A computationally feasible reduction of the O'Neill-Miller model of secondary frost heave

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Following previous work by Holden and co-workers, we have carried out an asymptotic reduction for the O'Neill-Miller model of frost heave, which is a quantitatively accurate representation of the original model, but which only involves the solution of two first order ordinary differential equations. This reduction can be made on the basis that the frozen fringe is thin, conduction of heat dominates advection of heat, and (importantly) the permeability of the frozen fringe is strongly sensitive to the water content. Solutions of the reduced model give predictions of heave versus time and of time of formation and thickness of discrete ice lenses, which agree with O'Neill and Miller's numerical results. Furthermore, the results can be interpreted in accord with the idea that clays heave large loads, but slowly; silts heave small loads, but fast; while sands may not heave at all. An explicit heaving rate parameter can be determined, whose size depends on the soil type through its permeability, characteristic suction, and on the applied load. The method of reduction can be applied to generalised versions of the O'Neill/Miller model, and in particular we have carried out similar analyses in respect of compressible soils, saline soils, and we have also initiated work on the extension of the model to allow for differential frost heave.

1 BACKGROUND

Frost heaving occurs when moist soil, particularly clay or silt, is frozen from the surface downwards (fig. 1). It is manifested by an upward heaving of the surface, which is due not so much to the expansion of water on freezing as to an upwards suction of the subsurface groundwater towards the freezing front. This suction is thought to be due to various effects which contribute to capillary-like forces; these include surface tension between ice and water phases, a chemical potential due to an electrical double layer on clay particles, and disjoining pressure effects due to short range forces in thin water films. A general description of frost heaving is given in the article by Miller (1980).

As the water is drawn upwards to the freezing front, it forms a series of discrete ice lenses in the soil. It is the formation and growth of these ice lenses which causes the heaving at the surface, which can be substantial. Frost heave is a natural phenomenon which occurs in most northerly regions. It is responsible for widespread damage to roads and pavements (even in England, where the construction of the M1 in 1959 was interrupted by heave of the road surface), since in the subsequent thawing, the melting of the ice lenses severely weakens the soil structure (thus facilitating the development of potholes in roads, for example).

Frost heave is also associated with various kinds of *patterned ground formation* in permafrost regions, such as sorted stone circles and soil hummocks, although the precise relationship is not clear. Descriptions of patterned ground are given by Washburn (1980) and Williams and Smith (1989).

The early development of observational, experimental and empirical accounts of frost heave is due to Taber (1929, 1930) and Beskow (1935). In the post-war period, there was a substantial development of physico-chemical theories, and a central figure in this development is R.D. Miller who, in collaboration with numerous colleagues and students (Koopmans and Miller 1966, Miller 1972, Romkens and Miller 1973, Snyder and Miller 1985), developed a consistent account of frost heave based on a number of different physical concepts. This account is outlined below. Reviews of this and other work have been given by O'Neill (1983) and Black (1991).

