

Structure and Dynamics of Partially Solidified Systems

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THEORIES OF MUSHY ZONES: APPLICATIONS TO ALLOY SOLIDIFICATION, MAGMA
TRANSPORT, FROST HEAVE AND IGNEOUS INTRUSIONS

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ABSTRACT

This paper attempts to review some of the different physical contexts in which partially molten (or partially frozen) zones occur, and to outline some of the problems which one encounters in attempting to model such zones realistically. The nature of averaged equations for reactive two-phase regions is first discussed for the case of a solidifying alloy, in which the fluid velocity is zero. Four principal examples are then discussed, in which fluid flow is important. These are alloy solidification, magma transport, frost heave, and the solidification of igneous intrusions. Various similar mathematical features in the models are pointed out, as well as some common difficulties in their formulation.

LIST OF SYMBOLS

Roman

c	liquid solute concentration (2.4)
c_1	interface solute concentration (2.6)
c_D	specific heat (2.5)
D_T	Taylor dispersion coefficient §4
D	solute diffusivity
D_s, D_l	solute diffusivity, in solid, in liquid (2.3), (2.4)
d^S	solute deposition rate (2.3)
d	dendrite spacing (3.3)
$f(W)$	effective pressure function (3.22)
g	gravity
h	depth of lens/fringe boundary (3.38)
I	ice volume fraction (3.22)
J	ice flux (3.22)
k	thermal conductivity (2.5)
k_h	hydraulic conductivity
k_0	reference value for k_h
k^*	dimensionless hydraulic conductivity (3.32)
$[k]$	scale for k_h
\tilde{k}	ratio of k_0 to $[k]$
K	permeability (3.1)

K_0	permeability coefficient (3.2)
Le	Lewis number. = κ/D
L	latent heat (2.5)
m^s	rate of freezing (2.1)
m^l	rate of melting (2.2)
m	exponent in permeability law (3.2)
n	void fraction of soil (3.23)
N	size distribution of crystals (3.54)
p, p_l	liquid pressure (3.1), (3.9)
p_s	solid pressure (3.13)
p_i	ice pressure (3.22)
p_w	water pressure (3.22)
p_e	interfacial pressure jump
p	overburden pressure (3.23)
q	dimensionless heating (1.1)
Q	dimensionless heat flux into frozen fringe (3.36)
r	density parameter (2.7)
R	Rayleigh number (3.4) ₅
St	Stefan number (3.4) ₄
s	solid solute concentration (2.3)
t	time
T	temperature (3.53)
T_0	reference temperature (2.6)
T_s	dimensionless surface temperature (3.37)
u	dimensionless temperature (1.1)
\underline{u}	barycentric velocity (3.4)
\underline{u}^s	solid velocity (3.13)
\underline{U}	water flux (3.22)
\underline{v}	liquid velocity (relative to \underline{u}) (3.9)
\underline{V}	solidification rate (2.11)
v_i	ice velocity (3.22)
$v(\theta)$	crystal growth rate (3.56)
$.W$	water volume fraction (3.22)
x	length coordinate (1.1)
x_l	phase change boundary (1.5)
x_s	phase change boundary (1.3)
y	vertical coordinate

z	vertical coordinate (3.22)
$[z]$	length scale (3.31)
z_f	frozen fringe/subsoil boundary
z_ℓ	fringe/lens boundary
<u>Greek</u>	
α	void fraction (3.1)
β	Clapeyron slope
γ	hydraulic conductivity exponent (3.29)
Γ	liquidus slope (2.6)
δ	small parameter in magma transport model (3.15)
$\delta_1, \delta_2, \delta_3$	small parameters representing hydrostatic pressure variation, heat advection, density difference (3.31)
ϵ	small heat conduction parameter (3.15)
ζ	bulk viscosity (3.13)
η_b	representative grain shear viscosity (3.14)
θ	undercooling (3.59)
κ	thermal diffusivity
λ	partition coefficient (2.8)
Λ	dimensionless latent heat (1.2)
μ	compaction parameter (3.15)
ν	surface area fraction (3.55)
Π	osmotic pressure (3.22)
ρ	density (2.1)
ρ_ℓ	liquid density
ρ_s	solid density (2.7)
σ	Soret coefficient (after (3.62))
τ	pressure scale (3.31)
ϕ	pressure 'head' driving magma flow (3.15)
Φ	mass fraction of crystals (3.54)
χ	mass fraction (1.3), (2.1)
ψ	weighting factor (3.24)

§1. INTRODUCTION

Mushy zones, zones of coexisting phases of the same material, are commonly encountered in multi-component flows. Such flows occur in a wide variety of applications, particularly in geophysics and metallurgy. For instance, the problems of alloy solidification (Flemings and Nereo [1]), transport of magma by porous flow in the deep mantle (Frank [2], Turcotte [3]), water flow through and beneath glaciers (Nye and Frank [4], Weertman [5]), frost heave (Everett [6]), solidification of the earth's inner core (Jacobs [7], Loper [8]), solidification of magma chambers (McBirney and Noyes [9]), volcanic eruption (Spera [10]), and steam-water flows in reactor cooling

systems (Fowler [11]), are all situations where a two-phase model incorporating a change of state is necessary: one could mention more.

In this paper, I will attempt to review some mathematical models that have been developed to a greater or lesser degree to describe such situations. In particular, section 3 dwells in some detail on four of the above mentioned topics: A, alloy solidification; B, magma transport in the deep mantle; C, frost heave; and D, layered igneous intrusions. A mathematical model should be designed with the aim of solving a definite problem, or understanding a particular phenomenon. In the above contexts, the particular motivating problems are these: A, how and why does freckle formation occur (or the formation of V and A type (segregates) (McDonald and Hunt [12])? B, how does the generation of magma at great depth in the earth's mantle facilitate the rupture of the lithosphere which enables it to penetrate to the surface without freezing? C, how can one quantitatively predict when frost heave will occur? D, how and why does intermittent or cyclic layering occur in, e.g., the Skaergaard intrusion? It will be evident that some of these topics overlap with interests of other participants at this conference (e.g. Loper, Hellowell, McBirney, Worster, Stevenson, Scott, Ribe, Jaupart ...), and more detail on them will doubtless be found elsewhere in these proceedings.

In studying models of such widely differing systems, one finds analogies, both mathematical and physical. For example, all the models described below have difficulties with the appropriate boundary conditions. The alloy problem is typical: it is missing one (Worster [13]). One can use a heuristic assumption, but ideally, the extra boundary condition should be derivable. This (or its equivalent) is in fact a major topic of investigation in the study of growth of single dendrites (cf. Xu and Cole's presentation at the conference). One finds (for varying reasons) different (average) pressures in the two phases, for example in B (magma transport) and C (frost heave). In B, differing pressures are an assumption of the model, and occur for dynamic reasons: their effect (see section 3) is uncertain, possibly facilitating lithosphere fracture; in C, they represent a thermodynamic equilibrium state (due to interfacial free energy); yet fracture on freezing can occur here too. Indeed, fracture (hot tearing) also occurs in alloys (A) and magma chambers (D) (Norton et al. [14]), and is thus a common feature which can result from the change of density with phase. Frost heaving occurs either as primary heave (no mush), or as secondary heave (when there is an ice/water/soil mush). The distinction seems analogous to the generation of a mush in alloys via instability of a planar interface (Mullins and Sekerka [15]): this may prove useful in future studies of this phenomenon.

We will be primarily concerned with two component systems. However, the simplest type of mushy zone can occur in a single component model, the Stefan problem with internal heating. The governing equations are just those of heat conduction, but even a simple problem in which a cold steel plate is heated volumetrically by Joule heating till it melts on the inside is not well-posed (Elliott and Ockendon [16], Tayler [17]) in the following sense: there does not exist a solution having distinct liquid and solid regions separated by a moving boundary unless the solid becomes superheated (which is unphysical).

Before the onset of melting, the temperature is described by the following dimensionless equations (Tayler [17]):

$$u_t = u_{xx} + q, \quad 0 < x < 1, \quad (1.1)$$

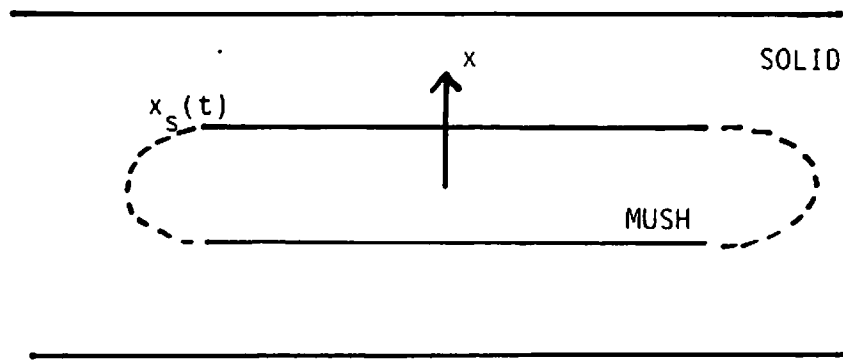


Fig.1. Joule heating of a metal plate. The metal begins to melt in the interior (near $x = 0$), forming a mushy zone.

with

$$\begin{aligned} u_x &= 0 \quad \text{on } x = 0 \\ u &= -1 \quad \text{on } x = 1, \end{aligned} \quad (1.2)$$

and $u = -1$ for $t = 0$. Here u is (dimensionless) temperature, q the internal heating parameter. When u first reaches the melting point ($u = 0$), a finite time must elapse before the solid acquires enough heat to melt. During this time a region of mixed phase develops, in which $u = 0$ for $x < x_s(t)$, and the mass fraction χ of liquid satisfies

$$\Lambda \chi_t = q, \quad x < x_s, \quad (1.3)$$

where Λ is the latent heat parameter, while (1.1) applies in $x > x_s$, together with

$$u = 0, \quad u_x = 0 \quad \text{on } x = x_s^+. \quad (1.4)$$

When $\chi = 1$ at $x = 0$, the mushy zone becomes liquid there, and then we have a further region $0 < x < x_\ell(t)$ ($< x_s(t)$), in which (1.1) applies, with boundary conditions (1.2)₁, and

$$\begin{aligned} u &= 0. \\ [u_x]_-^+ &= \Lambda(1-\chi_+) \dot{x}_\ell \end{aligned} \quad (1.5)$$

on $x = x_\ell(t)$. Notice that χ cannot be assumed to be equal to one at $x = x_\ell$ (when $x_\ell > 0$), in contrast to (1.4)₂, which is equivalent to $\chi = 0$ on $x = x_s$. This apparently arbitrary assumption that mass fraction is smooth at $x = x_s$ is justified on physical grounds (i.e. otherwise the solid would be superheated). A similar prescription occurs in magma transport (§3B), but when freezing (of alloys) is considered, no such simple argument works, and we must forage for the extra boundary condition on χ .

The equation (1.3) only makes sense if χ is a local volumetric average. Thus, implicitly, we must average the system over its heterogeneous microscopic state, and in so doing, distinguish the local structure from the averaged global structure. This is common policy in turbulent flows, and

in two-phase flows. and we expect to have a solve some microscopic problem (approximately) in order to constitute fully the macroscopic one of interest. For the Stefan problem, this has been done by Lacey and Tayler [18].

§2. TWO COMPONENTS - NO MOTION

Molten alloys and aqueous solutions are examples of multi-component liquids which, almost invariably, freeze in a morphologically complex way. This phenomenon, the so-called cellular or dendritic solidification, occurs because of supercooling of the liquid ahead of a solidification front and, as such, is analogous to corresponding superheating which would occur in the Joule heating problem of §1. Multi-component fluids do not have a single melting/freezing temperature, rather the solidus and liquidus temperature depend on relative concentration of the species (as well as pressure) (Flemings [19]). Thus, in front of a solidifying interface, there is a variation of both temperature and concentration, through thermal and chemical boundary layers. Moreover, if κ is thermal diffusivity, and D is chemical diffusivity, (in the liquid), the ratio of the thicknesses of these layers is large whenever the Lewis number $Le = \kappa/D$ is large (which is the typical case). The concentration of the liquid, and consequently its liquidus (freezing temperature), will thus change much more rapidly than the temperature, and in such a way as to make the liquid 'constitutionally' supercooled. In actual fact, the planar interface is then morphologically unstable (Mullins and Sekerka [15], and the interface becomes convoluted: at first cellular and then, for larger growth rate, the growing cells are themselves unstable and one finally obtains a very convoluted, 'dendritic' region of coexisting solid and liquid - the mushy zone. This situation is depicted in Fig. 2.

