

A compaction model for melt transport in the Earth's asthenosphere. Part I: the basic model

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Abstract

At mid-ocean ridges and hot spots, magma is created by the partial melting of hot ascending mantle rocks as they adiabatically decompress. The study of the dynamics of the transport of melt through the resulting partially molten rocks has led several authors to pose models for the concurrent melting, melt transport by buoyancy, and compaction of the solid residue. While these several models are of the same form, there is some discrepancy concerning the precise underlying physical basis. In addition, several studies of these models make further simplifying assumptions, the merits of which need to be evaluated. The purpose of this chapter, and of its sequel, is to re-present the basic model equations, together with a critical discussion of these different points of view. In particular, we develop a model for magma transport in partially molten rocks based on conservation of mass and momentum for both solid and liquid phases. This two-phase model involves different pressures in the solid matrix and liquid melt, and a compaction relation between the two is proposed. The formal relation of the compaction rate to a bulk viscosity is pointed out, and its physical significance as regards thermodynamic equilibrium is discussed. Possible realistic prescriptions for grain viscosity and permeability are given, and the importance of large-scale circulation is emphasized. Dynamic and thermodynamic boundary conditions for the equations are posed, and, in particular, the importance of a condition of 'macroscopic equilibrium' is emphasized.

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Notation*		Units			
				η	shear viscosity of matrix (10) Pa s
				η_m	mantle viscosity scale (15) Pa s
				η_b	compaction viscosity coefficient (14) Pa s
C	interphase drag coefficient (6)	$\text{kg m}^{-3} \text{s}^{-1}$		κ	permeability coefficient (20) $\text{m}^3 \text{s kg}^{-1}$
E	viscous dissipation in the melt (18)	Pa s^{-1}		μ_l	liquid viscosity (7) Pa s
G	Gibbs free energy (36)	J kg^{-1}		ρ^l, ρ^s, ρ	liquid, solid, mixture densities (volume-averaged) (1) kg m^{-3}
I	interphase pressure work rate (18)	Pa s^{-1}		ρ_s, ρ_l	solid and liquid densities (phase-averaged) (5) kg m^{-3}
L	latent heat (18)	J kg^{-1}		σ^l, σ^s	liquid and solid stress tensors (4) Pa
M	interphase drag (4)	Pa m^{-1}		τ_{ij}	matrix deviatoric stress tensor (13) Pa
S	melting rate (1)	s^{-1}		τ_m	mantle deviatoric stress scale (15) Pa
S_l, S_s	liquid and solid entropy (36)	$\text{J kg}^{-1} \text{K}^{-1}$		χ	liquid mass fraction (2)
T	temperature (16)	K			
T_0	reference temperature (16)	K			
W	matrix viscous dissipation (18)	Pa s^{-1}			
X	tortuosity coefficient (20)				
a	grain diameter (17)	m			
b	tortuosity (17)				
c_p	specific heat at constant pressure (18)	$\text{J kg}^{-1} \text{K}^{-1}$			
g	gravity (vector) (4)	m s^{-2}			
g	gravity (scalar) (7)	m s^{-2}			
h_l, h_s	liquid and solid enthalpies (36)	J kg^{-1}			
\bar{k}	permeability (7)	m^2			
k	thermal conductivity (18)	$\text{W m}^{-1} \text{K}^{-1}$			
n	creep exponent (15)				
p_s	solid pressure (11)	Pa			
p_l	liquid pressure (6)	Pa			
r	shrinkage ratio (20)				
t	time (1)	s			
u^l, u^s	liquid and solid velocities (1)	m s^{-1}			
u	mixture (barycentric) velocity (3)	m s^{-1}			
y	vertical coordinate (7)	m			
Γ	slope of Clapeyron curve (16)	K Pa^{-1}			
α	pore volume fraction (14)				
β	thermal expansion coefficient (18)	K^{-1}			
γ	surface energy	Pa m			
δ_{ij}	Kronecker delta (= 1 if $i = j$; = 0 if $i \neq j$) (10)				
ζ	bulk or compaction viscosity (10)	Pa s			

* The equation number is given in parentheses.

1. Introduction

Frank (1968) first introduced the idea that magma in partially molten regions of the Earth could flow as in a porous medium. This idea was developed by Sleep (1974), who realized that the resulting problem was fundamentally one of two-phase flow, involving the interacting motions of solid rock and interstitial melt. Significant advances were made by Turcotte and Ahern (1978) and Ahern and Turcotte (1979), who were able to provide simple but effective solutions for the description of melting in upwelling zones.

The basic dynamical phenomenon is that when melt is created, it will tend to rise through its own buoyancy. There is thus a 'melt production rate', and this melt presumably is that which is ultimately involved in volcanism at the Earth's surface. Both the observed quantity and geochemistry of these erupted melts have an important bearing on processes in the deep Earth, and thus the dynamics of their transport from the asthenosphere is of considerable importance. Most studies to date have concentrated on the physics of this process, reserving the chemical aspects for future study.

All the models are posited on the principle that interstitial melt in a partially molten granular rock will exist at three-grain intersections (tubules) that will ultimately interconnect to form a porous

three-dimensional network through the rock. The thermodynamic reasons for this are explained by McKenzie (1984), and are consistent with the experimental measurements of Cooper and Kohlstedt (1986), and Vaughan and Kohlstedt (1982).

