi),$$

$$\tilde{\Psi} = p(x) f(\xi),$$

$$\phi = g(\xi), \qquad \xi = \tilde{Y}/q(x), \tag{4.33}$$

where

$$g'' + fg' = 0,$$

$$h' = g,$$

$$f'' = he^g,$$

$$f(-\infty) = 0, \qquad g(\infty) = 0, \qquad h(\infty) = 1, \tag{4.34}$$

and

$$q = \frac{T(1+\mu)}{1-\mu},$$

$$p = \frac{T^3(1+\mu)^2}{(1-\mu)^3},$$

$$Y_s'^2 = \mu,$$

$$\left\{ \frac{T^3(1+\mu)^2}{(1-\mu)^3} \right\}' = \frac{1-\mu}{T}. \tag{4.35}$$

The last equation of (4.35) defines $\mu = Y_s'^2$ in terms of $T(x)$ (which is given from the corner flow).

If this similarity solution is to match to the thermal boundary layer, then we need $\mu \to 0$ as $X \to \infty$. This requires $(T^3)' \sim T^{-1}$ as $X \to \infty$ in (4.35), i.e. $T \sim X^{1/4}$, which is at odds with $T \sim X^{-1/2}$. In any case, $\mu \to 0$ yields $q \neq$ constant, which gives the same problem as before.

It does not seem that the boundary layer equations and boundary conditions (4.13) and (4.14) are consistent with reasonable initial similarity solutions. Indeed, one can reasonably doubt that (4.13) and (4.14) with *constant* B have a solution at all. A proof of this statement would be of some interest.

5. Discussion

Let us first summarize the status of the present analysis, and then compare it with other relevant results. We have presented an analysis of convection of a fluid with exponential dependence of viscosity on temperature, in the combined asymptotic limits $\sigma \to \infty$, $R \to \infty$, $\varepsilon \to 0$, these being the Prandtl number, the "Rayleigh" number, and the dimensionless inverse activation energy, respectively.

In terms of ε, R, the Nusselt number is given by

$$N \sim \bar{C}a^{-2/5} \varepsilon R^{1/5}. \tag{5.1}$$

This Rayleigh number effectively uses the basal viscosity [strictly the internal (isothermal) viscosity], analogous to Christensen's [4] Ra_T. On transcribing to Morris and Canright's [9] parameters R_Y and $\theta[R_Y = \varepsilon Re^{\beta}$, $\theta = (1+\varepsilon\beta)/\varepsilon$, $\theta_h = \beta]$, and taking into account the factor $(1+\varepsilon\beta)^{-1}$ in (3.2), we find that Morris and Canright's [9] prediction can be written in the form $N \sim C_m \varepsilon R^{1/5}$, where $C_m \approx \varepsilon^{1/5} C$, C being given by their table 1; this is because their analysis is essentially that of section 4. By comparison, $\bar{C}a^{-2/5}$ and C are quite close, and the main difference is the factor $\varepsilon^{1/5}$, which gives a multiplicative difference of about 1.5 at values of $\varepsilon \sim 0.1$. This would make the present prediction more in line with results of Lux and Sacks [8], Christensen [4], and with laboratory experiments of Richter et al. [12], but one should also point out that this better agreement may be illusory, given the nature of the approximations involved.

It is a salutory exercise to examine the errors involved in the derivation of (5.1). There are four principal ones:

(1) Neglect of plume temperature in the thermal layer. The plume thermal profile $\phi \sim \delta_p/\delta$ is swept across the top layer outside the primary thermal layer [6, 13]. Thus to correct for this error, we should solve the top thermal boundary layer equations with

$$\phi \to R^{-1/10} \varepsilon^{-1/4} \Phi_0 \quad \text{as} \quad Y \to \infty, \tag{5.2}$$

where Φ_0 is the plume centerline temperature at $y = 1$, which can be found by solving the plume-bottom-layer-plume diffusion equation (3.15) and (3.26) for Φ. This gives a relative error of $O(R^{-1/10}\varepsilon^{-1/4})$.

(2) In writing a similarity solution for the top thermal layer, we neglect (at leading order) the external stress in the core, which is [from (3.33)] of relative $O(\varepsilon^{1/2})$, and the same error occurs in the heat flux; since the lid base retards the core flow, the core stress must accelerate the thermal layer, therefore (presumably) decreasing the heat flux.

(3) In order to compute the heat flux, the lid was supposed thin, that is, $\gamma \ll 1$. This introduces errors of $O(\gamma^2) = O(R^{-2/5}\varepsilon^{-2})$ into the heat flux.