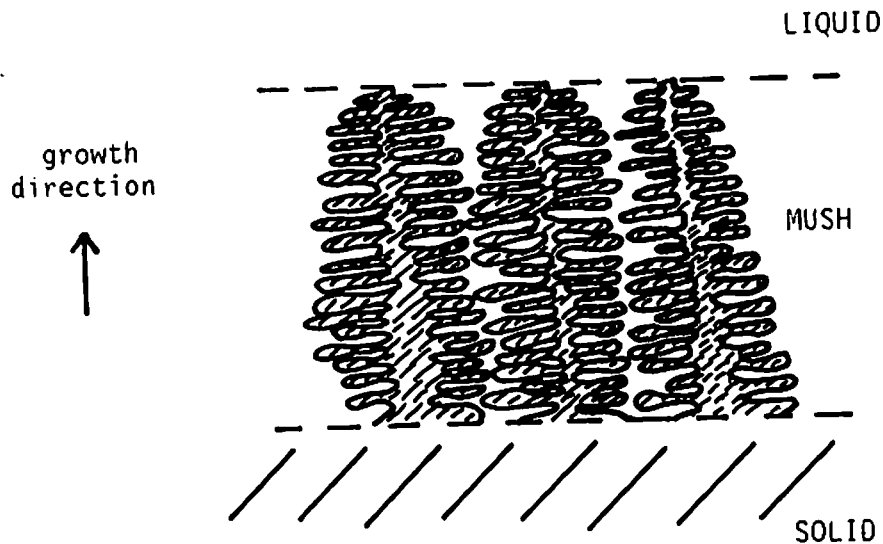


Fig. 2. A dendritic mushy zone in a freezing alloy.

A simple model to describe a mushy dendritic region involves the solute concentrations s and c in the solid and in the liquid respectively, the liquid mass fraction χ , and the temperature T . These variables represent local volumetric averages. Appropriate governing equations, if no motion is assumed and pressure effects are neglected, are (compare Hills et al. [20], Fowler [21], Worster [13], Taylor [17])

$$\frac{\partial}{\partial t} [\rho(1-\chi)] = m^S, \quad (2.1)$$

$$\frac{\partial}{\partial t} [\rho\chi] = -m^S = m^L, \quad (2.2)$$

$$\frac{\partial}{\partial t} [\rho(1-\chi)s] = \nabla \cdot [\rho(1-\chi)D_S \nabla s] + d^S, \quad (2.3)$$

$$\frac{\partial}{\partial t} [\rho\chi c] = \nabla \cdot [\rho\chi D_L \nabla c] - d^S, \quad (2.4)$$

$$Lm^L + \rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T, \quad (2.5)$$

$$T = T(c_i), \quad \text{e.g. } T = T_0 - \Gamma c_i. \quad (2.6)$$

In these equations, the mixture density ρ is given by

$$\rho = \rho_S / (1 + r\chi), \quad r = (\rho_S - \rho_L) / \rho_L, \quad (2.7)$$

and we must assume $r = 0$ (zero solidification shrinkage) to be consistent with $\rho = \text{constant}$, as implied by adding (2.1) and (2.2); here ρ_S and ρ_L are solid and liquid densities (in isolation); (2.1) and (2.2) are equations of conservation of total liquid and solid mass, m^S is the freezing rate; (2.3) and (2.4) are conservation equations for solid and liquid solute; (2.5) is the energy equation, and (2.6) is the prescription of local thermodynamic equilibrium. The constants are D_S and D_L : solid and liquid diffusion coefficients; L : latent heat; k : thermal conductivity; Γ : liquidus slope; c_p : specific heat. The process is supposed slow enough that supercooling is insignificant; then the liquidus temperature T is a function of c_i , the average liquid concentration at the (local) solid/liquid interface. The term d^S in (2.3) and (2.4) is a deposition rate, given by

$$d^S = \lambda c_i m^S, \quad (2.8)$$

where λ is the partition coefficient. More general treatments could include more terms, but practically, the above set is as complicated as one might want to get. The unknowns $\rho, \chi, m^S, s, c, d^S, T, c_i$ (eight) are thus apparently given by the eight equations, but (2.1) and (2.2) are not independent (remembering $r = 0$). In fact, (2.1) is redundant. If $r \neq 0$, solidification shrinkage occurs and we require the barycentric velocity $u \neq 0$. Thus we require one additional constitutive law to relate c_i to c . The simplest assumption is that c is microscopically uniform, and then

$$c_i = c. \quad (2.9)$$

If we neglect both solid and liquid diffusion ($D_S = D_L = 0$), then it is a simple matter to show that the Scheil equation

$$c = c_0 \chi^{-(1-\lambda)} \quad (2.10)$$

applies (Flemings [19]).

Neglect of D_s is quite reasonable, and so also is that of D_ℓ , in most cases. This is because the purpose of the mushy zone is to extend the length of the diffusively-controlled, supercooled concentration boundary layer, which occurs in planar solidification, such that no supercooling occurs. The mush is thus a kind of extended boundary layer, and in particular it is controlled by thermal diffusion (Worster [13], Fowler [21]). Therefore the importance of the solute diffusion term is measured by $Le = \kappa/D_\ell$, where κ is the thermal diffusion. Almost invariably, $Le \gg 1$, so that D_ℓ is negligible. In fact, it is possible that neglect of diffusion is a formal requirement of an averaged theory, i.e. the microscale is determined by the size of D_ℓ : see below. It should, however, be added that in a complete theory, the interfacial surface energy must appear.

Notice the curious structure of the energy equation, which is standard in these problems. T is determined by the thermodynamic constraint, so that the energy equation determines the rate of freezing m^s .

It is possible to write down travelling wave and similarity solutions for these equations (Fowler [21], Worster [13]; the latter compare favourably with experiments on aqueous solutions. Metallurgists are, however, much more interested in situations involving fluid flow, which will be considered later.

Having written our model equations, let us now worry about them a little. We have not appended boundary conditions for reasons of space: these are the usual continuity of flux conditions, etc., and serve to completely specify the problem, except that one condition is missing, probably at the mush/liquid 'interface'. Two possible choices are that $\chi = 1$ there, or that the front is marginally thermodynamically stable to constitutional supercooling (Worster [13], cf. Langer [22]). There is no a priori justification for such a choice. It is attractive, because it makes some sense. But an adequate justification for it constitutes a major problem for theoretical endeavour (Taylor [17]). In practice, marginal stability theory for single dendrites seems to work well both for isolated dendrites (Trivedi [23]), and for an array (Somboonsuk et al. [24]). The relation to the marginal supercooling constraint is not clear.

A related problem is our assumption that $c_i = c$, i.e. that the liquid solute concentration is microscopically uniform. This implies that the particle solute Peclet number $Pe = Vd/D$ is small, where V is solidification rate, and d is primary dendrite spacing. Typical values for metals of $V \sim 10^{-2} \text{cm s}^{-1}$, $d \sim 10^{-2} \text{cm}$, $D_\ell \sim 10^{-4} \text{cm}^2 \text{s}^{-1}$ give $Pe \sim 1$. Thus this assumption is not likely to be realistic. In fact, one might hypothesise that primary dendrite spacing is controlled by the restriction that $Pe \sim 1$, so that c is not microscopically uniform, but varies in such a way that a local marginal instability criterion is satisfied. This is, of course, pure speculation, but is consistent with measurements of d which indicate

$$d \propto (\kappa/V^2)^n \quad (2.11)$$

for a given material (fixed κ), with $n \approx 1/2$ (Flemings [19]). The pore liquid would then be undercooled by $\sim r(c_i - c)$. However, it is still feasible that $c_i \approx c$, since only a very small amount of undercooling in metal alloys will cause high solidification rates. A recent discussion is given by Trivedi [25], who finds (with his theory) reasonable, but not wholly satisfactory agreement with experiment. As with the single dendrite problem, the rôle of the surface energy is not completely clear (cf. Pelce and Pomeau [26]).

§3. SLOW REACTIVE FLOWS

When we come to include velocity, we have all the pitfalls inherent in obtaining averaged equations for two-phase flow (Ishii [27], Drew [28]). The worst of these is that the inertial momentum terms cause a variety of interfacial interaction terms, whose determination is not yet finalised. Further, a proper description of these terms is necessary to avoid having an ill-posed problem since, with no interactive terms and equal pressure between the phases, the inviscid, non-interactive, problem is not necessarily hyperbolic (it can be elliptic in time). That is to say, the characteristics of the system may be complex, arbitrary small perturbations will be amplified, and the initial-value problem is not well-posed (e.g. Klebanov et al. [29]). Examples of such fast reactive flows (involving a change of phase) are steam/water flow in nuclear reactors and heat exchangers, and the eruption of phenocryst bearing or vesicular magma through a fracture in the earth's crust. For such flows, typically turbulent, simple one-dimensional averaged pipe-flow models may be sufficiently useful for practical purposes. For slow flows however, we have the advantage that inertial terms vanish (though not the interactive drag), and the problems of ill-posedness do not (seem to) arise. In this section, we present discussion of four problem areas of current interest to metallurgists and geophysicists.

A. Alloy solidification

Let us reconsider the problem studied in §2, but now with a liquid velocity included. As a realistic simplification, we suppose the solid has zero velocity. Advective terms are put into (2.1) and (2.8) in a straightforward manner (see Hills et al. [20], Fowler [21]). If V is a typical solidification rate (this will be controlled by the rate of cooling), then we define dimensionless variables in terms of a length scale κ/V , velocity scale V , time scale κ/V^2 , and pressure scale $\rho_s \kappa \eta_l / K_0 \rho_l$, where η_l is the liquid viscosity, κ is the thermal diffusivity, and K_0 is related to the permeability K defined (dimensionally) by

$$\underline{v} = -(K/\alpha\eta_l)[\underline{\nabla}p + \rho_l g \underline{j}] , \quad (3.1)$$

where α is liquid volume fraction ($= \rho_x/\rho_l$), \underline{j} is the unit upward vector, p is pressure, and \underline{v} is the liquid velocity through the dendrites. The permeability K for the Darcy flow assumed in (3.1) is usually taken as

$$K = K_0 x^m , \quad (3.2)$$

with a prescribed value of m , typically taken as $m = 2$, and then

$$K_0 = d^2/\lambda , \quad (3.3)$$

where d is dendrite spacing, and the numerical factor λ is of order 100 (Mehrabian et al. [30]). Other values of $m > 2$ are popular. The case $m = 3$ gives the Carman-Kozeny equation (Bear [31]).

The simplest model neglects solidification shrinkage, but retains a buoyancy term in the momentum equation, in keeping with the Boussinesq approximation; in addition, we neglect both solid and liquid solutal diffusion. The model can then be written in dimensionless form

$$\begin{aligned} \underline{\nabla} \cdot \underline{u} &= 0 , \\ \frac{d}{dt} [xc + (1-x)s] + \underline{\nabla} \cdot [(1-x)(c-s)\underline{u}] &= 0 , \\ \frac{\partial}{\partial t} [(1-x)s] &= -\lambda c \frac{\partial x}{\partial t} , \text{ (using (2.9))} , \end{aligned}$$

$$\frac{dc}{dt} - St \frac{\partial x}{\partial t} = \nabla^2 c, \quad (\text{using (2.6)}) .$$

$$u = \chi^m [-\nabla p + Rcj], \quad (3.4)$$

where u is the barycentric velocity, equal to χv (since the solid is stationary). The material derivative $d/dt = \partial/\partial t + u \cdot \nabla$. As before, λ is the partition coefficient, St is the Stefan number

$$St = L/c_p r, \quad (3.5)$$

and R is the dendritic Rayleigh number

$$R = \frac{(\Delta\rho)gK_0\rho_\ell}{V\eta_\ell\rho_s} \approx \frac{(\Delta\rho)gK_0}{V\eta_\ell}, \quad (3.6)$$

where $\Delta\rho$ is the rate of change of density with composition on the liquidus. Positive R corresponds to releasing light fluid when freezing. If, for example, $V \sim 10^{-2} \text{ cm s}^{-1}$, $\eta_\ell \sim 10^{-2} \text{ gm cm}^{-1} \text{ s}^{-1}$, $\Delta\rho \sim 1 \text{ gm cm}^{-3}$, $g \sim 10^3 \text{ cm s}^{-2}$, $d \sim 10^{-2} \text{ cm}$, so $K_0 \sim 10^{-6} \text{ cm}^2$, then we find $R \sim 10$. Thus this buoyancy term can be quite important. Values of L/c_p are typically around 300 K for metals, and of the same order as the melting range, so that $St \sim 1$ is typical.

A phenomenon of practical interest in metallurgical solidification processes is that of freckle formation (Copley et al. [32]) and channel V- and A- type segregates (Mehrabian et al. [30]). These phenomena are understood to be due to convective interdendritic fluid flow driven by buoyancy forces, and occur when the local fluid velocity is in the direction of, and greater than, the solidification rate. In order to understand when this should occur, consider the situation when an alloy is cooled from below, and releases light fluid as it solidifies (thus $R > 0$ in (3.4)). We can analyse the stability of the steady state $u = 0$ to convective motion. Unfortunately the boundary conditions for (3.4) are coupled to equations governing motion in the liquid. With the ansatz that such motion is purely vertical, Fowler [21] found that convection occurs for $R > 4.7$ (if $\Delta\rho$ in (3.6) is the density difference between the initial liquid and that at the eutectic point).

A more realistic approach treats convection in the liquid zone properly. Compositional variations at the mush/liquid interface drive (double-diffusive, finger type) convection in the liquid, which can be seen in experiments (Sample and Hellawell [33]), as illustrated in Fig. 3(a). Because of the low resistance to flow in the melt, it is (essentially) mechanically uncoupled from the mush flow (i.e. it has a no-slip condition at the interface), and the condition on mush pressure at the interface for (3.4) is $p = 0$. Additionally, this implies that motion in a melt is vigorous, even turbulent, and consequently a linear analysis of the melt motion is not very useful, rather the two-parameter bifurcation approach of Proctor [34] is indicated. We parametrise the melt convection either by methods of boundary layer theory (Roberts [35], [36], also see Howard [37]) or by computing time-averaged turbulent fluxes (Turner [38]). The latter is probably more realistic, and a good deal easier. It is possible that Fowler's [21] result will not be significantly changed.