Providing this idea is accepted, then the basic physics of partial melting in a moving porous medium is represented by the model of Turcotte and Ahern (1978). The novel feature that has been introduced more recently is to take explicit account of the effect of *compaction* of the solid (residue) matrix. This will be discussed in more detail later, but essentially, it means the squeezing of the crystalline matrix as the melt is simultaneously evacuated. Turcotte and Ahern (1978) have in fact considered this compaction in passing, but note only that its effect will be noticeable over a length of some 100 m, and they then ignore it. More recently, other authors have arrived at similar results (e.g., Ribe 1985a) when compaction is explicitly included.

Discussion of the compactive process is, to date, entirely theoretical. Despite this, at least three independent investigations (Fowler, 1984, 1985; McKenzie, 1984; Scott and Stevenson, 1984, 1986) have deduced—for rather different reasons—the same constitutive law describing compaction. In view of the theoretical nature of its derivation, and because of its possible application in other areas (e.g., sedimentary basins and magma chambers), it is important that the underlying physical process be described clearly. This chapter aims to elucidate some features of this description.

2. Mathematical Model of Magma Transport

2.1. Mass conservation

Equations for partially molten zones have been discussed exhaustively by many authors, and it is not our purpose to repeat all this description here. Sleep (1974) and McKenzie (1984) pointed out that the proper context for a complete model is that of two-phase flow, such as is commonly encountered in the nuclear industry (Drew, 1983). It is straightforward to pose equations for such a medium, particularly when the inertial terms are negligible, as is the case here. For example, the

conservation of liquid and solid mass may be written in the mixture theory form

$$\frac{\partial \rho^l}{\partial t} + \text{div}[\rho^l u^l] = \rho S \quad (1a)$$

$$\frac{\partial \rho^s}{\partial t} + \text{div}[\rho^s u^s] = -\rho S \quad (1b)$$

where ρ^l , ρ^s and $\rho = \rho^l + \rho^s$ are, respectively, the *phase-averaged* densities of liquid, solid and mixture, and u^l and u^s are their corresponding phase-averaged velocities. The term ρS represents the rate of melting (mass per unit volume per unit time), and it is determined from the energy equation. We define

$$\rho^l = \rho \chi, \quad \rho^s = \rho(1 - \chi) \quad (2)$$

so that χ is the liquid mass fraction, and the barycentric velocity, u , is defined by

$$\rho u = \rho^l u^l + \rho^s u^s \quad (3)$$

2.2. Momentum conservation

The general forms of the momentum equations for a slow two-phase flow (Ishii, 1975; Drew, 1983) may be written as

$$\nabla \cdot [\alpha \sigma^l] + \rho^l g = M \quad (4a)$$

$$\nabla \cdot [(1 - \alpha) \sigma^s] + \rho^s g = -M \quad (4b)$$

where g is gravity, and α is the liquid volume fraction. Thus

$$\alpha \rho_l = \rho^l, \quad (1 - \alpha) \rho_s = \rho^s \quad (5)$$

where ρ_l and ρ_s are the ordinary liquid and solid densities. Here σ^s and σ^l represent the phase-averaged solid and liquid stress tensors, and M is the interactive drag term. Equations (4a) and (4b) are simply (A7) and (A8) of McKenzie (1984). Since we anticipate that Darcy's law should apply to the liquid flow, it is reasonable to constitute the drag term M as

$$M = -\rho_l \nabla \alpha + C(u^l - u^s) \quad (6)$$

(McKenzie, 1984; Fowler, 1985), where ρ_l is the liquid pressure, which leads us to express equation (4a) in the form

$$\alpha(u^l - u^s) = -\frac{\tilde{k}}{\mu_l} \nabla [p_l + \rho_l g y] \quad (7)$$

where y is taken vertically upwards, \tilde{k} is the

permeability, μ_l is the liquid viscosity, and

$$C = \alpha^2 \mu_l / \bar{k} \quad (8)$$

Built into Darcy's law here is the assumption that the deviatoric part of the liquid stress is negligible, which seems evident. The derivation of the momentum equations in this way may seem overly pedantic, as an overall force balance together with Darcy's law will suffice (e.g., Scott, 1988). It is a matter of taste: the present discussion places the mathematical model for magma transport within the context of two-phase flow.

2.3. Differential pressures

As we move towards a description of compaction, the various derivations in the literature begin to deviate. The nature and significance of these deviations has not been discussed previously (but see Scott and Stevenson (1986, appendix A)).

As a preliminary step, let us note that in rapid two-phase flows, the prescription of $p_s = p_l$ (where p_s is the phase-averaged solid pressure) can lead to problems of ill-posedness. Furthermore, it is standard, for example in soil mechanics, that one has differential pressures, so that the effective pressure (that is, the overburden minus the interstitial pressure) is non-zero. Finally, in so far as a partially molten rock is a generalized porous medium, one expects to have to solve for p_l , whereas we can expect p_s to be approximately equal to the overburden pressure.

It is then reasonable to suppose that $p_l \neq p_s$, so that a constitutive relation for p_l and p_s needs to be posed. In the formulation of Fowler (1985) and Scott and Stevenson (1986), this constitutive relation is the compaction relation, and is determined by a local granular deformation model. That is, it is implicit in this description that, if $p_s \neq p_l$, then the individual grains are non-lithostatically stressed, and they respond to this state of stress by viscous deformation. If $p_s > p_l$, the pores will then tend to close up, for example. This physical description has important quantitative ramifications.