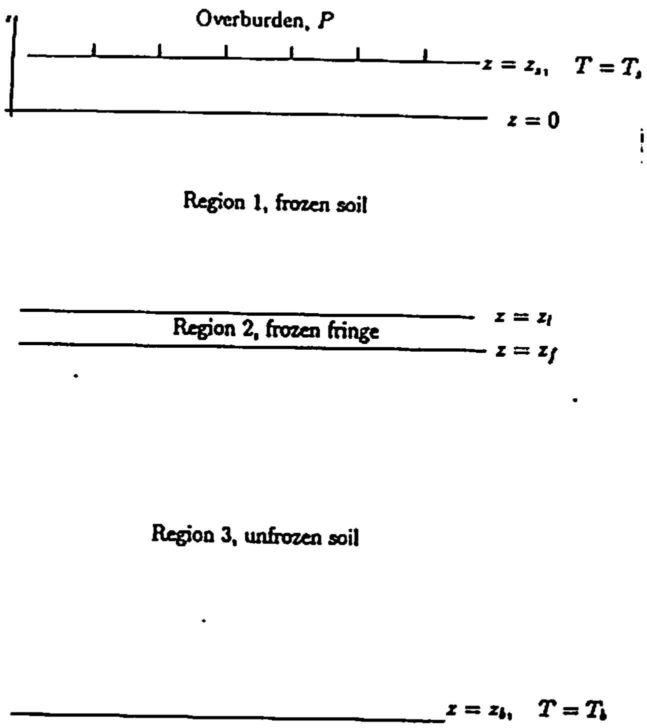

Figure 1: Schematic picture of a heaving soil

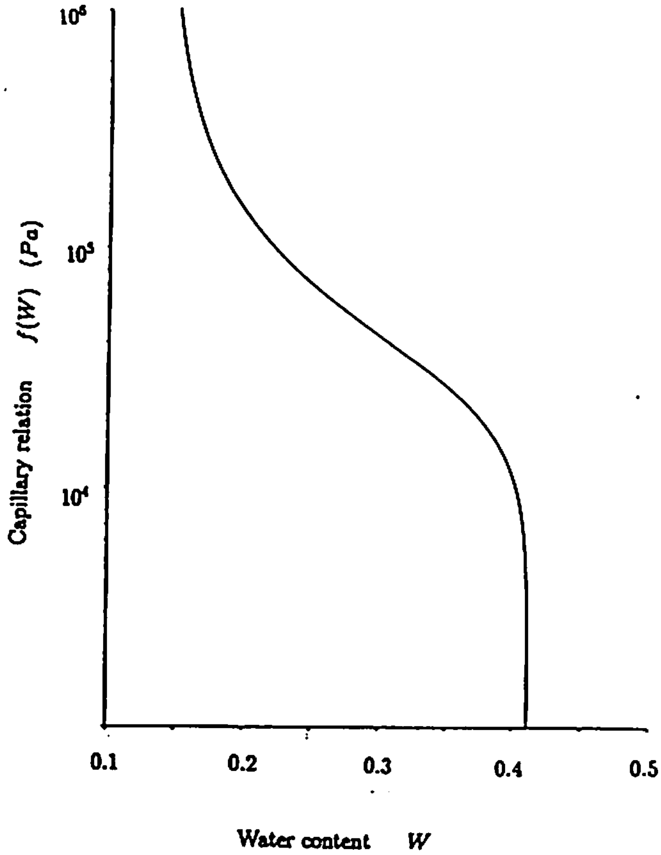

Figure 2: Capillary relation $f(W)$

2 THE MILLER MODEL

Miller (1972) distinguishes between primary heave and secondary heave. In the latter, a thin partially frozen fringe of ice exists below the lowest ice lens. This region of soil contains both pore ice and pore water, and is considered to be in thermodynamic equilibrium. Moreover, the freezing temperature T_f within the fringe is given by a generalisation of the Clapeyron relation, such that T_f is related to the pore pressure exerted by each of the ice and water phases; this relation can be theoretically derived (Loch 1978).

In allowing separate ice and water pressures p_i and p_w, a constitutive relation between them must be prescribed. Miller (1980) takes this in the form $p_i - p_w = f(W)$, where W is the volumetric water fraction in the fringe (fig. 2). As mentioned, this can be ascribed to surface tension as well as other effects, although the physical interpretation of what this means for ice-water re-distribution is less clear. In fact, Miller's 'rigid ice' concept assumes that the pore ice is rigidly connected to the lowest ice lens, and in order for heave to take place due to water freezing on to the lowest lens, this requires the rigid ice to move past the equally 'rigid' (i.e. undeformable, in the model) soil particles.

This is accomplished by the process of thermal regelation, which causes soil particles to move through ice due to an imposed temperature gradient by a melting-refreezing process. Again, this phenomenon has been experimentally observed (Romkens and Miller 1980).

