Toward a Description of Convection with Temperature-and-Pressure-Dependent Viscosity

By *A. C. Fowler*

In seeking a description of convection in the Earth's mantle, one is led to the study of convection in a fluid at large Rayleigh number, and with large activation energy and volume for viscosity. This paper presents an initial attempt to understand the asymptotic behavior in the combined triple asymptotic limit where these parameters are all large. The model is a simplified set of mean field equations. We show that it mimics successfully both constant and temperature-dependent viscosity. We then show that temperature-and-pressure-dependent viscosity gives an internal isoviscous structure. There are tempting inferences that might be suggested for the convective style in the earth's mantle.

1. Introduction

Convection in the earth's mantle is a much studied, but little understood, subject. Such issues as to whether convection is singly or multiply layered are controversial and unresolved, and often fundamental facets of the flow, such as subduction at plate boundaries, go more or less unconsidered, despite their importance. Part of the reason for this lies with the failure of computational convection models to reproduce adequately realistic flow configurations.

The early concept of plate tectonics (Turcotte and Oxburgh [1]) rests on the idea that the flow is broadly akin to high Rayleigh number convection. When the Rayleigh number *Ra* is large (and the viscosity is constant), thermal boundary layers form at the boundaries of the circulation cell. Turcotte and Oxburgh analyzed the boundary layer structure and concluded

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that the results of their analysis were consistent with observed tectonic plate viscosities and oceanic heat flow measurements. Indeed, this theory was remarkably successful and forms the basis for much of current thinking about plate tectonics: the lithospheric plates are the thermal boundary layers of the convection cells, for example.

However, this picture already contains logical inconsistency in its interpretation for the earth's mantle. Tectonic 'plates' are thought to behave more or less rigidly (other than at subduction zones) exactly because they are the cold thermal boundary layers of convection cells. Specifically, it is well known (e.g., Kirby [2]) that the effective viscosity of mantle minerals such as olivine is an *extremely* strong function of temperature. Hence, the coolth of the lithosphere is consistent with its 'rigidity'; but equally, it is dramatically *inconsistent* with the classical assumption of a constant viscosity material. This would, of course, not matter too much if a variable viscosity made only a qualitative difference; but in fact it is worse than that. It is intuitively obvious that for a fluid of sufficiently variable (with temperature) viscosity, 'vigorous' convection at high Rayleigh number will take the form of a rapid isothermal circulation of relatively low viscosity warm fluid beneath a stagnant, cold viscous lid. That is, the effect of including a realistic dependence of viscosity on temperature is to *suppress* the lithosphere plate motion. This idea is confirmed by laboratory experiment (Nataf and Richter [3]) and both numerical and analytical solutions of the governing equations (Christensen [4], Fowler [5]).

We thus have a conundrum. Constant viscosity convection seems to represent adequately the inferred characteristics of mantle convection, but variable viscosity convection yields a scenario in which subduction does not occur. The cause of subduction in the earth is actually a fundamental feature of the flow mechanics which has not been addressed to any useful degree; one consequence of this is an inability of mantle convection theory to make any prediction at all about the style of Venusian tectonics.

Another long-standing 'observation' concerning the earth is that the sublithospheric mantle is inferred to have a *relatively* constant viscosity, of the order of 10^{22} poise (Cathles [6]). It has to be said that the study of the mantle's interior has led to a long and continuing controversy concerning precise values. Nevertheless, the arguments concern whether there is a ten-fold or hundred-fold increase of viscosity in the lower mantle, for example. The pointed fact is that the internal viscosity *seems* to be relatively constant, compared with the many orders of magnitude variability one might expect from mineral flow laws.