2.4. Total momentum equation and matrix rheology

Summing equations (4a) and (4b), we have

$$-\nabla[\alpha p_l] + \nabla \cdot [(1 - \alpha)\sigma^s] + \rho g = 0 \quad (9)$$

We now need to pose a constitutive equation for the matrix stress tensor.

McKenzie (1984) supposes $p_s = p_l$ and writes the general viscous relation as

$$\sigma_{ij}^s = -p_l \delta_{ij} + \left(\zeta - \frac{2}{3} \eta \right) \nabla \cdot u^s \delta_{ij} + \eta \left(\frac{\partial u_i^s}{\partial x_j} + \frac{\partial u_j^s}{\partial x_i} \right) \quad (10)$$

where η is a shear viscosity and ζ a bulk viscosity. Note that this is equivalent to

$$\sigma_{ij}^s = -p_s \delta_{ij} + \eta \left(\frac{\partial u_i^s}{\partial x_j} + \frac{\partial u_j^s}{\partial x_i} - \frac{2}{3} (\nabla \cdot u^s) \delta_{ij} \right) \quad (11)$$

if

$$p_s - p_l = -\zeta \operatorname{div} u^s \quad (12)$$

Scott and Stevenson (1986) prefer the combination of equations (11) and (12), and derive (from a local grain deformation model) an expression for ζ (see below).

An important point that can be made is that the prescription of a *bulk* viscosity in equation (10) has nothing to do with whether the medium is compressible or not. In fact, McKenzie's requirement of compressibility of the whole medium is apparently a statement concerning the mixture strain rate, and not the matrix strain rate. Compressibility of the matrix could only occur via grain deformation, since the grains are in interfacial contact at melt levels of a few per cent.

Actually, the idea of a bulk viscosity (Batchelor, 1967, p. 154) represents the notion that the instantaneous dynamic pressure may be different from the local thermodynamic equilibrium pressure. In view of this, it is noteworthy that the former (total) pressure is $\alpha p_l + (1 - \alpha)p_s$, whereas the local thermodynamic equilibrium pressure is p_l (via the Clapeyron curve in a non-isotropically stressed system (Fowler (1985), and see below), so that the prescription of a bulk viscosity would actually be of the form of equation (12), although which velocity should appear is not clear.

If we adopt this latter approach, then we derive a compaction relation, such as equation (12), independently of the matrix rheology, and the latter can be prescribed separately. With this viewpoint, *matrix* compressibility seems unlikely, and we would therefore propose that

$$\sigma_{ij}^s = -p_s \delta_{ij} + \tau_{ij} \quad (13a)$$

$$\tau_{ij} = \eta \left(\frac{\partial u_i^s}{\partial x_j} + \frac{\partial u_j^s}{\partial x_i} \right) \quad (13b)$$

where, however, the viscosity is likely to be reduced due to the state of granular stress (see below).

2.5. Compaction relation

As noted above, McKenzie (1984) and Scott (1988) proffer a compaction model based on bulk viscosity. These differ slightly in detail, but are essentially of the form of equation (10). Alternatively, Fowler (1985) and Scott and Stevenson (1986) propose a detailed mechanistic relation between p_s and p_l (see also Sleep (1988)). When the melt fraction χ is small, both these relations take the form of equation (12), where (at small melt fraction (χ)) $\zeta \propto \eta/\chi$. Since χ is expected to be small (several per cent), the precise form is inessential. However, it is quantitatively important that this microscopic model predicts that $\zeta \sim 1/\chi$, whereas McKenzie (1984) takes it as a constant. Here we follow the micromodel-derived result. Specifically, with the assumption $\chi \ll 1$, we take

$$p_s - p_l = -(\eta_b/\chi)\nabla \cdot \mathbf{u}^s \quad (14)$$

and η_b is proportional to the matrix viscosity.

The result in equation (14) is consistent with Batchelor's (1967) discussion (p. 253f), where he in fact derives the same result from the same physics. He observes that the 'observed macroscopic properties of a medium often become more intelligible if one can imagine a microscopic structure which would give rise to a given kind of macroscopic behaviour, and the devising of microscopic models is a common exercise in the science of rheology....'

2.6. Grain viscosity

A further distinction must now be made. We need to choose a value of η_b consistent with experimental knowledge. It is often thought that the shear viscosity in the mantle is determined by the process of dislocation creep (e.g., Kirby and Kronenberg, 1987), and in this case it is reasonable to suppose (if $|p_s - p_l| \geq \tau_m$) that

$$\eta_b = \eta_m [\tau_m / |p_s - p_l|]^{n-1} \quad (15a)$$

where $n = 3$ for the process of dislocation creep, and where τ_m is a typical mantle deviatoric stress (e.g., 10 bars) and η_m is a corresponding viscosity. A relation like equation (15a) has been deduced

by Sleep (1988) from a consideration of viscous dissipation due to compaction and regional shearing; he suggests (omitting various constants) that (equation (D6))

$$\eta_b \approx \eta_m [\tau_m^2 / \{\tau_m^2 + \chi(p_s - p_l)^2\}]^{(n-1)/2} \quad (15b)$$

which would be essentially equivalent to (15a) were it not for the factor χ . This difference may be due to Sleep's derivation of equation (15b) from a macroscopic point of view, while (15a) is explicitly based on the grain-scale variations in stress.