(4) Finally, the corner heat flux is locally higher than in the rest of the thermal layer, as follows from Section 4.G; if we carry this analysis across (at least in terms of scales), we obtain an enhanced heat flux contribution of (relative) order $\delta_p^{1/6} \sim R^{-1/20}\varepsilon^{-1/24}$.

Thus, we expect a heat flux of the form

$$N \sim C\varepsilon R^{1/5}\left[1 + O\left(R^{-1/10}\varepsilon^{-1/4}, \varepsilon^{1/2}, R^{-2/5}\varepsilon^{-2}, R^{-1/20}\varepsilon^{-1/24}\right)\right]; \quad (5.3)$$

it is readily apparent that, whereas (5.1) may give reasonable agreement for moderate R and ε, one really needs $\varepsilon^{1/2} \leq 0.1$ for an accuracy of 10%; this is a viscosity contrast $\geq 10^{40}$! Similarly, we require $R \gg 1/\varepsilon^5$ for (5.1) to be valid. For $\varepsilon \sim \frac{1}{16}$ (viscosity contrast $\sim 10^7$), this means $R \gg 10^6$, but even for $R = 10^{10}$, one only has $R^{-1/10}\varepsilon^{-1/4} = 0.2$, $R^{-1/20}\varepsilon^{-1/24} = 0.35$. Some (perhaps all) of the corrective terms in (5.3) can be calculated semianalytically, and one might then hope to get better comparison with experiment. However, the main point to make vis-à-vis the simple parametrization (5.1) is that one should not expect high precision at moderate viscosity contrasts and Rayleigh numbers: here, moderate means (let us say) $R < 10^{10}$, $\varepsilon > \frac{1}{15}$, or viscosity ratio $\eta_0/\eta_b < 10^7$. Ideally, one wants much more extreme values.

More important at these moderate parameter values is the *structure* of the flow: the importance of plumes in driving the interior flow, etc., is revealed at values lower than those at which the asymptotic limit is truly reached. Of significance here is the prospect that the heat flow is controlled (at leading order) entirely by the flow in the boundary layer beneath the lid. If this observation could be extended to convection in the terrestrial planets, it would make for a certain simplicity in analysing such flows.

Our analysis is incomplete in the following two respects: we have no rigorous justification for supposing that the flat roof case, $\lambda = \delta$, of Section 4 has no solution—although that does seem plausible. On the other hand, we have not really shown, either, that the solution of Section 3 is self-consistent: for this we would need a corner flow analysis similar to Sections 4.F, H for the case $\lambda = \gamma$. This is not pursued here.

From what has been said above, one should be wary of seeking confirmation of the analysis in direct numerical comparison. Nevertheless, certain facets of the solution are worthy of a comparison. For example, we predict a surface heat flux proportional to $x^{-2/5}$ away from the "ridge." This prediction should be at least grossly confirmed, if the solution given here is correct. Indeed, in Figure 10(c) of Christensen's [4] paper, one does see slanting isotherms in the lid. The parameters are (approximately) $R \sim 10^7$, $\varepsilon \approx 0.12$. Other predictions could be at least qualitatively confirmed (or not) by suitable examination of numerical results. For example, the shear stress just beneath the lid should be $O(1/\varepsilon^{1/2})$ compared to that in the core flow, but $O(1/\varepsilon^{5/2})$ within the lid (e.g. at the top surface). The (negative) surface shear stress should decrease with increasing x like $x^{-1/5}$, away from $x = 0$. Recent specific examination of the vertical stress profiles in numerical work by David Yuen and Francesca Quareni does confirm the curious skin effect analysed in Section 3.

An examination of such predicted structural features would give a clearer picture of the relation of analysis and experiment, since the Nusselt number versus Rayleigh number relationship, while of practical interest, is not a very good test of the appropriateness of the asymptotic theory.