Just as surface plate velocity found a wrong explanation in constant viscosity convection, so constant (sub-lithospheric) viscosity has a wrong explanation in terms of variable temperature-dependent viscosity. In fact, the vigorous convecting region beneath the rigid lid is approximately isothermal (just as for constant viscosity), and so if the viscosity η depends only on temperature T, then it will also be constant. Now in fact, it is thought that the viscosity of crystalline rocks varies significantly with pressure (Sammis et al. [7]) and hence depth, and if this is the case, then a rapidly convecting isothermal flow will have a (significantly) variable viscosity with depth, which is presumably discounted by the studies of post-glacial rebound, polar wander, geoid anomalies, etc. There is a partial comeback to this, insofar as a rapidly convecting medium will actually have an *adiabatic* (rather than isothermal) temperature, due to compressibility. In the earth, this does indeed work to reduce viscosity variation with pressure; nevertheless, the likely magnitude of the variability of viscosity is such that it would require an extraordinary benign deity to arrange matters to give a constant viscosity.

In fact, such an outcome is extremely unlikely for a different reason. Fowler [8] used best estimates of the variation of mantle parameters with depth to infer viscosity variation with depth and found a viscosity rising from an asthenospheric value of 10^{20} poise to 0.5×10^{22} poise at 800 km depth, and then back down to 10^{20} poise at the core-mantle boundary. Hence, relatively constant viscosities are indeed possible for adiabatic temperatures; but then, one finds an (adiabatic) temperature at the CMB of 2300K. Now, constraints on the melting point of iron alloys require a basal temperature of at least 3000K and most probably a good deal more. As a consequence, Jeanloz and Richter [9] and many, many others postulate a thermal boundary layer at the base of the mantle, across which the temperature jumps by $\geq$ 1000K.

Unfortunately, this latter concept is a remnant of constant viscosity thinking. It is a fabrication that has little theoretical basis for variable viscosity convection, where the motto for basal thermal boundary layers is "no more change than allows an order of magnitude variation of viscosity." This precludes a basal temperature rise of more than a few hundreds of degrees Kelvin.

So, finally, we come to another basic conundrum of mantle convection: how can a fluid with strongly temperature-and-pressure-dependent viscosity have an interior with relatively uniform viscosity? Fowler [8] suggested that convecting fluids of this type will in fact adopt a flow that achieves this, although quite how was not clear. Here, laboratory experiments are not obviously of any help (although perhaps an ingenious experiment could be devised to simulate depth dependence of viscosity). Nor, indeed, are numerical computations. Partly, this is due to the difficulty of achieving realistic parameter values, but also it must be said that the importance of this particular problem has not been grasped and pursued.

The final possibility that we turn to in this paper is that of an analytic, boundary layer approach. For temperature-dependent viscosity, such analyses were given by Morris and Canright [10] and Fowler [5]. The comparable study with temperature-*and*-pressure-dependent viscosity raises several difficulties, and we begin our pursuit of a boundary layer description by first attempting to analyze a simpler problem which, however, may well exhibit the same general features. This is a model based on a mean field approach (Quareni et al. [11]) with the additional assumption of high lateral wave number. This model is presented in Section 2. As it represents a severe modification of actual convection, we illustrate in Section 3 that it has the same qualitative boundary layer characteristics as constant viscosity convection and in Section 4 that it similarly mimics temperature-dependent viscosity. Finally, in Section 5 we analyze the case of a temperature-and-pressure-dependent viscosity. We show that 'convection' in this case does indeed achieve a relatively constant viscosity. In Section 6, we discuss how the variables adjust so as to allow a relatively isoviscous core, and in Section 7, we offer some potential conclusions for the earth's mantle.

2. Mean Field-Type Model

The mean field approach for two-dimensional convection has been discussed by Quareni et al. [11]. Essentially, the temperature field is written as the sum

$$\bar{T}(z,t) + \theta(z,t)\cos kx, \tag{2.1}$$

where $\bar{T}$ is the horizontally averaged temperature, and θ is a measure of buoyancy. More specifically, the single mode mean field approach expands the variables as a Fourier series and then retains only a single mode in the temperature field. Put a different way, the field variables are projected on to a single horizontal mode. In this way, the two-dimensional partial differential equations are projected on to partial differential equations in one space variable. For steady states, the problem becomes one of ordinary differential equations. The advantage of the mean field approach is that it is computationally simpler, while apparently retaining a reasonable representation of the vertical thermal structure. For this reason, it is a good indicator of the true convective state of the system.