If diffusional creep is the controlling process, then $n = 1$ would be an appropriate value. This could be important for the shear viscosity of partially molten rock (Stocker and Ashby, 1973; Cooper and Kohlstedt, 1984). However, it is important to realize that the large-scale shear viscosity of the wet polycrystalline matrix could be controlled by processes other than the intra-granular deformation of single crystals that controls compaction. In particular, fluid-enhanced diffusional creep is unlikely to be relevant to the compaction process. Bearing this in mind, it is felt that equation (15a) is a reasonable best guess, in the absence of relevant data. Pressure solution creep (Cooper and Kohlstedt, 1984) also gives $n = 1$, but may be more applicable to compacting sediments, or in hot-pressing, where the pore geometry is more irregular.

It has been suggested (as with the 'bulk' viscosity in equation (14)) that the adoption of such a complicated relation as equation (15) is unwarranted in view of the lack of any experimental constraint, and that adoption of constant values for η_b and ζ would be just as worthwhile. This viewpoint has obvious merits, but since we wish to make quantitative statements about the solutions of the model presented here, we feel constrained to make the best guess at the constitutive parameters, in order to have some control over the physical meaning of the numbers we use.

As an instance, suppose that $\eta_m = 10^{20}$ poise, and imagine that $|p_s - p_l| \sim 100$ bars (this is in fact what we find to be reasonably typical in part II). If $\tau_m \sim 10$ bars, then $\eta_b \sim 10^{18}$ poise; in this case $\eta_b \ll \eta_m$, and the external mantle flow will perceive the partially molten zone as a buoyant, relatively inviscid 'bubble-like' domain.

Since the different approaches of three different sets of authors all lead (when $\chi \ll 1$) to the common recipe that is equivalent to equation (12),

one may now wonder why the differences matter. There are two reasons. Firstly, quantitative solution and approximation may be significantly different depending on how ζ is chosen, and our present discussion is aiming towards an opinion on this. Secondly, the nature of the derivation of equation (12) has serious implications for the thermodynamic boundary conditions. To date, these are discussed only by Fowler (1985, 1987, 1989) and imprecisely at that. Here, we aim to give a more complete description of appropriate thermodynamic boundary conditions (for a pure material). Their importance is attested to by the effect they have on the solutions, and a proper understanding of them is therefore necessary.

2.7. Thermodynamic 'equilibrium'

The discussion of the boundary conditions consequent upon the thermodynamic nature of the partial melt is deferred to section 3. Here we recall the important thermodynamic implications of having $p_s \neq p_l$. Thermodynamic equilibrium between the solid and liquid implies no heat flow, no momentum flow, and no mass flow. If a solid and liquid coexist with a common (grain boundary) interface, then heat, momentum and mass will be transported across the boundary in such a way as to reach equilibrium. The nature of our assumption about the partially molten medium is that it responds virtually instantaneously to a heat imbalance, so that the temperature is locally constant, but the momentum flux is not. That is, the interfacial temperature is at local equilibrium, while the stress field is not. This assumption underlies the inclusion of compaction in a large-scale model: otherwise, we should adopt Turcotte and Ahern's (1978) assumption $p_s = p_l$. For a non-hydrostatically stressed interface, the appropriate Clapeyron relation is that $T = T_0 + \Gamma(-\sigma_n)$ (Kamb, 1961), where σ_n is the normal stress. Since the liquid supports no significant deviatoric stress, we therefore pose the *local thermodynamic equilibrium constraint* that

$$T = T_0 + \Gamma p_l \quad (16)$$

where Γ is the slope of the Clapeyron curve. We emphasize that the local equilibrium is only *at* the grain/pore interface. The grains as a whole are not in equilibrium. Whether equation (16) does indeed apply depends on how much superheating

can be sustained. Melting rates in the presence of superheating are extremely unlikely to be greater than corresponding crystal growth rates. The lowest values quoted by Dowty (1980) for supercooled growth are $\sim 10^{-12} \text{ m s}^{-1} \text{ K}^{-1}$. A pore of dimension 10^{-4} m (melt fraction 1%, grain size 1 mm) subject to a superheat of $\Delta \text{ K}$ would thus be expected to melt in a time $\ll (1/\Delta) \text{ y}$. Significant values of Δ are those comparable to the variation of Γp_l in equation (16). With $\Gamma \sim 10^{-2} \text{ K bar}^{-1}$, and a hydrostatic pressure drop of $O(100 \text{ bar})$ (corresponding to a partial melt depth of $O(1 \text{ km})$), then Δ is significant if $\Delta \sim 1 \text{ K}$, but the associated relaxation time ($\ll 1$ year) is then much less than a relevant dynamic time-scale, e.g., the convective time-scale for transport across the partially molten zone. For example, if mantle rock rises at 1 m y^{-1} , then the convective time-scale is 10^3 y (for a thickness of 1 km). Thus equation (16) is expected to be a good approximation.

Apart from its effect on the equation, the adoption of equation (16) means that the thermodynamic boundary conditions are the boundary conditions for p_l .

2.8. Permeability

The choice of the permeability \bar{k} in Darcy's law (equation (7)) will quantitatively affect the results we obtain, particularly as \bar{k} depends on χ , and $\chi \ll 1$. If $\bar{k} \propto \chi^m$, then the precise form of m is crucial. Scott and Stevenson (1986) consider the form

$$\bar{k} = a^2 \chi^m / b \quad (17)$$

where a is grain diameter, and b is a numerical constant. Nye and Frank (1973) derive $m = 2$, and b varies from 100 to 1000 for an interconnected tubular network. McKenzie (1984) prefers $m = 3$ (the Kozeny-Carman relation for small χ). However, in keeping with the precise assumption about the location of melt at three-grain intersections, which lies behind our assumption of interconnectedness, it seems $m = 2$ is to be preferred.