In a vertically solidified casting, with high fluid velocities in the direction of solidification (R large), channels form in the mush, through which the upwards convection is focussed (see Fig.3). In this case, a modification of the appropriate boundary conditions is necessary, as indicated by Roberts and Loper [39]. Industrially, one wishes to suppress

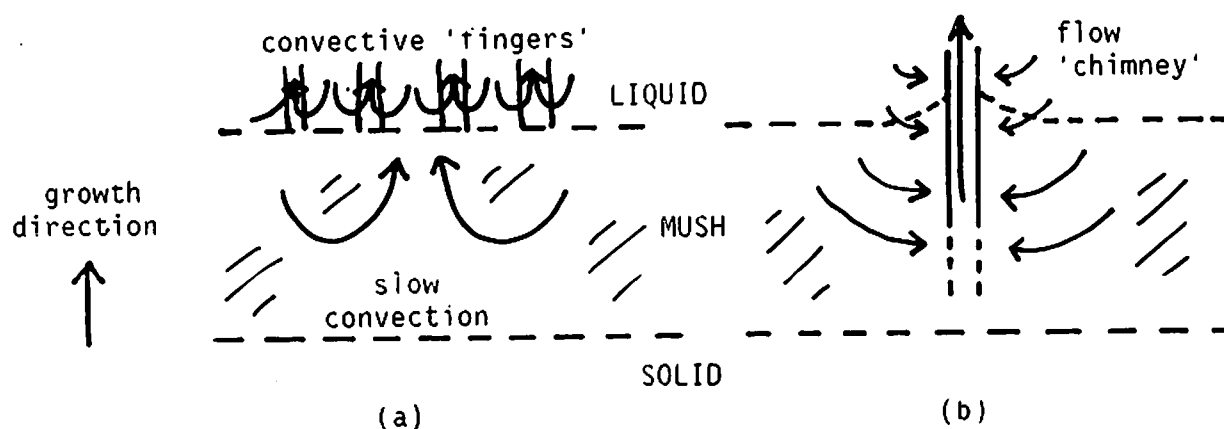


Fig. 3. The onset of channel formation in $\text{NH}_4\text{-Cl}$ solution frozen from below.

(a) Prior to channel formation: vigorous convective 'fingers' in the melt are driven by the release of light fluid at the boundary; at the same time, more sedate convection occurs within the mush.

(b) The combined convection systems become vigorous enough that 'melt-back' occurs. Channels form in the mush, and cause the flow system to alter.

freckling, and one tries to reduce R . In nature, values of R may be very large. One interesting possible application of this model is to the solidification of the earth's inner core, which appears to be solidifying at the expense of the outer liquid core, generally considered to be an alloy of iron with sulphur or oxygen (Loper [40]). For simplicity, let us ignore pressure effects, and let us suppose that solidification occurs dendritically (and as a slurry, but cf. Loper and Roberts [41]). If the dendrite spacing is $[d]$ millimetres, one finds using estimates from Loper [40] that

$$R \sim 10^9 [d]^2 . \quad (3.7)$$

A high Rayleigh number boundary layer theory is relevant, whatever size $[d]$ is. We also find that, with $D \sim 10^{-9} \text{ m}^2 \text{ s}^{-1}$ (Loper and Roberts [42]), the particle Peclet number is

$$Vd/D \sim 10[d] . \quad (3.8)$$

For other reasons, Loper [40] suggested $[d] \sim 3$ (in our terms), which indicates that (3.8) is reasonably consistent with the discussion in §2.

If the mushy zone theory is to apply to the earth's core, it must show that the inner core is virtually solid at a depth of a few kilometres below the inner core boundary, as this is constrained to be solid by seismic data. In fact, an (unpublished) asymptotic analysis of the model at large values of R does seem to confirm that in this case the mushy zone is much thinner, and that χ drops rapidly in a near surface boundary layer. This resolves the problem of the inner core's solidity, but the model has yet to include the presumed flow channels explicitly. However, for other reasons (see below), it may be doubted that dendritic growth does occur.

As is the case for the situation with no flow, the determination of the dendrite spacing requires further prescription. This is of particular importance when there is fluid flow, since then the permeability coefficient K_0 is obviously of major importance. Moreover, the effect of dendrite

coarsening (Kattamis [43]) should also be included in an evolutive way, as it also will affect the permeability.

A further phenomenon of industrial (and possibly geologic) significance is the columnar-equiaxed transition (Flemings [19], Kattamis [43]). The later stages of a casting often exhibit a transition from columnar growth to equiaxed growth, that is, the growth of isolated (but still dendritic) grains in the melt, and this is known to be at least partly due to the ability of strong convective flows to break off or erode dendrite tips. It is also facilitated by the existence of nuclei in the melt. A transition occurs when the dendrites are unable to continue growth, and their shards (and other nuclei) serve as nucleation bases for equiaxed crystal growth. The process is not fully understood, and a model to explain the transition would be useful (Flood [44]).

Since this transition is enhanced by convection, it seems likely that the solidification of the earth's core occurs in this way rather than dendritically, so that the alloy model would need some modification to accommodate this. A 'slurry' model has been put forward by Loper and Roberts [45]. However, the crucial process which needs to be explicitly included is an evolutive model for the (mean) crystal size, and some further understanding of the columnar-equiaxed transition might be necessary before this can be done.

B. Magma transport

A problem of interest in geophysics is that of determining how molten magma is able to get to the surface of the earth, and how its chemistry is determined by conditions at depth. Very little had been done on the dynamic aspects of this problem, until very recently. However, the implications are quite important for predicting features of volcanism, batholith emplacement, as well as the dynamic effects of magma propagation on the geochemistry of igneous rocks.

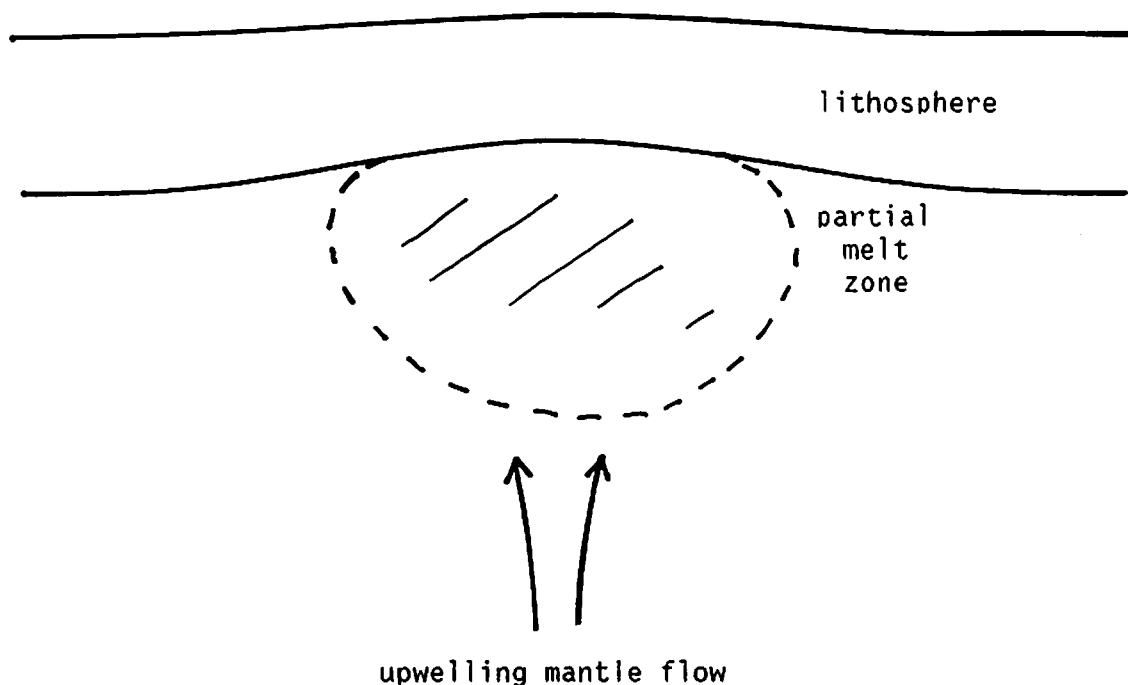


Fig. 4. Geometry of partial melt zone.

Magma is generated (e.g. beneath mid-ocean ridges and at hot spots (for example, Hawaii) by the upwelling of hot mantle rock as part of the general circulation of mantle convection. As this rock rises, at rates of the order of centimetres per year, it intersects the Clapeyron solidus curve which varies steeply with pressure. Partial melting at grain boundaries results, and the resulting melt is considered to form an interconnected, permeable network (McKenzie [46]) at a depth of a hundred kilometres, or less. At higher levels in the lithosphere, the temperature decreases sharply (as in a thermal boundary layer), and the rock becomes once more cold. Most estimates indicate a melt volume fraction of at most a few per cent.

The challenge to a model of the partially molten zone is to understand how magma segregates from its parent rock, and the effect on the chemistry, and perhaps more importantly, to explain how molten magma is able to penetrate up to perhaps fifty kilometres of cold mantle rock (e.g. under hot spots), and how volcanoes and magma chambers then form. This last topic is essentially a dynamic one, and in our discussion we will ignore chemical effects, and in particular, we will ignore the multi-component nature of the flow. In a sense, the role of concentration in alloy solidification is taken here by pressure, which has an equally important effect.

Early work on this problem was done by Frank [2] and Sleep [47] and a comprehensive analysis was done by Turcotte and Ahern [48] and Ahern and Turcotte [49]. A simple model assumes Darcy flow ((3.1) and (3.2), with $m = 2$), and constant mixture density ρ and solid density ρ_s :

$$\begin{aligned} d\chi/dt + \text{div}(\chi\mathbf{v}) &= m^{\ell}/\rho , \\ \mathbf{v} &= -(K_0\rho_{\ell}/\rho\eta_{\ell})\chi\nabla(p_{\ell} + p_{\ell}gy) , \\ Lm^{\ell} + \rho c_0 dT/dt &= k\nabla^2 T , \\ T &= T_0 + \beta p_{\ell} , \end{aligned} \quad (3.9)$$

here written in dimensional terms. The geometry of the situation is represented in Fig. 4. There are some similarities to (3.4) (but beware variants in notation), which may be opaque at first glance. An easier comparison is to (2.1)-(2.7), where (2.2), (2.5), (2.6) are the equivalents of (3.9)₁, (3.9)₃, (3.9)₄. The variables in (3.8) are the liquid mass fraction χ , the rate of melting m^{ℓ} , the liquid velocity \mathbf{v} , the liquid pressure p_{ℓ} and the temperature T . The parameter β is the slope of the Clapeyron curve. The equations are respectively those of liquid mass, liquid momentum, energy, and the thermodynamic equilibrium constraint. The material derivative, however, is $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$, where \mathbf{u} is the barycentric velocity (i.e. the average velocity of the solid-liquid mixture). Approximations involved are that the solid (only) velocity is much smaller than the liquid velocity, χ is small, and various (small) interactive terms have been ignored (Fowler [50]). Extra defining equations are required for \mathbf{u} and p_{ℓ} . The bulk velocity \mathbf{u} must be determined from the mantle convection problem. We suppose it is kinematically prescribed, e.g. $\mathbf{u} = (0, V)$ in an upwelling (two-dimensional) region, and in addition the solid pressure is essentially lithostatic, i.e.

$$p_s = p_0 - \rho_s gy . \quad (3.10)$$

p_{ℓ} and p_s have a close analogy to c_i and c in (2.4) and (2.6), and one simple assumption is that

$$p_l = p_s \quad (3.11)$$

(Turcotte and Ahern [48]). Magma flow is then vertically upwards, and a suitable dimensionless reduction of (3.9) yields

$$x_t + 2\chi x_y = 1, \quad (3.12)$$

with an assumed boundary condition $\chi = 0$ at the base of the partially molten zone $y = 0$. The steady state solution is $\chi = y^2$, and perturbations to this are stable, but form shocks (or fronts). A similar phenomenon (for similar reasons) is found in the melt run-off from polluted snow (Colbeck [51]). It should be mentioned that the partial melt zone terminates in the lithosphere, where refreezing occurs according to the Stefan condition. In particular, magma cannot escape to the surface by porous migration. The assumption of vertical mantle flow is not restrictive. More generally, if $\underline{u} = V(u,v)$, then the right hand side of (3.12) is replaced by v : however, χ behaves similarly.