The Boussinesq equations of base-heated thermal convection at infinite Prandtl number can be written in the form

$$\begin{aligned} \nabla\cdot\mathbf{u} &= 0, \\ 0 &= -\nabla p + \nabla\cdot\tau + Ra\,T\mathbf{k}, \\ \frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T &= \nabla^2 T, \end{aligned} \tag{2.2}$$

where $\mathbf{u}$ is velocity, T is temperature, p is pressure (with lithostatic pressure subtracted off), τ is the deviatoric stress tensor, $\mathbf{k}$ is a unit vertical vector, and Ra is a suitably defined Rayleigh number. For variable viscosity, it proves relevant to choose the basal (smallest) viscosity as the reference value, whereas the applied temperature drop gives the appropriate temperature scale. Some slight variation in the latter is possible, which makes for added convenience. The full single mode mean field equations are given by

Quareni et al. [11]. We write the velocity field as

$$\mathbf{u} = \left(-\frac{1}{k}w_z \sin kx, w\cos kx\right) \tag{2.3}$$

in terms of cartesian axes $(x,\ z)$ with z pointing *upwards*. Then the derivative components of the stress tensor τ_1 and τ_2, defined by $\tau = \begin{pmatrix}\tau_1 & \tau_2\\ \tau_2 & -\tau_1\end{pmatrix}$, are given by

$$\begin{aligned} \tau_1 &= -2\eta w_z \cos kx, \\ \tau_2 &= -\frac{\eta}{k}\left(\frac{\partial^2}{\partial z^2} + k^2\right) w \sin kx, \end{aligned} \tag{2.4}$$

so that the momentum equation (2.2) reduces to

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right)\left[\eta\left(\frac{\partial^2}{\partial z^2} + k^2\right)w\right] - 4k^2\frac{\partial}{\partial z}(\eta w_z) - Rak^2\theta = 0, \tag{2.5}$$

where the mean field approximation takes the viscosity in the form $\eta = \eta(\bar{T}, z)$. (Equation (2.5) differs from the equation of Quareni et al. [11] because w in that paper is the *downwards* velocity.) The energy equation gives the two equations for $\bar{T}$ and θ:

$$\begin{aligned} \bar{T}_t + \tfrac{1}{2}(\theta w)_z &= \bar{T}_{zz}, \\ \theta_t + w\bar{T}_z &= \theta_{zz} - k^2\theta. \end{aligned} \tag{2.6}$$

It is suitable to rescale w and θ with a factor $\sqrt{2}$ to remove the $1/2$ from $(2.6)_1$ and to take z pointing downwards, so that $\partial/\partial z \to -\partial/\partial z$, and the top and bottom of the convecting layer are defined by $z=0$ and $z=1$, respectively.

Lastly, we reduce the form of these equations even further by supposing that (*i*) k is 'large' in (2.5), so that

$$w = R\theta/\eta, \qquad R = Ra/k^2, \tag{2.7}$$

and (*ii*) horizontal conduction is 'small' in (2.6), i.e., we neglect the term $-k^2\theta$ (which is not necessarily asymptotically consistent with the reduction of (2.5)). Thus, we have

$$\begin{aligned} T_t - (\theta w)_z &= T_{zz}, \\ \theta_t - wT_z &= \theta_{zz}, \end{aligned} \tag{2.8}$$

where we have dropped the overbar on the average temperature. We emphasize that the simplifications we have made to the mean field model are not necessarily ones that we consider to be good *approximations*, rather we have sought the simplest form of these equations that is likely to be consistent with the basic physical prescription. In particular, (2.7) represents the fact that positive buoyancy ($\theta > 0$) in a convecting flow is associated with upwelling ($w > 0$).