Ahern and Turcotte (1979) and Maaløe and Scheie (1982) suggest that \bar{k} should depend on $\chi - \chi_c$ with $\bar{k} = 0$ for $\chi < \chi_c$, if some critical melt fraction is necessary for percolation. The assumption on dihedral angles for a pure material negates this, but of course real mantle rocks are multi-component, so that one can expect that a perco-

lation threshold may exist in practice. Several per cent of melt may be necessary before percolation is possible (Nicolas, 1986; Toramaru and Fujii, 1986). This is one reason why the present model is too simple to apply conscientiously to realistic cases.

2.9. Circulation

Scott (1988) and Scott and Stevenson (1989) emphasize the importance of coupling the large-scale mantle flow (circulation, u^s) to the local melt percolation velocity u^l . This is done through advection of heat in the energy equation. Typically $u^l \gg u^s$ in Darcy's law, so that u^s uncouples, and of course u^s depends on u^l through the position of the partial melt boundary. This in turn depends on u^l , since the matrix viscosity will be different, due to equation (15).

While this view is meritorious, it is also problematical. The circulation is properly part of a much larger scale flow, and to some extent it may be kinematically prescribed away from the melt zone. However, buoyancy effects within the melt zone will be significant, and will be enhanced by a reduction of viscosity. Scott and Stevenson (1989) have shown in this case that significant local secondary circulation can develop within the melt zone, leading to a narrow concentration of melt upflow.

2.10. Basic equations; conservation of energy

The basic equations are the conservation of mass equations (1) with (2), momentum conservation in the form of (7) with (17) (and $m = 2$) and (9), matrix rheology (13), the compaction relation (14) with (15) (the latter form also needs to be used for the matrix viscosity), and the local thermodynamic equilibrium condition (16).

There remains conservation of energy. This has been discussed in some detail by McKenzie (1984, appendix A) and Fowler (1985), following standard two-phase flow treatments (Ishii, 1975). Fowler's form, which involves separate adiabatic terms, is

$$\rho LS + \rho c_p \frac{dT}{dt} - \beta T [\alpha \frac{dp_l}{dt} + (1 - \alpha) \frac{dp_s}{dt}] = k \nabla^2 T + W + E + I \quad (18)$$

where $d/dt_s = \partial/\partial t + u^s \cdot \nabla$, $d/dt_l = \partial/\partial t + u^l \cdot \nabla$, β

is the thermal expansion coefficient, k is the thermal conductivity, L is latent heat, c_p is specific heat at constant pressure. We suppose for simplicity that c_p , β , k are the same for each phase. The terms W , E , I represent viscous heating in the matrix, viscous heating in the melt, and a pressure work term occurring via melting. This last term is absent from McKenzie's formulation since $p_l = p_s$. The usual single phase form of equation (18) applies outside the melt.

These basic equations are summarized in equations 19, where some realistic simplifications are made, particularly that the melt fraction χ is small.

(i) Conservation of mass of the melt:

$$\chi_t + \nabla \cdot [\chi u^l] = S \quad (19a)$$

(ii) Darcy's law for the melt velocity:

$$u^l - u^s = -\chi \chi \nabla [p_l + \rho_l g y] \quad (19b)$$

(iii) The viscous compaction relation:

$$p_s - p_l = - \left(\frac{\eta_m}{\chi} \right) \left[\frac{\tau_m}{|p_s - p_l|} \right]^{n-1} \nabla \cdot u^s \quad (19c)$$

(iv) Conservation of energy:

$$\rho_s L S + \rho_s c_p \frac{dT}{dt} - \beta T \left[\frac{\rho_s}{\rho_l} \chi \left\{ \frac{\partial p_l}{\partial t} + u^l \cdot \nabla p_l \right\} + \left(\frac{1 - \chi}{1 + r\chi} \right) \left\{ \frac{\partial p_s}{\partial t} + u^s \cdot \nabla p_s \right\} \right] = k \nabla^2 T \quad (19d)$$

(v) The Clapeyron relation:

$$T = T_0 + \Gamma p_l \quad (19e)$$

In equation (19a-e), χ , X , r , d/dt are a permeability coefficient, a tortuosity coefficient, the shrinkage ratio, and the material derivative, respectively, and are given by

$$\chi = a^2 / \mu_l X, \quad X = b \rho_s / \rho_l \approx 10^2 - 10^3$$

$$r = (\rho_s - \rho_l) / \rho_l, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \{\chi u^l + (1 - \chi) u^s\} \cdot \nabla \quad (20)$$

The circulation equations for the deforming matrix are the following laws of mass and momentum conservation, together with the stress/strain rate relation:

$$\nabla \cdot u^s = -\nabla \cdot [\chi (u^l - u^s)] \quad (21a)$$

$$-\nabla p_s + \nabla \cdot \tau + \nabla \cdot [(\rho_s / \rho_l) \chi (p_s - p_l)] + \rho g = 0 \quad (21b)$$

$$\tau_{ij} = \eta \left(\frac{\partial u_j^i}{\partial x_j} + \frac{\partial u_i^j}{\partial x_i} \right) \quad (21c)$$

and where the matrix viscosity is likely to be of the order of

$$\eta \sim \eta_m [\tau_m / |p_s - p_l|]^{n-1} \quad (22)$$

provided $|p_s - p_l| \geq \tau_m$ (otherwise $\eta \sim \eta_m$, and similarly in equation (19c)). The matrix viscosity may in fact be less than equation (22), if other creep processes (such as fluid-assisted diffusional creep (Stocker and Ashby, 1973)) are more effective than intra-granular dislocation creep. The equations (19a-e) are those for the melt, while (21a-c) describe the matrix flow. They are coupled via the appearance of u^s in equation (19), so that if u^s is kinematically prescribed, the melt flow problem is uncoupled. They are also coupled via the boundary conditions to the matrix flow outside the melt zone, and these are considered in the following section.