Turcotte and Ahern [48] state that 'crystal deformation can readily occur at the high temperatures associated with partial melting. It is therefore appropriate to assume that the pressures of the solid and the liquid are equal'. Ahern and Turcotte [49] give an estimate of the length scale over which the matrix will collapse; this length, since called the compaction length (McKenzie [46]), they find to be of order 100 m. It is nevertheless an assumption of their analysis that this compaction will occur. More recently, several authors have attempted to explicitly include this compaction process in the model (McKenzie [46], Scott and Stevenson [52], Fowler [50], Ribe [53]). At least the first three of these papers are independent, but remarkably they all come to essentially the same conclusion. McKenzie [46] proposed that $p_l = p_s$, but that the solid pressure was not necessarily nearly lithostatic, and required a bulk viscosity term to be included, to represent the compressibility of the porous matrix. Since a bulk viscosity only has an interpretable meaning (Batchelor [54]) as a difference between the dynamic pressure and the static (thermodynamic) pressure, prescription of such a bulk viscosity may be equally well considered to be a prescription of the pressure difference $p_l - p_s$:

$$p_l - p_s = \zeta \operatorname{div} \underline{u}^s, \quad (3.13)$$

where ζ is the bulk viscosity. McKenzie [46] chose the solid matrix velocity in (3.13), presumably as it is this which is compacting. Fowler [50,55] studied a microscopic model of the deformation mechanism, and concluded (with previous approximations, notably small χ) that (3.13) applied, with $\zeta = \eta_b/\chi$, η_b being a representative grain shear viscosity. In terms of v , one finds

$$p_l - p_s = -(\eta_b/\chi) \operatorname{div} \chi \underline{v}. \quad (3.14)$$

Scott and Stevenson [52] adduce energetic arguments in favour of (3.13), and also [56] provide a microscopic model.

The uses to which these augmented models have been put is very varied. Scott and Stevenson [52,56] ignore melting; that is to say, they put $m^2 = 0$ in (3.9)₁, and ignore (3.9)₃. In this case, they find travelling wave solutions in one and two dimensions which possess some of the properties of solitons, and these can be reproduced in an analogue laboratory system (Scott, Stevenson and Whitehead [57]). Similar results have been given by

Richter and McKenzie [58].

Applications to surface geochemistry and magma production have been given by McKenzie [59,60], notwithstanding the difficulties associated with transporting the magma through the lithosphere (Emerman et al. [61], Campbell [62], Huppert and Sparks [63]). McKenzie uses a prescribed melting, and so essentially ignores (3.9)₃, or where he does not, takes χ as constant, seemingly tantamount to ignoring (3.9)₂.

Ribe [53] is the only author who attempts to justify Ahern and Turcotte's [49] statement about compaction by including the full equations, and in fact he comes to the same conclusions as they did, with a compaction length of ~ 10 -100 m. However, his analysis is based on the prescription of S , and the neglect of (3.9)₃. Thus all these papers include a complication (compaction), which is shown to be small (Ribe [53]), at the expense of losing an entire equation! Ribe [64] applies his model to the determination of melt composition. He treats the energy equation correctly, but ignores compaction (i.e. assumes $p_\ell = p_s$) on the basis of his earlier result (Ribe [53]). Evidently, a confirmation that $p_\ell \rightarrow p_s$ on a short length scale is still lacking from these results.

The set (3.9) and (3.14) can be written in the dimensionless form (Fowler [50])

$$\begin{aligned} \delta(dx/dt) + \text{div}[\chi \underline{v}] &= m^\ell, \\ \underline{v} &= -\chi \nabla \phi, \\ -\mu \text{div}[\chi \underline{v}] &= (\phi + ry)\chi, \\ m^\ell + \frac{d}{dt}(\phi - y) &= \epsilon \nabla^2 \phi, \end{aligned} \quad (3.15)$$

where ϕ is the liquid head, χ has been scaled, r is given by (2.7), and $p_\ell - p_s \propto \phi + ry$. Moreover, for earth-like values, we find

$$\delta \sim .05, \quad \mu \sim 1, \quad \epsilon \sim .03, \quad r \sim 0.25. \quad (3.16)$$

However, μ is proportional to η_b in (3.14), taken here as 10^{21} Poise (10^{20} Pa s); lower values may be appropriate for partially molten rock, particularly if under internal stress, so that μ may be realistically small. One would then expect $\phi + ry \sim 0$ (i.e. $p_\ell \approx p_s$), in keeping with Turcotte and Ahern's assumption. We assume a one-dimensional steady state with scaled mantle velocity equal to one vertically upward, so $d/dt = d/dy$, and we consider δ and ϵ to be small. An appropriate boundary condition for χ could be

$$\chi = 0 \quad \text{on} \quad y = 0, \quad (3.17)$$

as may be argued for thermodynamic reasons (cf. Turcotte and Ahern [48]). This is the equivalent of the arbitrary $\chi = 1$ dendrite solidification boundary condition, but here it is seemingly based on more secure reasoning.

The total pressure $(1-\alpha)p_s + \alpha p_\ell$ (α being void fraction) in the partially molten zone must balance the lithostatic pressure p_s . Neglecting (small) deviatoric stress, this suggests that

$$\chi(p_\ell - p_s) = 0 \quad \text{at} \quad y = 0, 1, \quad (3.18)$$

and hence, supposing $\chi \neq 0$ at $y = 1$, $p_\ell = p_s$ there, that is

$$\phi = -r \quad \text{at } y = 1. \quad (3.19)$$

It is also tempting to apply $p_\ell = p_s$ at $y = 0$, that is

$$\phi = 0 \quad \text{at } y = 0; \quad (3.20)$$

however, this cannot be deduced from the force balance (3.18); an alternative argument in favour of (3.20) is based on thermodynamic equilibrium: that is, since the melting temperature in the solid is $T = T_0 + \beta p_s$, but it is $T_0 + \beta p_\ell$ in the melt zone, the condition that the solid feels itself approaching the melting point as it nears the melt/solid boundary implies $p_\ell = p_s$ there, i.e. (3.20).

If we neglect the terms in ϵ and δ in (3.15)₄, then $m^\ell \sim \phi_y^{-1}$, and the equations reduce to

$$\begin{aligned} \chi^2 &= (\phi - y)/\phi_y, \\ \mu(\phi_y - 1) &= (\phi + ry)[(\phi - y)/\phi_y]^{\frac{2}{3}}, \end{aligned} \quad (3.21)$$

after integrating once, and using (3.17) and (3.20). The neglect of ϵ and δ is responsible for the reduction from a second order equation for ϕ to a first order one. In consequence, we cannot expect to satisfy both postulated boundary conditions for ϕ . Boundary layer theory suggests that it is consistent to neglect (3.19) along with $\epsilon = 0$, since advection of ϕ along subcharacteristics is towards $y = 1$.

Following this assumption, we then expect that if μ is indeed small, (3.21) will imply $\phi + ry \sim 0$, and indeed, there is such a solution. Evidently, (3.21) implies $\phi \sim -ry + \phi$, where $\phi < 0$. This solution ought to be approached in a boundary layer (the compaction zone), but no such composite solution exists. The equation (3.21)₂ has another physical solution, however, where $\phi > y$, and ϕ grows on a length scale $y \sim \mu^{2/3}$. Since $\phi + ry \sim 1$ corresponds to $p_\ell - p_s \sim 3 \times 10^3$ bars, this implies very large excess pressures indeed.

The rudimentary analysis outlined above was used by Fowler [50] to suggest that massive fracturing (due to $p_\ell > p_s$) would occur above regions of partial melt. This conclusion appears to be false, since further consideration of the structure of solutions of (3.15) indicates that the solution with $\phi > y$ is not compatible with a boundary layer at $y = 1$. The following appears to be the correct analysis. The 'outer' equations (3.21) are appropriate away from the base $y = 0$ and the top $y = 1$, except that an arbitrary constant $\phi_0 < 0$ appears in $\phi - y - \phi_0$ in place of $\phi - y$: we do not (indeed, cannot) suppose (3.20), that is $p_\ell = p_s$ at $y = 0$. There is a solution satisfying (3.18), in particular (3.17) and (3.19) apply. In the boundary layer at $y = 0$, of thickness δ , χ jumps from zero to a value $\mu/|\phi_0|$ (it should be emphasised that χ in (3.15) has been scaled, so that $\chi = O(1)$ represents a volume fraction of less than one percent); in the layer near $y = 1$, of thickness $\delta^{\frac{2}{3}}$, χ jumps again from $O(1)$ to a value $(\mu\pi/2r\delta)^{\frac{2}{3}}$ at the boundary ($\pi = 3.14 \dots$). The solutions are represented in Fig. 5. The analysis assumes $\mu \sim 1$, $\delta \sim \epsilon \ll 1$, $r \sim 1$. The pressure difference $p_s - p_\ell$ decreases to zero at $y = 1$, but $p_\ell + \rho_\ell gy$ decreases upwards, so that melt flows upwards towards $y = 1$. In a steady state, the contraction of melt veins by excess solid pressure is balanced by their opening by melt-back, just as for water flow through ice (Röthlisberger [65]). Melt is generated at $y = 0$ when the solid temperature ($T_0 + \beta p_\ell|_{y=0}$)

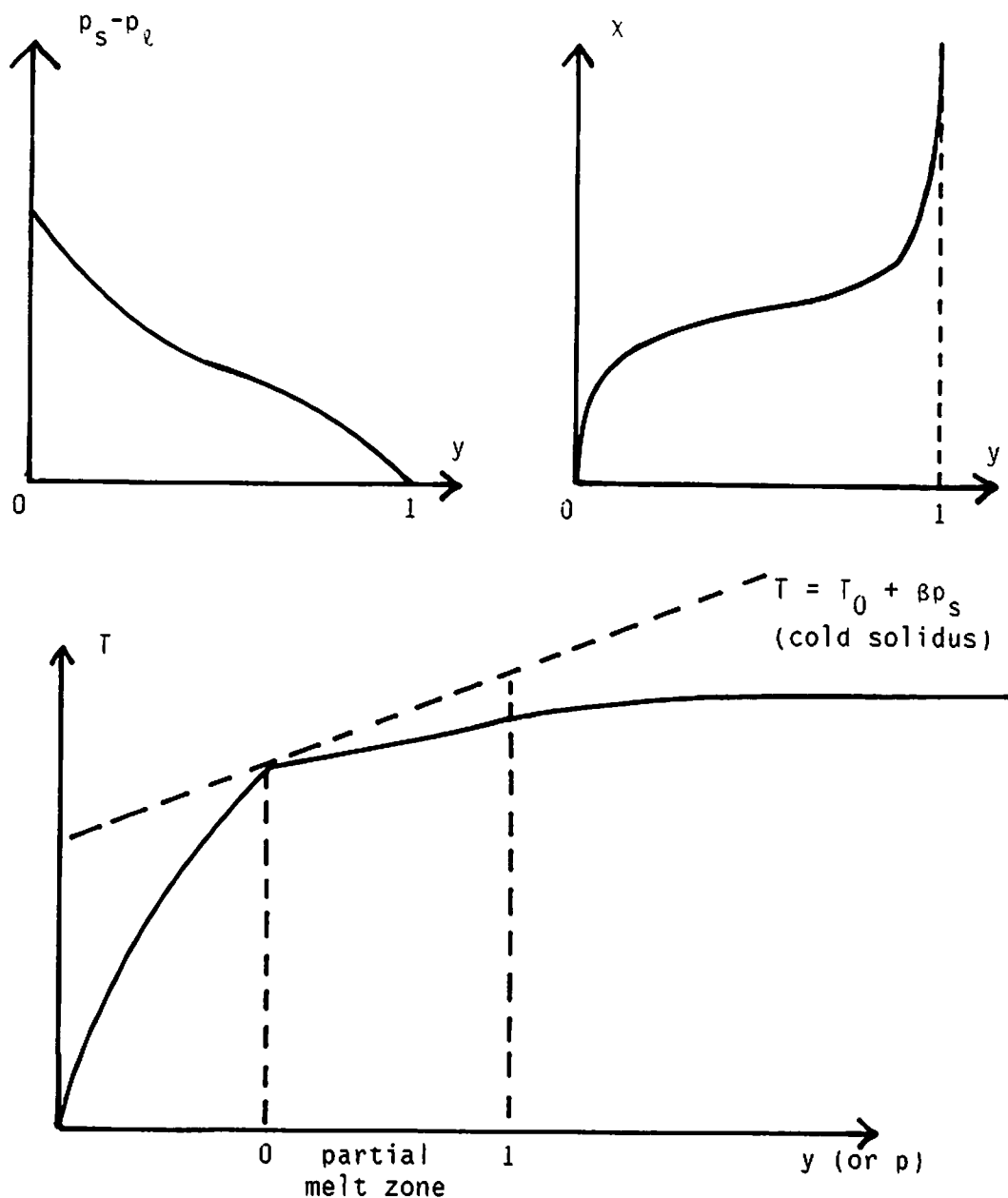


Fig. 5. Schematic solutions of the model (3.15), or (3.9). Melt fraction increases upwards and flows upwards. Liquid pressure deficit ($p_s - p_l$) decreases to zero at the refreezing boundary, thus facilitating fracture. Notice the odd inference that the upwelling solid begins to melt when it is supercooled by some tens of degrees: an untrustworthy result!

is less than the solidus ($T_0 + \beta p_S|_{y=0}$) by an amount $\sim \beta[\phi]$, where $[\phi] = \rho_\ell g d$ is the pressure scale, and d is the depth scale. If we take $\beta \sim 10^{-2} \text{K bar}^{-1}$, $[\phi] \sim 3 \times 10^3 \text{ bar}$ (Fowler [50]), this is an undercooling of about 30K. It seems likely that the assumption of local thermodynamic equilibrium is erroneous, at least at $y = 0$.