The last feature of Miller's model is a criterion for ice lens formation. This is considered analogous to the formation of cracks in unsaturated soils, and is due to the effective pressure transmitted between soil grains becoming zero. At this point, the soil loses its integrity, and water can push in to form a lens. It is found that lens formation occurs *within* the fringe, and thus the model can explain the formation of a sequence of discrete lenses.

These four concepts (generalised Clapeyron equation, capillary suction, thermal regelation, lens initiation) are central to the Miller model, and are not without their detractors. However, they form the basis for the most conceptually advanced frost heave model, and provide a coherent basis for the study of frost heave.

3 MATHEMATICAL FORMULATION

The Miller model was implemented mathematically by O'Neill and Milller (1982,1985). They treat the frozen, unfrozen and partially frozen fringe as three separate continua, and formulate conservation laws of heat and mass transport in each region. Because of the fine spatial and temporal scale of the model, it is, as Black (1991) says, "hardly ready for use on practical problems. Computational difficulties are unusually formidable ...". Both for this reason, and also because, notwithstanding its complexity, the Miller model is conceptually too *simple* (it considers soil to be rigid and saturated, pore water to be chemically pure, and is essentially limited to one-dimensional heaving), O'Neill and Miller's formulation needs simplification. The way to do this lies in making judicious approximations, which simplify the model without surrendering accuracy. Initial work by Holden (1983), Holden *et al.* (1985) and Piper *et al.* (1988) has been extended by Fowler (1989) and more recently Fowler and Krantz (1993), using four basic approximations: heat transport by advection is small, gravitational effects are small, the frozen fringe is thin (due to the fact that the freezing temperature variation in the Clapeyron relation is small), and the permeability variation with water fraction is large (specifically $k \propto W^{\gamma}$, with γ (≈ 9) (O'Neill and Miller 1985) being large).

These four approximations allow one to reduce the calculation of frost heave in the O'Neill/Miller model to that of solving two simple first order differential equations for the freezing front position and the soil surface elevation. We now detail the effect of these simplifications.

Firstly, the fringe is thin compared to the length of the soil column, and is consequently seen as a surface by the rest of the soil. Secondly, convective transport of heat is small compared with the conductive flux (i.e. the Peclet number, $Pe \ll 1$). The energy equations outside the fringe then reduce to

$$\nabla^2 T = 0 \tag{1}$$

in $z_s > z > z_f$ and $z_f > z > z_b$ with boundary conditions $T = T_s$ on $z = z_s$, $T = T_0$ on $z = z_f$ and $T = T_b$ on $z = z_b$. The solution to this equation in one dimension is then

$$T = T_s + \frac{z_s - z}{z_s - z_f}(T_0 - T_s) \tag{2}$$

in $z_s > z > z_f$ and

$$T = T_0 + \frac{z_f - z}{z_f - z_b}(T_b - T_0) \tag{3}$$

in $z_f > z > z_s$.

Using these expressions to evaluate the temperature gradient on each side of the fringe, the fringe equations may be solved to determine the locations of the moving boundaries z_s and z_f. This can be done using the fact that the effects of gravity are negligible, and the exponent in the permeability function is large. This fact gives rise to a boundary layer below the lowest ice lens in which the water pressure changes rapidly. Asymptotic methods applied to Darcy's equation lead to a uniformly valid expression for the water pressure as a function of water content. On the formation of a new lens, water content within the boundary layer relaxes rapidly to a steady state and so the water pressure can be considered to be quasi-static.

The remaining differential equations in the fringe express conservation of mass and energy. The small Peclet number and thin fringe allow these equations to be written as a set of algebraic equations for variables evaluated at each side of the fringe. Using the forms found for the temperature and pressure profiles and also the remaining fringe relations, it is possible to reduce these algebraic equations to the two (dimensional) ordinary differential equations

$$\dot{z}_f = -\frac{A}{z_s - z_f} + \frac{B}{z_f - z_b}, \tag{4}$$

$$\dot{z}_s = \alpha\left(\frac{\nu}{z_s - z_f} + W_l \dot{z}_f\right), \tag{5}$$

where A, B, ν and α are constants determined in terms of the parameters of the problem by complicated formulae (see Fowler and Krantz (1993) for details).