3. Boundary Conditions: External Flow

The previous section summarizes equations that have been derived in one form or another by several previous authors (McKenzie, 1984; Fowler, 1985; Scott and Stevenson, 1986). While the model equations proposed here in section 2 are not therefore likely to cause any *extreme* contention, the same cannot be said about the associated boundary conditions. The reason is simply that very few authors have actually discussed the appropriate boundary conditions for the governing differential equations.

To understand why this is, one has to realize that, although there is broad agreement on the physics of the compaction process, and on the appropriate model equations to be solved (basically those in section 2), different authors have used widely varying *implementations* of this basic model. In order to avoid confusion, we will use 'the model' to refer to the equations presented above in section 2, but will discuss below results of the differing implementations of this model used by differing researchers.

Several authors have used the model to predict the amount of crust generated at mid-ocean ridges, hot spots and island arcs. These include McKenzie (1984, p. 731f), see also Spiegelman and McKenzie (1987), and Ribe and Smooke

(1987). In addition, the trace element chemistry of the oceanic crust has been considered by McKenzie (1984), see also Richter (1986) and McKenzie (1985a,b), and major element chemistry has been considered by Ribe (1985b). (Note that in the physical model of section 2, we have purposely excluded chemical effects.)

In order to discuss such surface observations, one must suppose that magma can penetrate the cold lithosphere (via dyke propagation), and this is the tacit assumption made in the papers above. However, it is by no means obvious that this will always occur: an alternative is, that the upwelling magma in a partially molten zone solidifies when it reaches the base of the lithosphere. In the face of this possibility, conclusions concerning crustal formation must remain tentative, if plausible. Moreover, the issue of refreezing *must* be addressed in solving our model, since the lithosphere base will provide a thermal boundary condition for the melt zone. If all the upwelling magma solidifies, then the latent-heat release in the Stefan condition is in fact what determines the location of the solidification boundary, which is part of the problem to be solved.

How then has this thermal boundary condition been ignored in studying the dynamics of magma transport? Largely, because major simplifications are made first. In particular, many authors ignore the process of melting, either by taking the melting rate S to be zero (McKenzie, 1984; Scott and Stevenson, 1984) or prescribed (Ribe, 1985a), and neglecting the energy equation. These studies are, in a sense, kinematic, and in particular, the temperature plays no role. However, there is no reason to make such an arbitrary assumption, and in fact melting is central to the dynamics. Other authors have included melting, recognizing that it is essentially produced by the adiabatic decompression of the matrix (Scott and Stevenson, 1989). Recognizing in addition that compaction and heat conduction are 'small', these authors are able to give an explicit description of the melting rate. Thus in this case also, the temperature disappears from the problem.

We lay emphasis on these and similar simplifications here, because the approximations involved are *singular*, and in part II, we show that their introduction causes serious alterations to the solutions of the problem. Therefore, it is necessary to consider the boundary conditions to the model in some detail.

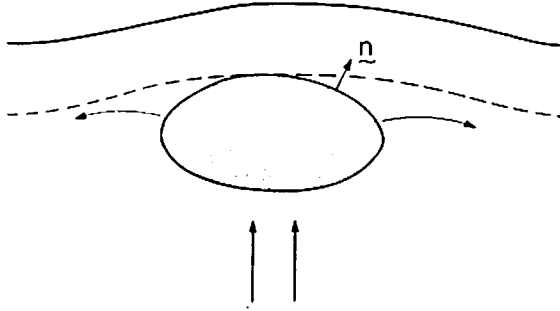


Figure 1. A partially molten zone underlying a lithospheric plate

For the moment, we ignore possible lithospheric fracture, and magma penetration of the lithosphere, and suppose that the partial melt forms an enclosed volume with unit outward normal \mathbf{n} , as shown in Figure 1. Let us re-emphasize that in this discussion, we consider only a chemically pure substance, for which refreezing can occur at a single boundary. Real magma freezes over a temperature range, and while it is important to include chemistry eventually, we feel that this may best be done by first fully understanding the purely physical problem. The boundary conditions that we seek to apply are those of mass conservation; the force (momentum) balance; surface energy balance (the Stefan condition); and two thermodynamic relations (the continuity of temperature and free energy).

Firstly, we write the equations governing the flow of the cold rock outside the partial melt volume, V . These are (with the Boussinesq approximation)

$$\nabla \cdot \mathbf{u} = 0 \quad (23a)$$

$$-\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g} = \mathbf{0} \quad (23b)$$

$$\tau_{ij} = \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \eta \sim \eta_m \quad (23c)$$

$$\rho c_p (T_t + \mathbf{u} \cdot \nabla T) - \beta T (\rho_t + \mathbf{u} \cdot \nabla \rho) = k \nabla^2 T \quad (23d)$$

where we have neglected viscous heating in (23d).