If the result portrayed in Fig. 5 is valid, then fracturing is not inevitable. Fracturing is most likely at $y = 1$, where $p_\ell = p_S$, and it will occur if $\tau_1 \geq \tau_0$, where τ_1 is the principal extensional deviatoric stress, and τ_0 is the yield stress, likely to be of order 200 bars. In regions of mantle upwelling, extensional stress is likely, due to the swelling of the lithosphere from beneath. Furthermore, we might expect the partial melt viscosity to be small, since the pressure difference will lead to a lower viscosity, due to nonlinear effects. In that case, the melt zone cannot sustain deviatoric stress, and the boundary condition representing force balance at the lithosphere boundary would be $p_\ell - p_S = \tau_1/\alpha$. If $[\chi]$ is a typical melt volume fraction, then again, fracture would be expected if $p_\ell - p_S \geq \tau_0$, that is if $\tau_1 \geq \tau_0[\chi]$. Thus, the effective fracture strength of the lithosphere may be very small, if $[\chi]$ is small.

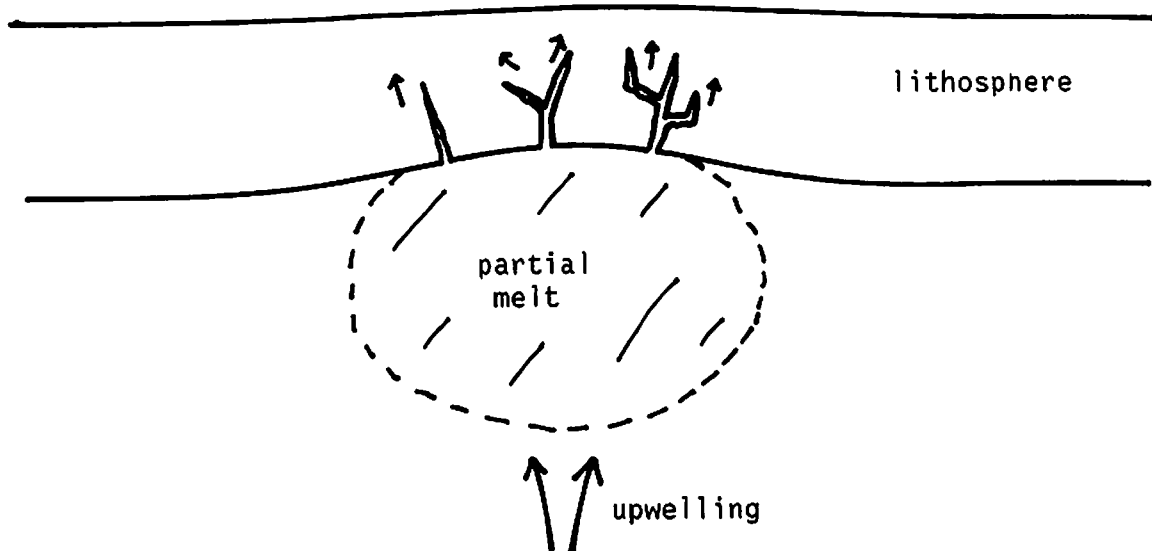


Fig. 6. Penetration of magma through lithosphere fractures.

The model discussed above suggests a possible way of generating fractures in the earth's lithosphere, which seem the only viable means of transporting magma to the surface. The subsequent migration of magma through the lithosphere has been studied by Emerman et al. [61]: the process is visualised in Fig. 6. The model discussed here is a steady state one. Time-dependent models may be prone to the kind of wave activity discussed by (e.g.) Scott and Stevenson [56], although whether this remains true when melting is included is not known as yet.

C. Frost heave

Frost heave is a phenomenon which occurs in freezing soils, when the surface is forced upwards by the accumulation of roughly horizontal ice lenses (Miller [66]) within the soil. The expansion of water on freezing is incidental to this process, which occurs because of an upwards water flux

from the deep soil. This flux is driven by a pressure gradient, which is a manifestation of the fact that the effective pressure ($p_i - p_w$, where p_i and p_w are ice and water pressures) increases as freezing progresses (i.e. as the water fraction W decreases). The problem is to predict the nature and extent of heave.

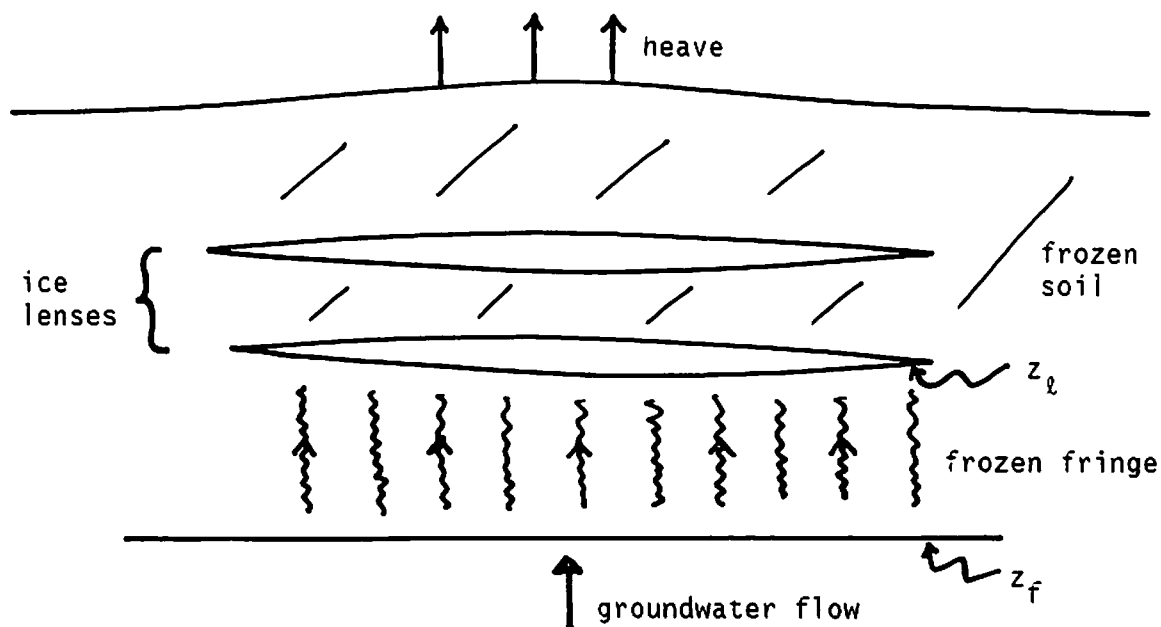


Fig. 7. Secondary frost heave.

In principle, this is quite a difficult task, since soils vary significantly in texture, from fine clays to coarse gravels, and a number of different models exist, incorporating various simplifications (O'Neill [67], Gilpin [68]). Furthermore, one must distinguish two types of heave: primary, in which unfrozen soil underlies the lowest ice lens, and secondary, where there is an intermediate partially frozen 'fringe'. This second case is of more interest, and we describe a model for it below.

The simplest model is one-dimensional, and is given by O'Neill and Miller [69]; the equations are

$$\begin{aligned}
 W_t + U_z &= -m^s / \rho_w , \\
 I_t + J_z &= m^s / \rho_i , \\
 U &= -\frac{k_h}{\rho_w g} \left[\frac{\partial p_w}{\partial z} - \rho_w g \right] , \\
 J &= I v_i , \\
 -L m^s + \rho c_p \frac{dT}{dt} &= k \nabla^2 T , \\
 L(T - T_0) / T_0 &= (p_w - \Pi) / \rho_w - p_i / \rho_i , \\
 p_i - p_w &= f(W) .
 \end{aligned} \tag{3.22}$$

These represent water and ice mass, water momentum, ice velocity (definition), energy, partially frozen thermodynamic constitutive law, and the interphase pressure relation. The variables are I and W , ice and water

volume fraction; U and J , water and ice volume fluxes per unit area upwards; m^S , volumetric rate of freezing; p_w and p_i , water and ice velocity and Π is the osmotic pressure (related to the presence of dissolved salts). L is latent heat, c_p is specific heat, k_h is the hydraulic conductivity, used in soil mechanics rather than the permeability, but easily compared, using (3.1). The equations (3.22) may be compared with (3.9), for example.

In soil mechanics, water is maintained at a lower pressure than the bulk (e.g. overburden) pressure due to interfacial surface energy (capillary effect), osmotic pressure, and the lower free energy associated with electrical double layers of water adsorbed at clay particles (of dimensions of order one micron). As water fraction decreases, the water pressure decreases, and consequently, one relates effective pressure (overburden minus water pressure) to water fraction, as a constitutive law. When ice and water coexist, it has been suggested (O'Neill and Miller [69]) that (3.22)₇ may be applied, to relate ice and water pressures. It should be emphasised that this relation is an equilibrium one. One may think of it as a generalised capillary law. The function $f(W)$ must be measured. It is typically found that $f(n) = 0$, where $W = n$ at saturation (usually $n \approx 0.4$); $f'(W) < 0$, and (near enough) $f \rightarrow \infty$ as $W \rightarrow 0$. Typical values of f in the intermediate range are $\sim O(1)$ bar.

The equation (3.22), generalises the Clapeyron law (3.9). If the right hand side of (3.22)₆ is written as a linear combination of p_w , $p_i - p_w$, and Π , then the three terms correspond to the Clapeyron effect, the Gibbs-Thomson relation, and the depression of melting point (liquidus variation) with solute concentration. The Clapeyron effect is of the order of $10^{-2} K \text{ bar}^{-1}$, whereas the liquidus can sometimes be depressed by many degrees. The temperature T_0 is a reference temperature.

We have seven relations for nine unknowns, so we are missing two extra relations. One comes from the ice 'momentum' equation: ice is considered rigid, but able to move past soil grains by regelation. Thus $v_i = v_i(t)$, and it is determined in the course of solution because it enters into some of the boundary conditions (see below). Notice v_i is the rate of heave. An additional constraint comes from the total momentum equation, which says (in the absence of granular creep, etc.) that

$$(1 - n)p_s + n[\psi p_w + (1 - \psi)p_i] = P, \quad (3.23)$$

where P is overburden pressure, p_s is soil grain pressure, n is porosity, and ψ is a weighting factor, which for simplicity would be

$$\psi = W/n, \quad (3.24)$$

although more complicated expressions may be used (O'Neill and Miller [69]). The relation (3.23) stems from the (averaging) assumption that the local averaged pressure is the weighted sum of the constituent pressures, as indeed makes sense. Then if n is the void fraction, $1-n$ is the volume fraction of soil grains, and thus $(1-n)p_s$ is the soil partial pressure; similarly $\psi n p_w$, $(1-\psi)n p_i$ give weighted water and ice pressures. For very slow motion, a force balance then yields (3.23).

To constitute p_s , we must make an assumption about the soil matrix. The simplest assumption is that the soil matrix is rigid (i.e. gravelly); then

$$I + W = n \quad (3.25)$$

is constant, and (3.23) serves to define p_s . A sensible lens formation criterion is then that $p_s \leq 0$, i.e.

$$\psi p_w + (1 - \psi)p_i \geq P/n . \quad (3.26)$$

At the opposite extreme, a colloidal compressible clay, in which the clay platelets are surrounded by a layer of adsorbed water, might have

$$p_s = p_w + p_e , \quad (3.27)$$

where p_e is related to the free energy of the adsorbed layer. (cf. Gilpin [68]). In this case (3.23) replaces (3.25), insofar as n is no longer constant. 'Lens' formation would then be if

$$p_w \leq 0 . \quad (3.28)$$

More generally, a mixture of these two may be appropriate. For what follows, we consider (3.25) with $n = \text{constant}$, and (3.26).

A dimensionless reduction of (3.22) can be written, anticipating that gravity and heat advection are negligible, and that the freezing temperature varies primarily with the effective pressure $p_i - p_w$. If we suppose τ is a typical pressure scale [e.g. $\tau \sim P \sim f(W)$], then neglect of gravity is consistent provided the length scale $[z] \ll \tau/\rho_w g \sim 10 \text{ m}$ if $\tau = 1 \text{ bar}$; this is appropriate for seasonal frost heave models. However, there is then no natural length scale (which suggests a similarity solution), and a balance of all the non-negligible terms in (3.22) requires a particular choice of scale $[k]$ for the hydraulic conductivity, usually taken as

$$k_h = k_0(W/n)^{\dot{\gamma}} . \quad (3.29)$$

The form of (3.29) is suggested by experiment. The dimensionless model is

$$\begin{aligned} W_t + U_z &= -m^S , \\ I_t + v_i I_z &= m^S / (1 - \delta_3) , \\ U &= -k^* \left[\frac{\partial p}{\partial z} - \delta_1 \right] , \\ -m^S + \delta_2 dT/dt &= \nabla^2 T , \\ T &= -f(W) - \delta_3 p , \\ I + W &= n . \end{aligned} \quad (3.30)$$

The parameters are defined by

$$\begin{aligned} \delta_1 &= \rho_w g [z] / \tau , \\ \delta_2 &= [k] \tau / \rho_w g \kappa , \\ \delta_3 &= 1 - p_i / p_w , \end{aligned} \quad (3.31)$$

where κ is the thermal diffusivity, $k/\rho c$, and $[k]$ is defined through (3.32) below. Here, p corresponds to p_w , other p variables stay the same. Typical choices of the relevant scales are $[z] \sim 1 \text{ m}$, $\delta \sim 10^{-1}$, $\tau \sim 1 \text{ bar}$,

$[T] \sim 10^{-1}K$, $[U] \sim 10^{-7}cm\ s^{-1}$, $[S] \sim 10^{-6}kg\ m^{-3}s^{-1}$, $\delta_2 \sim 10^{-3}$, $\delta_1 \sim 10^{-1}$. We have neglected the osmotic pressure Π in (3.22). The dimensionless hydraulic conductivity k^* is given by

$$k^* = (k_0/[k])(W/n)^Y ; \quad (3.32)$$

we have $[k] = \delta_1[u]$, so typically $[k] \sim 10^{-8}cm\ s^{-1}$, whereas k_0 can vary from $10^{-8}cm\ s^{-1}$ for a clay to $10^{-3}cm\ s^{-1}$ for a gravel. Thus generally, we may expect $k_0/[k] \geq 1$. The lens formation criterion (3.26) is

$$n[1 - \psi)f(W) + p] \geq 1 . \quad (3.33)$$

Evidently if $k^* \gg 1$, we can simply rescale $p \sim 1/k^*$ to balance (3.30)₃.