Boundary conditions associated with these equations, for prescribed top and bottom temperatures, are

$$T = -1, \quad \theta = 0, \quad w = 0 \text{ on } z = 0;$$

$$T = T_b, \quad \theta = 0, \quad w = 0 \text{ on } z = 1. \tag{2.9}$$

In keeping with what was said earlier about the scaling of temperature, T_b can be chosen to have any convenient value (of $O(1)$); this will be useful when we consider pressure dependence in Section 5. We define the (Nusselt) number N to be the value of $\partial T/\partial z$ on $z = 0$. Now, in a steady state, a first integral of $(2.8)_1$ is possible; carrying this out, rescaling θ with $(N/R)^{1/2}$, we finally obtain the following pair of ordinary differential equations:

$$\frac{\partial T}{\partial z} = N[1-\theta^2 f],$$

$$\frac{\partial^2\theta}{\partial z^2} + R\theta f T_z = 0, \tag{2.10}$$

where the fluidity $f = 1/\eta$. We require $\theta = 0$ on $z = 0, 1$; $T = -1$ on $z = 0$; $T = T_b$ on $z = 1$; and the extra condition helps to prescribe the unknown N. In the following sections, we consider successively the three rheologies given by

(i) (constant) $\eta^{-1} = f = 1$;
(ii) (temperature dependent) $\eta^{-1} = f = \exp(T/\epsilon)$;
(iii) (temperature and pressure dependent) $\eta^{-1} = f = \exp[(T - \mu z)/\epsilon]$. (2.11)

In the first and second cases, we take $T_b = 0$, whereas in the third case we choose $T_b = \mu$; these choices are in keeping with the idea that the reference viscosity is chosen to be the basal viscosity. In so far as the mantle is concerned, we note that values of activation energy $E^* = 125$ kcal mole^{-1}, activation volume $V^* = 10$ cm^3 mole^{-1}, and a temperature drop of 3000 K are consistent with the study of the asymptotic limits implied by numerical values of $\epsilon \ll 1$, $\mu = O(1)$. In addition, we suppose that the 'Rayleigh' number $R \gg 1$, and we anticipate that also $N \gg 1$. We begin our study by examining the constant viscosity case.

3. Constant Viscosity

With $f = 1$, we have

$$T_z = N(1-\theta^2),$$

$$\theta_{zz} + R\theta T_z = 0, \tag{3.1}$$

whence

$$\theta_{zz} + NR\theta(1-\theta^2) = 0, \tag{3.2}$$

and we have to satisfy

$$T = -1, \quad \theta = 0 \text{ on } z = 0;$$

$$T = 0, \quad \theta = 0 \text{ on } z = 1. \tag{3.3}$$

The basic conductive solution is $\theta = 0$, $T = -1 + z$, $N = 1$, convective instability sets in for $NR > \pi^2$, i.e., at $R = \pi^2$. The solution of (3.2) at large values of NR can be written as a quadrature, and N is determined by

$$N = \left[\int_0^1 (1-\theta^2)\,dz\right]^{-1}. \tag{3.4}$$

It is useful to construct the boundary layer solution of (3.2). The outer solution is just $\theta = 1$, and there are boundary layers near $z = 0$ and $z = 1$. Near $z = 0$, we put

$$z = (NR)^{-1/2}\zeta \tag{3.5}$$

so that, to leading order,

$$\theta_{\zeta\zeta} + \theta(1-\theta^2) = 0, \tag{3.6}$$

which is the full equation. We can infer from (3.5) that $\int_0^1 (1-\theta^2)\,dz \sim (NR)^{-1/2}$, and hence from (3.4) that

$$N \sim R, \tag{3.7}$$

which confirms that $N \gg 1$ when $R \gg 1$. It should be noted that (3.7) is distinct from the correct result for the full equations, $N \sim R^{1/3}$. This is to be expected; the important point is that the simple model gives the correct qualitative behavior when $R \to \infty$.

4. Temperature-Dependent Viscosity

New let us consider the simple model (2.10) with

$$\eta^{-1} = f = \exp(T/\epsilon), \qquad \epsilon \ll 1. \tag{4.1}$$

The equations are

$$T_z = N[1 - \theta^2 \exp(T/\epsilon)],$$
$$\theta_{zz} + R\theta T_z \exp(T/\epsilon) = 0, \tag{4.2}$$

with

$$T = -1, \quad \theta = 0, \text{ on } z = 0,$$
$$T = 0, \quad \theta = 0 \text{ on } z = 1. \tag{4.3}$$

We suppose $R \gg 1$, $\epsilon \ll 1$, with the proviso that in addition $N \gg 1$. The solution of this case provides a rehearsal for the temperature-and-pressure-dependent viscosity case of the following section, but it also illustrates how the simple model mimics the two-dimensional equations in qualitative detail.