The prescription of boundary conditions at the boundary ∂V of V has been discussed by Fowler (1985). Mass conservation requires

$$[\rho(\mathbf{u} - V) \cdot \mathbf{n}]^\pm = 0 \quad (24)$$

where $+$, $-$ refer to either side of ∂V , and V is the velocity of the boundary. If $\chi \ll 1$, this may be

written as

$$u_n \approx u_n^\pm + \chi u_n^\pm + r \chi V_n \quad (25)$$

where n signifies the normal component. If $\chi V_n, \chi u_n^\pm \ll u_n$, then $u_n \approx u_n^\pm$.

The force balance at ∂V implies

$$[\sigma_{nn}]^\pm = [\sigma_{nn}]^\pm = 0 \quad (26)$$

where on the solid ($+$) side,

$$\sigma_{ij} = -p \delta_{ij} + \tau_{ij} \quad (27)$$

and on the partially molten side,

$$\sigma_{ij} = \alpha \sigma_{ij}^\pm + (1 - \alpha) \sigma_{ij}^\pm \quad (28a)$$

or

$$\sigma_{ij} = \frac{1}{(1 + r\chi)} \left[-\{(1 - \chi)p_s + (1 + r)\chi p_l\} \delta_{ij} + (1 - \chi)\tau_{ij} \right] \quad (28b)$$

if $\chi \ll 1$, then the normal component gives, approximately,

$$p^+ \approx p_s - (1 + r)\chi(p_s - p_l) \quad (29)$$

(ignoring deviatoric terms), and the tangential stress balance is

$$\tau_{nt}^+ \approx \tau_{nt}^- \quad (30)$$

3.1. The Stefan condition

The usual balance law for energy fluxes equates the jump in heat flux across a phase change boundary to the latent heat released or absorbed by the freezing or melting process. In the present context, this is slightly altered by the pressure work term across the subsolidus-partial-melt boundary. By appealing to the conservation form of the averaged energy equation (Fowler, 1985), this boundary condition can be written in the form

$$-\rho \chi (e_l - e_s) (V_n - u_n^\pm) = - \left[k \frac{\partial T}{\partial n} \right]^\pm - [\sigma_n \cdot \mathbf{u}]^\pm \quad (31)$$

where σ_n is the normal stress (vector) at the interface. The latent heat at a solid-liquid interface (at pressure p_l) is defined by

$$L = e_l - e_s + p_l \left(\frac{1}{\rho_l} - \frac{1}{\rho_s} \right) \quad (32)$$

(rather than equation (3.11) of Fowler (1985)). We have $[\sigma_n]^\pm = \mathbf{0}$; ignoring deviatoric stresses,

equation (31) is then

$$\rho\chi \left[L - p_l \left(\frac{1}{\rho_l} - \frac{1}{\rho_s} \right) \right] (V_n - u_n^l) = \left[k \frac{\partial T}{\partial n} \right]_+ + \sigma_{nm} [u_n]^\pm \quad (33)$$

Finally, equation (25) gives, ignoring terms of relative order χ ,

$$\rho\chi \left[L - p_l \left(\frac{1}{\rho_l} - \frac{1}{\rho_s} \right) \right] (V_n - u_n^l) = \left[k \frac{\partial T}{\partial n} \right]_+ - p^+ \chi (u_n^s + rV_n) \quad (34)$$

At the solid–partially-molten boundary, we require the temperature to be continuous (because the thermal diffusion length is much larger than the grain size):

$$[T]^\pm = 0 \quad (35)$$

and that the temperature be ‘the’ melting temperature (for a multicomponent rock there would be a sequence of boundaries where the different components melt or solidify). We have already proposed (in section 2) that the melting temperature in the partially molten zone is related to p_l via equation (16). However, we might expect that the melting temperature in the solid zone (where no melt is present) would be $T_0 + \Gamma p$, where p is pressure. Ignoring slight deviations due to the non-hydrostatic stress, we would thus require $p^+ = p_l$ at the boundary, or (approximately) $p_s \approx p_l$ (from equation (29)) (ignoring surface tension—see below). Despite this, Fowler (1987) was unable to satisfy $p_s = p_l$ at all points of the boundary in his solution.

We propose the following resolution to this conundrum. The definition of melting temperature (and the derivation of the Clapeyron relation) rests upon the assumption of thermodynamic equilibrium between the liquid and solid, at constant pressures. Equivalently, we wish to propose that the solid–partial-melt boundary is at a thermodynamic quasi-equilibrium, in the sense that the average free energy is continuous across this boundary. Prescription of the continuity of free energy is the condition from which one *derives* the Clapeyron relation, and is thus the appropriate one here. Moreover, it is not equivalent to the local (microscopic) condition (equation (16)), because it is a macroscopic condition applied to the averaged equations. In fact, while equation (16) results from assuming G is continuous across

the solid–liquid pore boundary, the supposition that $p_s \neq p_l$ ensures that the *averaged* solid and liquid free energies are *not* equal.