We suppose (3.30) applied in a frozen fringe $z_f < z < z_\ell$. We simplify the equations by neglecting δ_1, δ_2 and δ_3 , since they are small (and are regular perturbation coefficients). Then

$$I = n - W, \quad T \sim -f, \quad m^S \sim \sqrt{2}f = f_{zz}, \quad U \sim -k^*p_z, \quad (3.34)$$

so that

$$-(k^*p_z)_z - v_i W_z \sim 0 , \quad (3.35)$$

$$W_t - (k^*p_z)_z \sim -f_{zz} , \quad (3.36)$$

are two second order equations for W and p .

To derive boundary conditions, we acknowledge that the large latent heat (small δ_2) forces the temperature field to be conductive in $z > z_\ell$ (frozen) and $z < z_f$ (unfrozen). At z_f , we prescribe

$$W = n , \quad -T_z \sim f_z = Q , \quad (3.37)$$

where Q is the dimensionless heat flux into the fringe. The first condition is the extra condition defining z_f , cf. the condition $\chi = 1$ at the mush/liquid interface (§2). We allow a jump in heat flux at z_ℓ , and then the Stefan condition is

$$U - W\dot{z}_\ell = (-f - T_s)/h - f_z \quad \text{on } z_\ell , \quad (3.38)$$

where T_s is the dimensionless surface temperature, and h is the depth of z_ℓ below the surface,

$$h = -z_\ell + \int_0^t v_i dt , \quad (3.39)$$

if $z = 0$ is the undisturbed surface. Conservation of mass at the lens fringe boundary gives

$$(v_i - \dot{z}_\ell)(1 - n + W) = U - W\dot{z}_\ell \quad \text{on } z_\ell , \quad (3.40)$$

and a force balance at z_ℓ gives

$$1 = p + f(W) \quad \text{at } z_\ell . \quad (3.41)$$

At this stage, we count equations and boundary conditions. Equations (3.35) and (3.36) require four conditions, and the boundaries z_0 and z_f , and the velocity v_i are unknown. (3.38), (3.39), (3.40) and (3.41) comprise five conditions, so we need two more. One such is a global mass balance equation, which yields

$$U = v_i \text{ on } z_f . \quad (3.42)$$

A second is a force balance at z_f . If the water pressure there is hydrostatic, then

$$p = 1 \text{ on } z_f . \quad (3.43)$$

Eliminating p between (3.35) and (3.36), W satisfies the nonlinear convective diffusion equation

$$W_t + v_i W_z \sim [(-f'(W))W_z]_z . \quad (3.44)$$

Then (3.44), (3.38) and (3.40) yield the three expressions for $U = W\dot{z}_\ell$ on z_ℓ ,

$$\begin{aligned} v_i(1 - n + W) - W\dot{z}_\ell & \\ &= (-f - T_s)/h - f_z \\ &= (v_i - \dot{z}_\ell)(1 - n + W) , \end{aligned} \quad (3.45)$$

which implies

$$z_\ell = 0 \text{ (since } \dot{z}_\ell = 0 \text{)} . \quad (3.46)$$

Since (3.37) gives two conditions for W on z_f , we can uncouple W from p if (3.41) can be simplified to eliminate p . In the physically relevant case where $\bar{k} = k_0/[k]$ (see (3.32)) is large, we put

$$p = p_r + \bar{p}/\bar{k} , \quad (3.47)$$

and then (3.41) is approximately

$$W = f^{-1}(1) = W_0 \text{ on } z_\ell , \quad (3.48)$$

say (and we really require $\bar{k}(W_0/n)^Y \gg 1$). A numerical solution of (3.44), with (3.48), (3.37), (3.45) should be relatively straightforward, although not trivial, but no lenses will form (cf. (3.3)). Apparently we need $\bar{k}(W_0/n)^Y \sim 1$ in order to generate sufficiently high pore stress. Then p and W are coupled; a similarity solution ($z_0 \propto \sqrt{t}$, etc.) is possible, providing $g \propto t^{-\frac{1}{2}}$, which is appropriate to time-dependent thermal diffusion below the fringe. The resulting equations must be integrated numerically, for realistic choices of $f(W)$. An idea of the resultant profiles for p and W is shown in Fig. 8, which is similar to a figure given by O'Neill and Miller [69] from their numerical results.

Various points arise from the preceding discussion. Our notion of regelation is a mechanical one, whereby, if the fringe ice is rigidly connected to the lens ice, it is dragged upwards by the heaving lens. The resultant stress on the soil grains may be estimated as $\sigma \sim \rho L a v / k \beta$ where a is grain

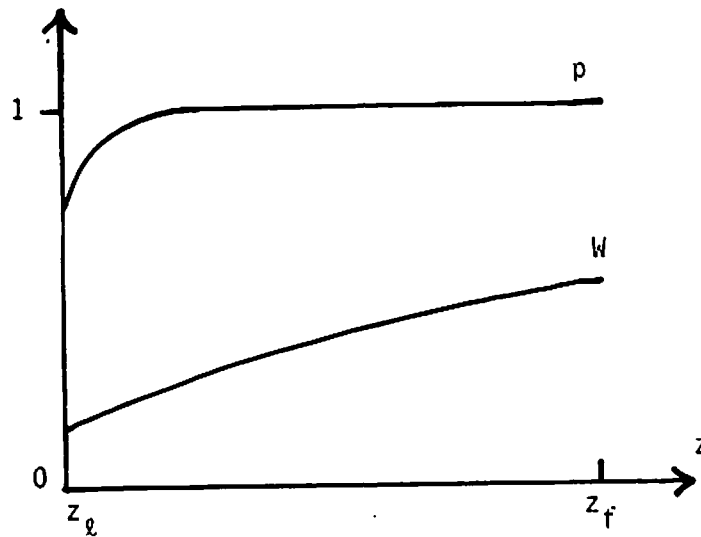


Fig. 8. Typical expected behaviour of p and W . The sharp rise in p would be due to a large permeability exponent γ .

radius, v is velocity, β is Clapeyron slope, k is thermal conductivity. At a grain radius of $50 \mu\text{m}$, and velocity 10^{-6}cm s^{-1} , this is $\sigma \sim .005 \text{ bar}$, which is insignificant.

Our discussion of effective pressure differs slightly from that of O'Neill and Miller [69]. Further extensions need to be made to include the effect of soil compressibility, as previously discussed. We can then expect $I + W$ to depend on p or W , so that (3.35) will include a further time derivative, and the system will be two coupled diffusion equations: positivity of the diffusion matrix is not automatically guaranteed. For example, if we use the colloidal (3.27) with (3.23), we would have (dimensionlessly)

$$p + n(1-\psi)f(W) = 1 - (1-n)p_e/P, \quad (3.49)$$

which shows how we may expect an extra constitutive law $n = n(W, p)$, so that the extra term n_t in (3.35) may contribute both W and p time derivatives.

Actually, we should not expect (3.49) to be appropriate; colloidal soils fracture when they freeze (Miller [66]), and heaving takes an altogether different form. In fact, that process bears strong similarity to the proposed fracture process when solid rock melts, which both involve expansion when the phase change occurs. A more realistic constitutive relation will involve a soil consisting of both fine and coarse particles.

The reader may wonder what causes the distinction between primary and secondary heave. The dendritic alloy solidification problem suggests an answer: the primary heave problem is unstable to the formation of dendrites, basically due to constitutional supercooling. Without actually doing the analysis, we can suspect that this will be the case if, in the primary case, the diffusivity for the water fraction W is small enough (since from (3.34), W replaces c in the thermodynamic law), i.e. if k^* is (probably) $> O(1)$: but one should actually do the problem. Models of primary heave are simpler to pose and analyse (Jackson et al. [70]).

A further cause for secondary heave, on this interpretation, will be solutes. If we take a diffusivity $D \sim 10^{-5} \text{cm}^2 \text{s}^{-1}$, velocity $v \sim 10^{-6} \text{cm s}^{-1}$,

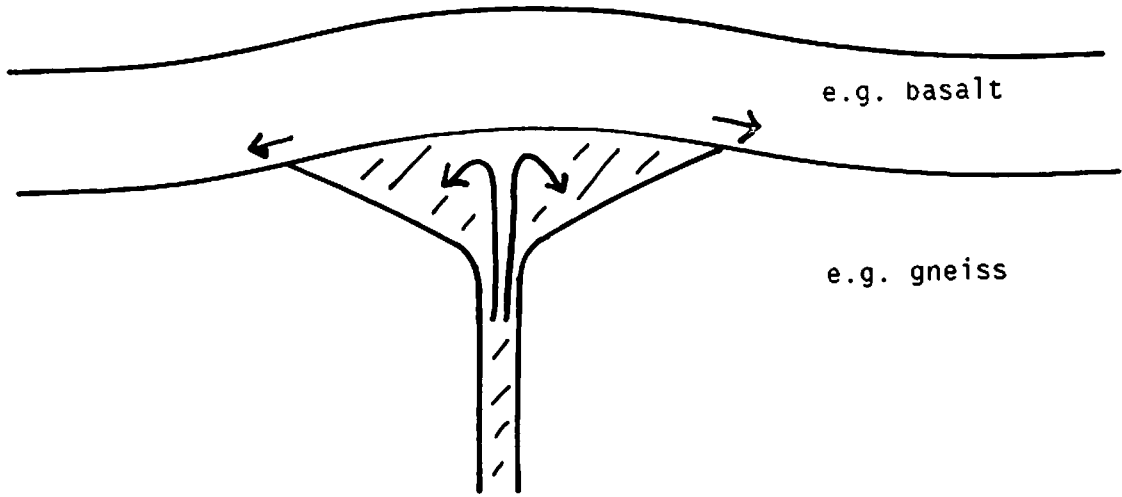
lengthscale $\ell \sim 1$ m, soil grain scale $a \sim 10$ μ m, then the pore Peclet number is $av/d \sim 10^{-6}$, and thus the Taylor transport coefficient (Bear [31]) is irrelevant, and solute transport is by diffusion only. Then $\ell v/D \sim 10$, and diffusion is moderate to small. We can expect c to increase as freezing proceeds, so that $\partial c/\partial w < 0$, and thus this will accentuate the variability of the freezing temperature. Since D is small, the presence of significant solute will lead to 'dendritic' freezing, just as for alloys, and may well be a more important effect than the capillary pressure drop.

A final point to mention is that the assumption of saturated soil is generally unrealistic. Below the frozen fringe, we expect a three phase soil/water/air mixture down to the water table. This can be treated in the same way as for the fringe. More importantly, the frozen fringe will be a four-phase air/ice/water/soil mixture. However, interfacial stability arguments (Miller [66]) suggest that a transition from air/water to ice/water pore constituents will occur over a comparatively narrow temperature range (thus air will be forced downwards), since the three coexisting pore phases are rather unstable. This suggests that a simple model is to take the distinction as a sharp one. However, it seems likely that a second 'mushy' zone will underlie the fringe, where the temperature is approximately constant at this transition temperature, and the air-ice partition gradually changes from pure air to pure ice.