The thermal structure of the convection equations exhibits four main regions as z increases (Fowler [5]): a cold *lithosphere*, where the main temperature drop occurs, and where the velocity field is exponentially small; a thin *asthenosphere*, in which the temperature change is $O(\epsilon)$ and which adjusts the flow to the main next region, the isothermal *interior*, which occupies most of the cell and in which flow is rapid and temperature is uniform. Finally, at the base there is a thin *thermal boundary layer*, where a small ($O(\epsilon)$) temperature jump occurs. As we shall see, these regions are exhibited by the solution of (4.2) also.

We begin the analysis by marching down through different asymptotic regimes as z varies.

Lithosphere

For small enough z, we have $T < 0$ and the exponential term in (4.2) is transcendentally small. Thus,

$$T \sim -1 + Nz,$$
$$\theta \sim bz, \tag{4.4}$$

where b and N are unknown.

Asthenosphere

The exponential term becomes important when $T = 0$, roughly speaking. More precisely, we identify this region by the change of variables

$$T = T^* + \epsilon\phi,$$
$$z = z^* + \delta\zeta,$$
$$\theta = \theta^* + \nu\chi, \tag{4.5}$$

where, to match to (4.4), we choose

$$\epsilon = \delta N,$$
$$T^* = -1 + Nz^*,$$
$$\nu = \delta b,$$
$$\theta^* = bz^*; \tag{4.6}$$

in addition, we define Λ by

$$T^* = -\epsilon \ln \Lambda. \tag{4.7}$$

With the definitions in (4.6), we have the matching conditions for ϕ, χ as

$$\phi, \chi \sim \zeta, \qquad \zeta \to -\infty. \tag{4.8}$$

If we suppose $\nu \ll \theta^*$, and choose

$$\theta^* = \Lambda^{1/2},$$
$$\nu = \delta\epsilon R\theta^*/\Lambda, \tag{4.9}$$

then, at leading order, the equations (4.7) are

$$\phi_\zeta \sim 1 - e^\phi,$$
$$\chi_{\zeta\zeta} \sim -e^\phi \phi_\zeta, \tag{4.10}$$

whence

$$\phi, \chi \sim \ln[1 + e^{-\zeta}], \tag{4.11}$$

and we have

$$\phi, \chi \to 0 \quad \text{as} \quad \zeta \to \infty. \tag{4.12}$$

At this point, let us collect the parameters: the eight unknowns are N, b, T^*, z^*, δ, θ^*, ν, Λ, and these are partly determined by the seven relations (4.6), (4.7), and (4.9). The final relation cannot be ascertained until the basal boundary conditions are satisfied.

We now write the equations in terms of ϕ and $\Theta = \theta/\theta^*$; thus we find

$$\frac{\epsilon}{N}\phi_z = 1 - \Theta^2 e^{\phi},$$

$$\frac{\Lambda}{\epsilon R}\Theta_{zz} + \Theta e^{\phi}\phi_z = 0, \tag{4.13}$$

and in fact, from (4.6), (4.7), and (4.9),

$$\frac{\Lambda}{\epsilon R} = \frac{\delta\theta^*}{\nu} = \frac{\theta^*}{b} = z^* = \frac{1-\epsilon\ln\Lambda}{N}, \tag{4.14}$$

thus we expect $\Lambda/\epsilon R \sim 1/N$.

Interior

The solution of (4.13) to all orders of ϵ/N and $\Lambda/\epsilon R$ that matches to (4.12) is simply the isothermal flow

$$\Theta = 1, \qquad \phi = 0.$$

Basal Thermal Boundary Layer

The basal boundary layer creates a small jump in ϕ in order to bring the buoyancy Θ down to 0. In fact, it has an inner and outer part; we consider these in turn.