We prescribe $[G]^\pm = 0$ across the partial-melt–solid boundary, where G is the average Gibbs free energy. This is relevant to a situation at constant pressure, which, while not appropriate locally, *is* appropriate at the *global* level. We now mimic the derivation of the Clapeyron relation. Denoting the void fraction of melt as α , consider a ‘reference state’ in which $p_s = p_l$; then, in the partial melt,

$$G = \alpha(h_l - TS_l) + (1 - \alpha)(h_s - TS_s) \quad (36)$$

where suffixes, l, s denote liquid and solid, h is enthalpy and S is entropy. In the solid (since $p^+ = p_s$),

$$G = h_s - TS_s \quad (37)$$

therefore the jump in Gibbs free energy across the boundary is

$$\Delta G = \alpha[\Delta h - T\Delta S] = 0 \quad (38)$$

whence

$$\Delta S = \Delta h/T \quad (39)$$

just as in the usual way. To express the variation of the ‘melting temperature’ with p_l or p_s , suppose we change p_l and/or p_s in the partial melt zone by small amounts δp_l and δp_s . From equation (28), we have

$$p^+ = (1 - \alpha)p_s + \alpha p_l \quad (40)$$

and therefore the change in pressure on the sub-solidus side is

$$\delta p^+ = \delta p_s - \delta[\alpha(p_s - p_l)] \quad (41)$$

From equation (36), using the usual thermodynamic relations for δG , we have

$$\begin{aligned} \delta G_- &= \delta\alpha[\Delta h - T\Delta S] + [v_s\delta p_s - S_s\delta T] \\ &= v_s\delta p_s - S_s\delta T \end{aligned} \quad (42)$$

from equation (39), whereas

$$\delta G_+ = v_s\delta p^+ - S_s\delta T \quad (43)$$

It follows that, since we require $\delta G_+ = \delta G_-$, the variation of T at the boundary is not prescribed at all (it is determined by the local microscopic equilibrium, equation (16)) and equality of equations (30) and (43) implies, using (41), that

$$\delta[\alpha(p_s - p_l)] = 0 \quad (44)$$

Since $p_s = p_l$ in the reference state, we see that satisfaction of the condition of continuity of

free energy across the boundary requires

$$\chi(p_s - p_l) = 0 \quad (45)$$

This is exactly the condition found *necessary* by Fowler (1987) in order to solve the problem. In this way we can resolve the apparent paradox suggested by a more naive description of the thermodynamics. Notice that this boundary condition is satisfied by *either* $\chi = 0$ or $p_s = p_l$; we shall find the former to apply at the lower part of the boundary, and the latter at the upper. Notice, also, that if $\chi = 0$ is selected, then the heat flux is continuous (see equation (34)), whereas if $p_s = p_l$, then freezing must occur.

In the above treatment, we have neglected surface energy, which also affects 'the' melting temperature (the Gibbs-Thomson effect). The role of surface energy has been considered by Stevenson (1986); it effectively gives rise to an additional pressure in the liquid of magnitude $-\gamma/a\chi^{1/2}$, where a is grain diameter, and γ is surface energy. If we take $a = 10^{-3}$ m, $\gamma = 2 \times 10^{-7}$ bar m (Dowty, 1980), $\chi = 10^{-2}$, this is of the order of 10^{-3} bar. Thus surface tension is generally negligible. Of course, the term is singular as $\chi \rightarrow 0$, but this problem has more to do with the initial formation of melt, when large differential pressures caused by volume changes (as well as surface tension) may cause rapid transient creep (see Fowler, 1985, p. 73). In writing equation (45) as a *macroscopic* boundary condition, we are implicitly assuming that such processes are more rapid than the relaxation to a free energy balance. In reality the existence of a threshold porosity for connectivity will remove the possible singularity anyway.

4. Summary

This chapter has attempted to resolve certain differences between various previously defined models of magma transport. While the particular choice of physical laws used here will no doubt remain contentious, we hope to have drawn out some of the hidden physical assumptions on which the discrepancies are based. The importance of obtaining a consensus on what should be the 'right' model will be illustrated in the sequel, where we examine various applications on the basis of the model derived here.

For the moment, we emphasize the significant conclusions that may be drawn.

(i) If a porous network exists along three-grain intersections, then simple arguments suggest that the permeability is proportional to the square of the melt fraction, $k \propto \chi^2$, rather than the Kozeny-Carman relation $k \propto \chi^3$ (for small χ). For $\chi \sim 10^{-2}$, such a choice may have significant quantitative importance.

(ii) Whether compaction is conceptualized as being due to medium compressibility, or due to local differential stresses, leads to little difference in the equations, but has important consequences for the thermodynamics. Moreover, the ability of grains to maintain differential stresses over dynamic time-scales is fundamentally related to the small-scale geometry of the grains, and the choice of a viscous type compaction relation reflects the fact that the grain response must be that of slow creep. Some authors have used such viscous compaction rates in other situations where it is not similarly appropriate. A crystal pile in a magma chamber, for example, has a much larger void fraction, and the grains should form a more soil-like matrix. It then seems more reasonable that a consolidation curve should more appropriately represent the compaction process, rather than the viscous creep of the grains (cf. Sparks *et al.*, 1985).

(iii) If grains are stressed to any significant degree (we shall find levels of $\sim 10^2$ bar), then the non-linear stress dependence of viscosity implies that we must think of the partially molten zone as having a viscosity at least some 100 times lower than the surrounding cold rock. In this case, the partially molten zone behaves crudely as a bubble-like fluid inclusion in the mantle flow-field.

(iv) We have here posed two thermodynamic equilibrium conditions. One is microscopic: that the free energy is continuous at the microscopic solid-liquid interface. The other is that the *average* free energy is continuous across the *average* subsolidus-partial-melt interface. It is evident that this concept has potential applicability to other models of mushy zones, in particular to models of alloy solidification (Worster, 1986), which are typically missing a boundary condition. It does not seem to have been proposed in the two-phase flow literature before.

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