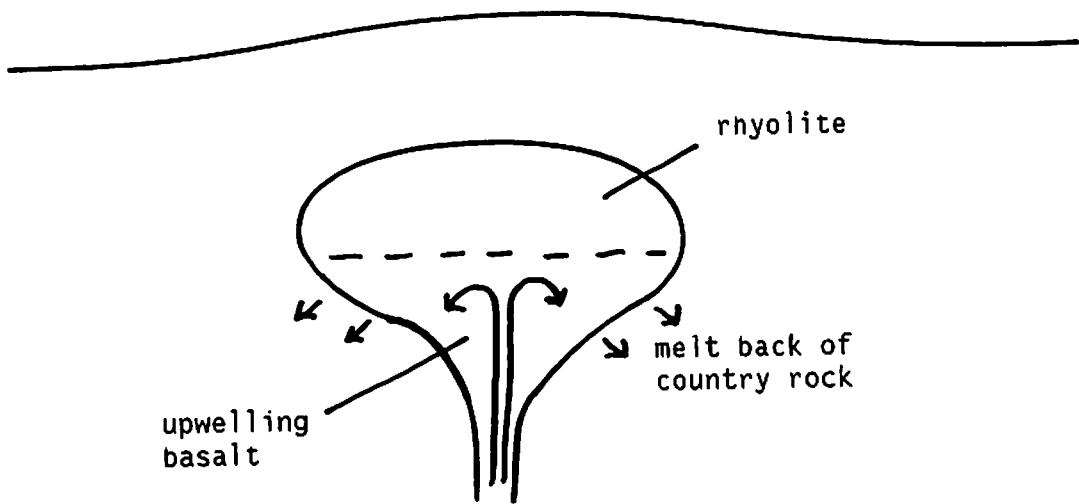
D. Layered igneous intrusions

The emplacement and solidification of magma chambers is a problem of considerable interest in geology and volcanism, but one in which dynamical features of the processes involved have only recently been seriously treated. Analyses of the propagation of magma-filled fractures (Emerman et al. [61]) show that transport of magma from deep in the lithosphere is dynamically feasible, and the magma transport model of (B) above suggests a way in which such fracture might be initiated. Magma flows upwards through fractures, probably being contaminated en route (Huppert and Sparks [63], Campbell [62]). There is nothing evident in the fracture mechanism to stop the fracture,* and presumably many such fractures permeate to the surface, and the eruption is exhausted only when the lava head is sufficient to cancel the excess buoyancy at depth, or the source region becomes depleted, or the lava is frozen. Presumably flood basalts are extruded in this manner. There are two possible ways of terminating a propagating fracture at depth (Fig. 9). If the fracture encounters a discontinuity in rock type, it may preferentially turn abruptly at an angle, and propagate along the discontinuity, forming a sill. This process will be facilitated by the existence of overlying low-density sediments (Holmes [71], p.163), and it has been suggested that primary picritic magma, from which continental flood basalts are then derived, ponds at the Moho in a similar sort of way (Cox [72]). With sufficient head, the magma may force the overlying rock upwards, forming a magma chamber (a laccolith). A possible example is the Skaergaard intrusion in East Greenland (Norton, Taylor and Bird [14]). The magma in this case would initially have a uniform composition. Alternatively, it is conceivable that as a fracture propagates into colder rock, the controlling process at the crack tip changes from being the formation of new surface, to being the melting of the side walls; that is,

*Other than cooling and crystallisation at the walls. However, at high Peclet numbers (see below) characteristic of magma ascent, a scaling analysis (unpublished) suggests that melt back of the walls is a more likely possibility. In any case (see below), crystal growth may be prohibitively small.



(a) Laccolith formation: overcapping basalt is pushed up by intruding magma, which propagates laterally along the unconformity.



(b) If a propagating fracture can be stopped by a switch over to melting of the country rock, then one would expect to pond rhyolitic magma at the top, with hotter, heavier basalt underneath.

Fig. 9.

a. sharp 'tip' cannot be maintained because the melting rate becomes more rapid. Justification of this idea requires some kind of theory of crack propagation with melting included, but the idea seems feasible. If melting becomes appreciable, then crack propagation effectively ceases and a magma chamber is 'blown up' by the melting of surrounding country rock, which continues until the Peclet number of the upwelling magma decreases to $O(1)$. Note that it is important to remember that this melting is consequent upon vigorous (forced) convection, that is, $Pe \gg 1$. Since $Pe = Re Pr$, and $Re \sim 10^2$, $Pr \sim 10^4$ are typical values for basaltic magma, (e.g. if ascent rate = 1 m s^{-1} , dyke width = 1 m , viscosity = 10^2 Poise, (Emerman et al. [16])), this process is likely to continue until the source region is depleted, or the overburden normal stress is sufficient to suppress the buoyancy force, as before. In this situation, the 'initial condition' for the magma chamber is that of a layer of hot, heavy basaltic magma underlying cold, but lighter and more viscous, rhyolite. A situation like this has been advocated by Holmes [73]. The subsequent evolution of such a system has been considered by Huppert and Sparks [74] and Rice [75].

The crystallisation of a magma chamber bears certain superficial similarities to the solidification of a casting. In particular, castings often reveal a transition from cellular/dendritic (columnar) growth near the margins, to equiaxed growth in the centre. The transition depends on the extent of nucleation sites in the melt, and the degree of initial superheating. In the so-called 'big-bang' type of nucleation (Jackson et al. [76]), many seed crystals are initiated almost immediately, and solidification takes place throughout the interior by crystal growth. Similar features are evidenced, e.g. in the Skaergaard, by the existence of skeletal or dendritic crystals in the margins, but tabular crystals in the interior. It is not surprising that the initial magma in the chamber contains lots of nucleation sites, both transported from below, and formed at the chilled margin. The importance of these for heterogeneous nucleation has been emphasised by Lofgren [77]. As crystallisation proceeds, crystals nucleate and grow, so that the magma is properly a two-phase medium. Theoretical estimates for the growth rate of the Skaergaard intrusion, $\sim 10^{-7} \text{ cm s}^{-1}$ (McBirney and Noyes [9], p.541), are sufficiently large that a planar interface would be unstable. It follows that a mushy region adjoins the solidifying boundary and, after a possible initial transient columnar structure, this will consist of a slurry of crystals. Modelling ideas previously discussed are therefore appropriate.

A major phenomenon of interest in many igneous intrusions is the formation of well-defined layers in the rocks (Wager and Brown [78]); the Skaergaard is a prime example of such features. These layers are distinguished visibly (light and dark gradations), and are chemically heterogeneous. It was originally thought (Wager and Brown [78]) that the layering was caused by irregular settling of crystals, but this explanation is now considered to be impractical. More obviously, the apparent regularity of the layers is highly suggestive of an origin in double diffusive convection (Turner [79]), and various attempts to relate this process to observations have been made (Turner [80], Chen and Turner [81], Irvine [82], McBirney and Noyes [9], Kerr and Turner [83], Rice [75,84]). The laboratory experiments of Huppert, Sparks, Turner and co-workers (Huppert and Sparks [85]) demonstrate a wealth of phenomena which occur in crystallising aqueous solutions. Many of these involve (double) diffusive layering, which is almost invariably produced in convective processes involving more than one component. For instance, Turner [80] crystallised sodium carbonate solution at a side wall (actually a central pipe). As crystallisation proceeds, light fluid is released and

rises to the top of the box, thus stratifying (stably) the bulk fluid, by the filling box mechanism (Baines and Turner [86]). Layering occurs because the side-wall cooling of the core fluid tends to drive boundary layer thermal convection downwards; however, the fluid can only obtain limited penetration because of the bulk stable density gradient. Cooling at the top has the same effect.

These experiments are, of course, idealisations, and proper thought needs to be paid to the question of which of these processes is likely to occur in molten magmas. Firstly, in an emplaced magma chamber, all four walls are cold, with the roof perhaps losing most heat (due to hydrothermal circulation) and the 'base' being warmest (if the feeder dyke is broad). At any rate, convective thermal boundary layers will initially stream down from the roof and side walls (Jaupart et al. [87]). Crystallisation of (buoyant) plagioclase together with heavy olivine at the margins would only aid this process. Therefore, a stable thermal stratification of the chamber will build up, analogously to Turner [80]. Release of buoyant liquid through crystallisation is doubtful, but in any event, simple boundary layer arguments may be misleading, since crystallisation must occur throughout a slurry (or perhaps a columnar zone, initially). We may visualise (analogously with (A) above) that the slurry is a region through which T and ϕ (crystal fraction) change rapidly. Component diffusivities are negligible, and the slurry width is scaled by the thermal diffusivity. Insofar as the slurry is a 'thermal' boundary layer, and one can define the viscosity as a function of crystal content, i.e. temperature, one might be able to consider the two phase flow as a one phase fluid with a suitable temperature-dependent viscosity (Spera et al. [89]). Such an approach needs to be further validated.

If crystals are present throughout most of the cooling history, as is suggested by the nature of cumulate rocks, then the magma is at or below its liquidus all the time, and temperature and crystal content are intimately linked throughout. The descending 'thermal' boundary layers at the walls should consist of an inner 'rigid' zone where the stress is below the yield stress (McBirney and Noyes [9]), and an outer crystal mush, which would most likely descend (even though the crystallised plagioclase is lighter than the liquid melt), since it seems unlikely (though not impossible) that the mush components would move in opposite directions, and the total mush density would be greater than that of the melt. In this way a thermally stable mushy liquid would build up at the chamber floor (see Fig. 10). The problem is now, what secondary convective mechanism can form diffusive layers? Although plagioclase is slightly lighter than the gabbroic liquid from which it crystallises, olivine is quite a bit heavier, and it seems likely that when crystallisation is occurring on the olivine-plagioclase cotectic, light

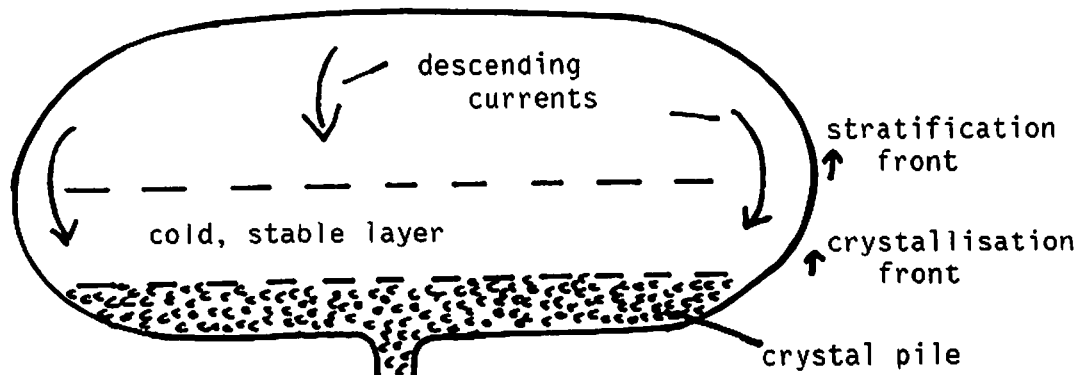


Fig. 10. Formation of a static crystal pile at the floor of a magma chamber.

liquid is being (volumetrically) released in the stratified magma. Consequently, convection could occur within the crystal suspension (compare section 3A). Whether this could be of double diffusive type is unknown. Brandeis and Jaupart [99] suggest that the intermittent deposition of convective plumes may cause layering, but this is speculative, as it rests heavily on Howard's [90] conceptual model of thermal turbulence which describes a rather different situation. In particular, it seems unlikely that a basal stagnant layer would not advance much more rapidly than the crystallisation front. (And hence the front would soon be shielded from plumes.) McBirney [91] has recently suggested that the tholeiitic parent magma of the Skaergaard lies in a part of parameter space inaccessible to double diffusive effects. His conclusion may well be right, although it is based on the linear stability diagram, which is known to be inaccurate at representing diffusive convection in the nonlinear regime (Proctor [34]): anything will convect at high thermal Rayleigh number and high Lewis number. A further complication is that the composition does not follow the temperature blindly - supercooling effects are important (Dowty [92]) and may be significant (McBirney and Noyes [9], Brandeis et al. [93]). Some experiments relating to convection of inter-cumulus liquid have been carried out by Kerr and Tait [94], but it would be nice to try and extend the Huppert/Turner experiments to crystallising slurries. It is not clear if this is possible in the laboratory.

McBirney and Noyes [9] distinguished two types of rhythmic layering, intermittent and cyclic. Intermittent layering has graded layers separated by layers of homogeneous rock. Cyclic layering is similar, but lacks the interspersed uniform layers. McBirney and Noyes associated intermittent layering with double diffusive convection, but ascribe cyclic layering to an oscillatory type of crystallisation which involves supercooling. This is a realistic possibility, since significant undercooling can occur in solidifying magmas. Dowty [92] summarises various experiments, which indicated maximum growth rates of $10^{-4} \text{ cm s}^{-1}$ to $10^{-8} \text{ cm s}^{-1}$ at subcoolings of up to $\sim 200 \text{ K}$. This is unlike the metallurgical situation, where any significant supercooling leads to rapid growth. Dowty ascribes the slow rate of crystallisation to the complexity of the melt phase. Variations in the olivine and plagioclase compositions in the Skaergaard seem to be consistent with a fluctuation on the order of ten degrees (compare data in McBirney and Noyes [9], fig. 12) with, for example, a Fo-Fa phase diagram).