4 RESULTS

It is simple to solve the equations (4) and (5) numerically (they can in fact also be solved analytically), and fig. 3 shows a typical sequence of results (Fowler and Noon 1993) for a range of overburden pressures. This sequence compares favourably with a similar figure of O'Neill and Miller's (1985). It is also possible to calculate lens spacing and thickness, and these also vary quantitatively as one might expect.

The form of equation (5) identifies a dimensionless heaving parameter α. This depends in a very complicated way on the applied overburden effective pressure N and the soil type, principally through the

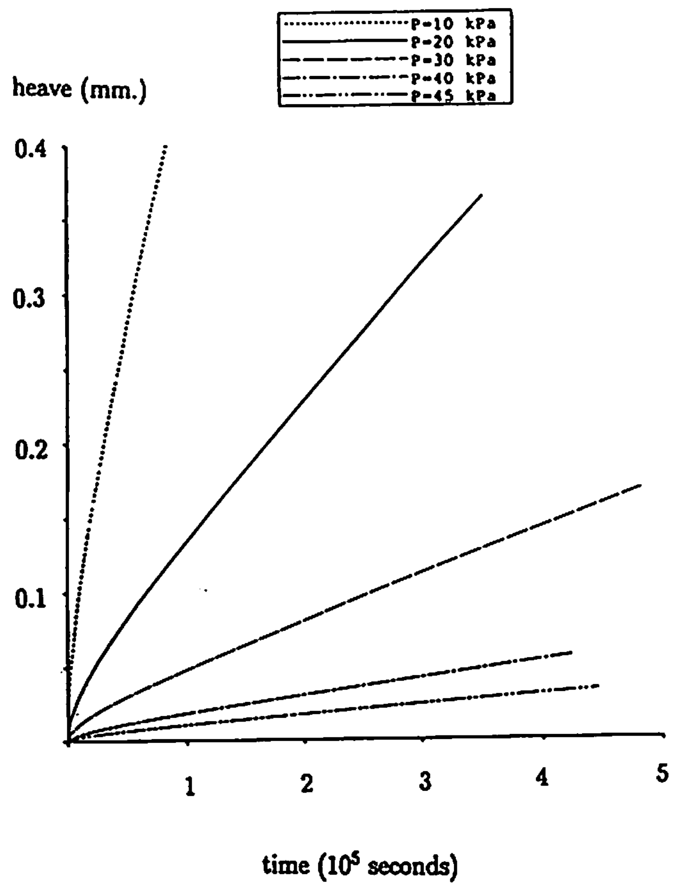

Figure 3: Heave against time for different overburdens

permeability and the characteristic function $f(W)$ in fig. 2. α is proportional to (saturated) permeability, which explains (obviously) why clays should heave more slowly than silts.

The dependence of α on load N is more complicated. Firstly, since N is an effective pressure, zero load corresponds to a positive value of $N = N_0 = p_a - p_\infty$, where p_a is atmospheric pressure, and p_∞ is the far field groundwater pressure. In fig. 4, we plot the variation of α with N for values $N \sim O(1)$ bar, for a characteristic relation $f \propto (1-S)^p/S^q$, where S is the saturation, and $p = 0.3$, $q = 0.3$. This choice of p and q gives a steeply increasing suction as S is reduced, and may be appropriate for finer soils. Depending on the size of N_0, we see that heave decreases more or less exponentially with increasing load, but is maintained for arbitrarily large load. Fig. 5, on the other hand, shows α versus N for $p = 0.3$, $q = 0.1$, the lower value corresponding to lower suctions at low S, and thus a coarser soil. Here there is an asymptote at a finite value of $N = N_c$, beyond which heaving in this mode does not occur, since no lens formation is possible. On the basis of fig. 5, it is possible to understand the absence of heave in sands, on the basis that $N_0 > N_c$, while if $N_0 < N_c$, then one expects heave to occur, but not for arbitrarily large loads. These observations are qualitatively similar to reported heaving characteristics, but further quantitative validatory work is necessary.