Outer layer

Assume $\epsilon \ln\Lambda \ll 1$, so $\Lambda/\epsilon R \sim 1/N$; then put

$$z = 1 - \omega/N, \tag{4.15}$$

so that

$$-\epsilon\phi_\omega = 1 - \Theta^2 e^{\phi},$$

$$[1-\epsilon\ln\Lambda]\Theta_{\omega\omega} - \Theta e^{\phi}\phi_\omega = 0, \tag{4.16}$$

and at leading order

$$\phi \sim -2\ln\Theta,$$

$$\Theta_{\omega\omega} + 2\Theta_\omega/\Theta^2 = 0, \tag{4.17}$$

with $\Theta \to 1$ as $\omega \to \infty$. Thus, $\Theta_\omega - (2/\Theta) = -2$, and the solution that equals zero at $\omega = 0$ is

$$(1-\Theta)e^{\Theta} = e^{-2\omega}. \tag{4.18}$$

The approach to zero is singular, however, and we have

$$\Theta \sim 2\omega^{1/2}, \qquad \phi \sim \ln(1/4\omega) \ \text{as} \ \omega \to \infty. \tag{4.19}$$

Inner layer

The outer thermal layer matches to an inner thermal layer where, from (4.16) and (4.19),

$$\omega = \epsilon\Omega, \quad \Theta = \epsilon^{1/2}\Gamma, \quad \phi = \ln(1/\epsilon) + \Psi, \tag{4.20}$$

whence

$$-\Psi_\Omega = 1 - \Gamma^2 e^{\Psi},$$

$$[1-\epsilon\ln\Lambda]\Gamma_{\Omega\Omega} - \Gamma e^{\Psi}\Psi_\Omega = 0, \tag{4.21}$$

with

$$\Gamma \sim 2\Omega^{1/2}, \quad \Psi \sim -2\ln\Gamma \ \text{as} \ \Omega \to \infty,$$

$$\Gamma = 0 \ \text{on} \ \Omega = 0. \tag{4.22}$$

We thus regain the full equations. A numerical integration of these gives the value of Ψ at $\Omega = 0$ as $\Psi_0 = -0.365$; then from (4.5), (4.7), and (4.20), we have

$$T_b = -\epsilon\ln\Lambda + \epsilon\ln(1/\epsilon) + \epsilon\Psi_0. \tag{4.23}$$

Application of the basal thermal boundary condition $T_b = 0$ thus finally determines Λ as

$$\Lambda = e^{\Psi_0}/\epsilon. \tag{4.24}$$

Therefore, $\epsilon \ln \Lambda \sim \epsilon \ln(1/\epsilon) \ll 1$ as assumed, and other assumptions we have made are borne out; for example, (4.9) gives $\nu/\theta^* = \delta\epsilon R/\Lambda \sim \delta N = \epsilon \ll 1$. Finally, we have from (4.14) that

$$N = [1 - \epsilon \ln \Lambda]\frac{\epsilon R}{\Lambda}, \tag{4.25}$$

whence

$$N \sim 1.44\epsilon^2 R[1 + O(\epsilon \ln(1/\epsilon))]. \tag{4.26}$$

The assumption that $N \gg 1$ is thus seen to require the ordered limit $R \gg 1/\epsilon^2 \gg 1$. The result does not apply quantitatively to the full convection equations, nor should we expect it to, but we have shown that the simple model exhibits the same thermal structure.

5. Temperature-and-Pressure-Dependent Viscosity

Now we turn to the temperature-and-pressure-dependent viscosity, that is,

$$\eta^{-1} = f = \exp[(T - \mu z)/\epsilon], \tag{5.1}$$

so that equations (2.10) become

$$T_z = N[1 - \theta^2 \exp\{(T - \mu z)/\epsilon\}],$$
$$\theta_{zz} + R\theta T_z \exp[(T - \mu z)/\epsilon] = 0, \tag{5.2}$$

with

$$T = -1, \quad \theta = 0 \text{ on } z = 0,$$
$$T = \mu, \quad \theta = 0 \text{ on } z = 1. \tag{5.3}$$

The analysis proceeds as in Section 4 by assuming $N \gg 1$, $R \gg 1$, $\epsilon \ll 1$, $\mu \sim 1$ and marching downwards through successive asymptotic regions. The procedure is very similar, but the pressure dependence leads to a dramatically different result, as we shall see.