The simplest model (Brandeis et al. [93], see Kirkpatrick [95]) assumes a stationary slurry and a single phase material. In one dimension, the temperature T satisfies

$$T_t = \kappa T_{zz} + (L/c)\phi_t, \quad (3.50)$$

where ϕ is the mass fraction of crystals. If $N\delta R$ is the number of crystals per unit volume of 'dimension' (e.g. radius) between R and $R + \delta R$, then

$$\phi = \int_0^{\infty} NV(R)dR, \quad (3.51)$$

if we neglect variations of density between crystals and liquid; here,

$V(R) \sim R^3$ is the volume of a crystal of dimension R . As we 'follow' a cohort of crystals, then $dR/dt = v(\phi, R)v(\theta)$, where θ is the undercooling, $v(\theta)$ is the growth rate, and v is the surface fraction of crystals of size R which is in contact with melt. One suitable assumption is $v = 1 - \phi$, independent of R . Additionally, $dN/dt = 0$; this is equivalent to

$$N_t + vN_R = 0. \quad (3.52)$$

If we neglect nucleation, and have an initial state of N_0 microscopic crystals ($R \approx 0$) per unit volume, then

$$N \approx N_0 \delta [R - \int_0^t v v(\theta) dt] \quad (3.53)$$

so that (with $V = aR^3$),

$$\phi \approx aN_0 \left[\int_0^t v v(\theta) dt \right]^3 \quad (3.54)$$

and

$$\phi_\tau \approx 3aN_0 \left[\int_0^t v v dt \right]^2 v v = 3(aN_0)^{1/3} \phi^{2/3} v(\phi) v(\theta) . \quad (3.55)$$

If we take

$$v = b\theta , \quad T = T_L - \theta , \quad (3.56)$$

where T_L is the liquidus temperature, then θ satisfies

$$\theta_\tau = \kappa \theta_{zz} - b(3L(aN_0)^{1/3}/c) \phi^{2/3} v(\phi) \theta , \quad (3.57)$$

together with suitable boundary conditions. A more complicated model, including nucleation, has been analysed by Brandeis et al. [93], who showed that oscillatory behaviour could occur (but only temporarily). For an alloy, we revert to (2.1)-(2.6), or approximately,

$$\phi s + (1 - \phi)c = \text{constant} ,$$

$$\frac{\partial}{\partial t} (\phi s) = \lambda c \frac{\partial \phi}{\partial t} , \quad (3.58)$$

to supplement (3.50), and (2.6) is replaced by

$$T = T_L(c) - \theta . \quad (3.59)$$

It would be interesting to see whether such a system could have oscillatory behaviour. However, the more interesting case is where near-eutectic solidification is occurring. This is the normal case in magmas, and it has been suggested that it can explain both rhythmic and intermittent layering in the Skaergaard (Maaløe [96]), as well as in other intrusions (Petersen [97]).

The role of double diffusive convection as an explanation for some of the layering in igneous intrusions remains to be elucidated. Chen and Turner's [91] experiments are very suggestive, but are analogues rather than models of the processes. Rice [75] suggests that oscillatory convection at high Rayleigh number will cause striations in the crystallisation front, just as solute bands form in crystal growth. Solute banding is considered (Hurle [98]) to be due to oscillations which can arise due to instabilities in the convecting melt, and banding in eutectics is strongly reminiscent of intermittent layering. Soret convection has been considered in this context (Jakeman and Hurle [99]), but it seems likely that it will only have a major effect on convection when the melt is otherwise compositionally homogeneous, since Soret coefficients σ (in the equation $\partial c/\partial t = D \nabla \cdot [\sigma c(1-c) \nabla T + \nabla c]$) are typically $\sim 10^{-3} \text{K}^{-1}$ (Walker and DeLong [100]), and thus σD is so small that it can generally be neglected (like D). The exception to this is when the melt is otherwise homogeneous; then very

small σ can radically alter the onset of thermal convection, because a very small Δc can have a large effect on buoyancy (Hurle and Jakeman [101]). However, it seems more likely that when large compositional variation is available due to other reasons (e.g. freezing), then Soret transfer will be irrelevant, except perhaps for small partition coefficients (Hurle [102]).

§4. DISCUSSION

There are a number of other situations in which multi-phase reactive models are appropriate. Temperate (at the melting point) glaciers flow as a mixture of ice and water. Water in temperate ice has many similarities to partially molten rock (§3B): water exists at three grain intersections, but also in moulines and crevasses: large, partially water-filled fractures which permeate the ice. From the dynamic point of view, water is largely of interest because its abundance at the bed causes rapid sliding (Lliboutry [103]), but crevasses occur easily due to ice stresses, so that the problem of water generation by partial melting within ice is less interesting. Most water at the base is supplied from the surface by rainfall, etc. Interesting problems do occur with regard to the subglacial drainage of this basal water (Röthlisberger [65], Nye [104]), in particular jökulhlaups are large-scale periodic floods in the outlet stream. The explanation of these (Nye [104]) involves a model of the drainage channel which bears strong resemblance to the model of magma transport (§3B). Excess pressure in the ice tends to close the channel, and this is balanced by melt-back of the channel wall.

Hydrothermal convection is the convection of meteoric water in the earth's crust driven by heating from below (typically from the cooling of a magma chamber). There are at least two interesting phenomena which occur. One is the oscillatory behaviour of hot springs or geysers at the earth's surface, which involves two-phase porous flow via the boiling of convecting groundwater (Norton [105]). An appropriate model of the matrix is as a fractured, permeable medium, and the fracturing may be due to the penetration of the water itself (Lister [106,107]). Such fracturing resembles Emerman et al.'s [61] magma fracturing process upside down, and Norton's description is suggestive of what may be appropriate for magma transport deep in the earth.

The other phenomenon is the existence of 'black smokers' and other hot water vents on the ocean floor, at mid-ocean ridges (RISE project group [108], Corliss et al. [109]). Large pressures prevent boiling, but chemical reactions between circulating oceanic water and the fractured crust cause remarkable black plumes to emerge at isolated vents on the ocean floor. The vents contain sulphides, which are precipitated out and form chimney-like enclosures at the vents. The existence of such isolated, upwelling vents is reminiscent of the freckling process in alloy solidification. Chemical reaction here plays a role analogous to partial melting.

As snow melts, water percolates downwards, to run off at the base. The simplest model relates water flux, via permeability, to water volume fraction (nonlinearly), so that (with gravity providing the draining force) a nonlinear first-order wave equation is relevant. Consequently, shocks (jumps) in water flux can occur, and this is of some interest if the snow is polluted. More general models would include air as well, and the overall description bears some relation to soil mechanics (Colbeck [51]).

The four situations we have discussed in §3, alloys (A), magma transport (B), frost heave (C), and igneous intrusions (D), share some common features, and it may be useful to point some of these out.

In approximating the dendritic zone of a solidifying alloy, we are led to

neglect solute diffusion ($D \rightarrow 0$). Although we lose a highest derivative term, the approximation is regular, as heat conduction is also essentially a $\nabla^2 c$ term. Furthermore, if the dendrite spacing does have anything to do with marginal stability, then it is feasible that the particle Peclet number dV/D be $O(1)$, and it is then virtually a requirement that the macroscale solute diffusion be neglected, if we are to self-consistently average over the microscale. Previously, we have only briefly mentioned Taylor dispersion (Bear [31]); this involves an enhanced (convective) 'diffusion' coefficient D_T if $dV/D \gg 1$. Then we have $D_T \sim dV$ (Bear [31]). The result is different from Taylor's [110] result since the pore flow is re-mixed at each pore intersection. But if $D_T \sim dV$, then the macroscale Peclet number is $\ell V/D_T$ (ℓ is the macroscopic length scale), which is $O(\ell/d) \gg 1$. Hence solute dispersion/diffusion can always be ignored.

In modelling dendritic growth, a degeneracy arises, in that one boundary condition appears to be missing. We have suggested, following Langer [22, 111], and Worster [13], that a condition of marginal supercooling might be appropriate, since Langer found that the condition of marginal stability gave good experimental results for the growth of dendrites in pure supercooled melts (cf. Trivedi [25]). The idea is that the degeneracy is real, but that the system preferentially evolves to its marginally stable state, just as the Fisher equation evolves to the marginally stable travelling wave solution (Murray [112]). It is completely open as to whether this is reasonable.

This condition is taken in the (approximate) form $\chi = 1$ at the mush/liquid interface, for (3.4). For the problems of magma transport and frost heave, we need precisely analogous conditions: $\chi = 0$ at the base of the partially molten zone, $I = 0$ at the base of the frozen fringe. For the one-dimensional magma problem, $\chi = 0$ is apparently required at the base of the zone to permit melting to occur (Turcotte and Ahern [48]). Apparently, this argument does not work for a freezing alloy; neither does it for frost heave. Elucidation of this boundary condition is of importance, both in itself, and because if marginal stability is relevant, then it is reasonable to expect similar considerations to determine dendritic spacing, which in turn determines the permeability (Nasser-Rafi et al. [113], Young and Kirkwood [114]).

A common feature to all these problems is the prescription of different pressures in the solid and liquid. At its simplest, this is forced by the expansion on change of phase. In frozen (or indeed unfrozen) soils, the effective pressure is an equilibrium concept, determined (in principle) by capillary and adsorption forces. Nevertheless, heaving soils grow ice lenses as a result of dynamic fracture of the soil, and freezing colloidal clays fracture as they freeze (Miller [66]). Fracturing due to contraction is manifested in the solidification of alloys as 'hot tearing' (Flemings [19]), and in magma chambers as pegmatite (and other) veins (Norton et al. [14]). Fracturing is also of importance in hydrothermal circulation, in glaciers, and, we hypothesise, in magma generation. In these cases, the effective pressure must be constituted dynamically. It is also of some importance as to whether the medium is open or closed. Expansion or contraction on freezing or melting can be accommodated, if liquid has free access, i.e. if the solid grains are free to move apart. This is generally not the case, at least when the solid fraction is large. The example of heaving does indicate one important modification which needs to be included in the magma transport problem. The earth's surface will swell in regions of magma generation, in order to conserve volume (or mass). Thus the lithostatic pressure p_s will be increased, and in two dimensions, this will

give rise to an extra normal stress as well. Potentially this has some ramifications for swelling above magma chambers (e.g. the Phlegraean Fields).

'Solutes' are an essential component of the problem of alloy solidification. Yet, we neglect their effect altogether in frost heave and magma transport. Arguably, this neglect is qualitative, insofar as pressure plays the role of concentration in these two problems. But it is likely that this effect is important in soil. Furthermore, the distinction between primary and secondary heave is reminiscent of planar/cellular alloy growth, in which case the small diffusivity of solutes may realistically be an essential ingredient for obtaining secondary heave. In the mantle, it is likely that pressure plays the primary role, though transport of different constituents will enter the problem in a complicated way, and it is certainly premature to make any authoritative statements about magma compositions near the earth's surface.

Mathematical modelling of these multi-component flows is an important constituent of the process of understanding the phenomena involved, and should serve to guide other experimental work. It is worth emphasising that these models, if complicated, are nevertheless the simplest such models which can claim to represent the basic physics involved. Their value lies in the fact that they can provide understanding, even though many of the phenomena are inaccessible to inspection, and can only be observed after the event. But it should also be emphasised that most of these models are as yet deficient in some way - boundary conditions are missing, or are not adequately justified, etc. One must therefore be wary: it does not do to validate a complex, but incompletely understood, model with laboratory experiments, if the model is then to be applied to phenomena occurring in a completely different context.

§5. CONCLUSIONS

This paper has attempted to summarise how mathematical models may be constructed of two-phase mushy zones, particularly in the context of solidifying alloys, magma generation and crystallisation, and freezing groundwater. Other situations whose modelling involves the same basic principles are hydrothermal convection, snow melting, subglacial hydrology, freezing of the earth's core, and (for fast flows) volcanic eruptions and steam-water reactor cooling flows (also in geysers).

In comparing these different situations, various common features emerge, which are discussed more fully in the preceding section. Since many of these topics relate to other presentations at this conference, we may here relate the emergent points to the contributions of other authors (for reference, see the Abstracts volume, available from D. Loper).

The main mathematical conundrum common to all these models is the necessary prescription of an extra boundary condition for solid (or liquid) fraction. For the magma generation problem, this can be derived from a force balance; for freezing problems, the same argument can be used to derive (3.42). However, alloy solidification seems to have no equivalent feature. Compare the model of Worster, and the problem of single dendrite growth, extensively discussed during the conference (compare R.A. Brown, R.F. Sekerka, J-J Xu).

Averaging approaches to two-phase flows typically assume local microscopic homogeneity (as, for instance, that solid and liquid temperatures are equal) on the basis that the corresponding relaxation processes, being related to the micro-scale, are rapid. However, a common feature of the models discussed here is that liquid and solid pressures are unequal. Associated with this is the phenomena of fracture, which is ubiquitous in these problems (crevasses, hot tearing, pegmatite veins, dike and sill propagation,

ice lens formation). Contributions of Stevenson, Scott, Ribe, discuss magma migration without reference to the generation of fractures. Compaction of crystal piles was discussed by Shirley (using McKenzie's (1984) bulk viscosity model), but the physics may be very different; solidification shrinkage is likely to drive intercumulus fluid flow, as discussed by Petersen, until the solid fraction is large enough to constrain the location of the liquid, due to interfacial effects.

Emerging from the comparison of different models and processes are various analogies, which may be suggestive and useful: the possibility that primary and secondary frost heave are akin to planar and dendritic solidification, and the consequent important role of solutes in soil water; the analogy between subglacial water flow and magma generation, which helps support the notion of a microscopic pressure differential between liquid and solid; the similarity between 'big-bang' nucleation in alloys and cumulus crystal growth in magma chambers, and the consequent importance of heterogeneous nucleation and non-equilibrium crystal growth. Some apparent analogies remain mysterious, for example, the possible connection between solute banding (e.g. in Czochralski growth), oscillatory zoning in plagioclase (Allegre et al. [115]), rhythmic layering, Liesegang rings, as well as the layers which occur in double diffusive convection. It is to be hoped that realistic mathematical models will be useful in explaining why some of these phenomena occur, and when.

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