Comparison with fluid experiments is more hazardous, since, apart from the above sources of error, laboratory fluids are limited (so far) to small viscosity ratios ($\leq 10^5$). Other complications arise, such as differing planforms (three-dimensionality), oscillations, etc. In addition, the application of the present version of the asymptotic analysis requires that the Prandtl number satisfy

$$\sigma \gg R^{3/5}\varepsilon^{1/2}. \quad (5.4)$$

For $\varepsilon = 0.1$, $R = 10^6$, this is $\sigma \gg 1260$, and already at the high end of some laboratory fluids' properties. Golden syrup has a (steeper than exponential) change in viscosity from 10^7 to 1 poise over a temperature range -20 to 80°C. At 20°C ($\eta \sim 10^3$ P) its Prandtl number is 4×10^5: thus a convection experiment with boundary temperatures of -20 and $+20$°C certainly can satisfy (5.4) for quite high R. However, if one increases the lower temperature to about 50°C, so $\eta_0/\eta_b \sim 10^6$ ($\varepsilon \sim 0.07$), then $\sigma \sim 4 \times 10^3$. Then (5.4) requires $R \ll 10^7$, but $R \gg 1/\varepsilon^5$ requires $R \gg 5 \times 10^5$. Obviously, this is not much of a range. When (5.4) is not satisfied, one starts to see viscous boundary layers at the side walls. Since the heat flux is determined from the thermal layer above, presumably no immediate effect occurs until the plume thermal layer becomes comparable to the viscous layer, that is, when [from (2.17)] $\delta_p\sqrt{\sigma} \sim \delta_p$. For $\sigma \gg 1$, this does not occur, so that no effect is felt. In the thermal layer, the ratio of buoyancy to inertia is, from (3.33) and (2.17), using (2.11), $\sigma^{-1}\varepsilon^{-1/2}$. Thus this layer, too, is unaffected, so long as

$$\sigma \gg \varepsilon^{-1/2}, \quad (5.5)$$

and the relative error in (5.3) is $O(\sigma^{-1}\varepsilon^{-1/2})$. The more relaxed restriction (5.5) is easily satisfied, so that we need hardly worry about the Prandtl number.

While the motivation for the present study is convection in the earth's mantle, there is no direct connection, insofar as the plates on the earth are not stagnant, but move: subduction occurs. The problem of the initiation of subduction in the earth is the principal problem in plate tectonics, one that has as yet no answer. One feature of the present solution which is intriguing in this connection is the enormous size of the stresses generated in the lid. To get some idea of these, recall from (2.5), (2.15), and (3.1) that the dimensional stresses (labelled with a superscript D) are defined as

$$\tau^D = \left\{\frac{\eta_0\kappa}{d^2}\right\}\varepsilon^{1/2}R^{3/5}\tau, \quad (5.6)$$

where τ is p, τ_1, or τ_2.

In the lid, (3.60) implies the largest deviatoric stress is the shear stress,

$$\tau_2^D \sim [\tau]\varepsilon^{-2}R^{3/5}\mathrm{T}_2, \tag{5.7}$$

where

$$[\tau] = \eta_0\kappa/d^2 \sim 10^{-3}\ \mathrm{bar} \tag{5.8}$$

for geophysical values $\eta_0 = 10^{22}$ P, $\kappa = 10^{-2}\ \mathrm{cm}^2\,\mathrm{sec}^{-1}$, $d = 3000$ km. For example, the shear stress just below the skin is

$$|\tau_2^D| \sim [\tau]\varepsilon^{-2}R^{3/5}\{\tfrac{2}{15}k^2x^{-1/5}\}; \tag{5.9}$$

its average value (for $a = 1$) is

$$\left|\overline{\tau_2^D}\right| \sim 0.1[\tau]\varepsilon^{-2}R^{3/5}. \tag{5.10}$$

With $\varepsilon \sim \frac{1}{15}$, $R \sim 10^7$, $[\tau] \sim 10^{-3}$ bar, this is about 400 bars. The deviatoric longitudinal stress in the skin is larger. From (3.60) and (3.84),

$$\begin{aligned}\tau_1^D &\sim [\tau]\varepsilon^{-2}R^{4/5}\tilde{\mathrm{T}}_1\\ &\sim [\tau]\varepsilon^{-2}R^{4/5}\left[0.19\{\tfrac{3}{5}a^{4/5}x^{-2/5} - \tfrac{7}{5}x^{2/5}\}\right].\end{aligned} \tag{5.11}$$

With $\varepsilon \approx \frac{1}{15}$, $R \sim 10^7$, $a = 1$, this gives $|\tau_1^D|_a| \sim 14$ kbar, which is quite a bit in excess of the yield strength of rock. The effect of introducing a viscoplastic rheology is therefore suggested, and one tantalizing possibility is that the yielding upper part of the lid may be able to force its way down into the mantle, thus initiating subduction. Indeed, preliminary numerical work by Quareni and Yuen seems to suggest that the stresses are even higher when pressure dependence is introduced into the flow law.

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