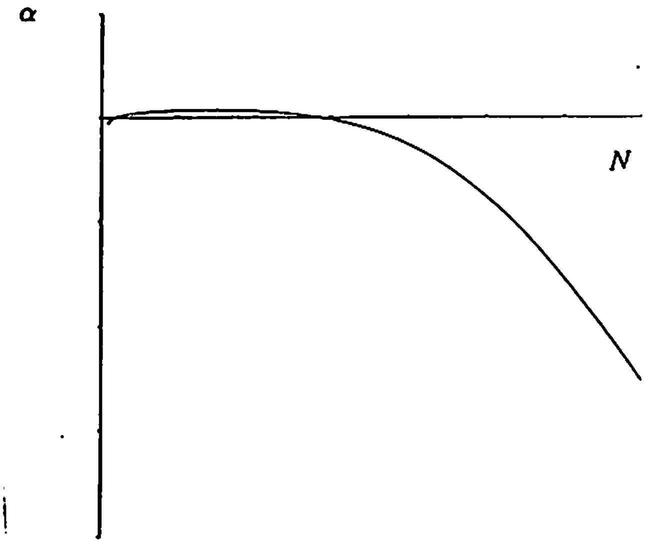

Figure 4: Heave parameter α versus effective overburden N (log-linear scale), clay-type soil

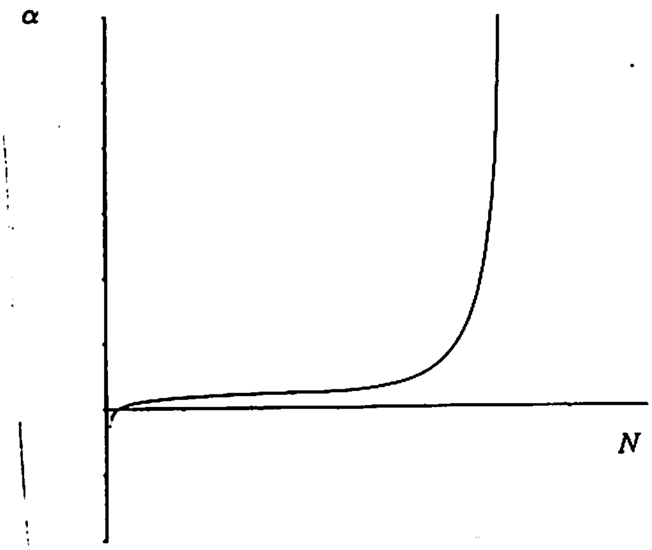

Figure 5: Heave parameter α versus effective overburden N (log-linear scale), sand/silt soil

5 CONCLUSIONS

O'Neill and Miller's model was grid locked because its extraordinary numerical complexity did not allow the possibility of making any realistic extensions to it. In particular, it is conceptually wrong, insofar as the assumption that the ice lens velocity v_i is spatially constant is (generally) inadmissible for a three-dimensional process. Our previous work not only renders the model tractable, it allows the possibility of extensions by adding new physics to the simplified model, and thus obtaining generalisations of this simple model. As an example, frost heave of saline soils leads to a similar reduction as for pure pore water, the difference lying in that the coefficients in the resulting equations are more complicated.

The application of the model simplifications outlined here to saline and compressible soils is outlined in the thesis by Noon (1993), and further developments to include unsaturated soils, as well as three-dimensional modelling of differential frost heave, are in progress.

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