Lithosphere

For $T < 0$, the exponential term is transcendentally small, and thus

$$T \sim -1 + Nz,$$
$$\theta \sim bz, \tag{5.4}$$

with b, N to be determined.

Asthenosphere

The validity of (5.4) breaks down when $T \approx 0$, where we put

$$z = z^* + \delta X,$$
$$T = T^* + \epsilon\phi,$$
$$\theta = \theta^* + \nu\chi, \tag{5.5}$$

and define Λ by

$$T^* = \mu z^* + \epsilon \ln(1/\Lambda); \tag{5.6}$$

choosing

$$\epsilon = \delta N,$$
$$\nu = \delta b,$$
$$T^* = -1 + Nz^*,$$
$$\theta^* = bz^*,$$
$$\Lambda = \theta^{*2}, \tag{5.7}$$

the equations become

$$\phi_X = 1 - \left[1 + \frac{\nu}{\theta^*}\chi\right]^2 \exp\left(\phi - \frac{\mu}{N}X\right),$$
$$\chi_{XX} + \frac{R\theta^*\epsilon\delta}{\nu\Lambda}\left(1 + \frac{\nu}{\theta^*}\chi\right)\phi_X \exp\left(\phi - \frac{\mu}{N}X\right) = 0, \tag{5.8}$$

with

$$\phi, \quad \chi \sim X \text{ as } X \to -\infty. \tag{5.9}$$

We now assume in anticipation that $\nu/\theta^* \ll 1$, $R\theta^*\delta/\nu\Lambda \ll 1$.

Whereas the former is a natural assumption, the latter seems rather arbitrary and is distinct to the preceding case; it was finally adopted on the basis of numerical integrations, although as we shall see it is unlikely to be valid! Nevertheless, it does in fact give the correct result.

At leading order,

$$\phi_X \sim 1 - e^\phi,$$
$$\chi_{XX} \sim 0, \tag{5.10}$$

so that

$$\chi \sim X, \quad \phi \sim -\ln[1+e^{-X}], \tag{5.11}$$

and $\chi \sim X$, $\phi \to 0$ as $X \to \infty$.

Convection Cell

The preceding approximation breaks down when $\chi \sim N$, and we write

$$1+(\nu/\theta^*)\chi = \theta, \qquad X = N\zeta, \tag{5.12}$$

so that henceforth θ is related to the preceding version by a scale θ^*. The equations are

$$\frac{1}{N}\phi_\zeta = 1-\theta^2\exp(\phi-\mu\zeta),$$

$$\theta_{\zeta\zeta} + \frac{\nu N}{\theta^*}\left(\frac{R\theta^*\epsilon\delta}{\nu\Lambda}\right)\theta\phi_\zeta\exp(\phi-\mu\zeta) = 0. \tag{5.13}$$

Now, whatever the value of the coefficient in the second equation, the first gives $\phi \sim \mu\zeta - 2\ln\theta$, so long as $\phi_\zeta \gg N$, i.e., $\theta > 0$. Since we expect θ to tend to zero at the base of the cell (only), we can expect this approximation to apply throughout the cell. Therefore, we revert to the original z variable, and put

$$\zeta = (z-z^*)/\epsilon,$$

$$\phi = \mu\zeta + \psi, \tag{5.14}$$

so that

$$\frac{1}{N}(\mu+\epsilon\psi_z) = 1-\theta^2\epsilon^{\psi},$$

$$\theta_{zz} + \frac{a\theta}{\mu}(\mu+\epsilon\psi_z)\epsilon^{\psi} = 0, \tag{5.15}$$

where, using (5.7),

$$a = \mu R/\Lambda. \tag{5.16}$$

Now the point is, that as long as $\psi_z \sim O(1)$, then

$$\psi \sim -2\ln\theta, \tag{5.17}$$

and so, irrespective of the value of a, θ approximately satisfies

$$\theta_{zz} + a/\theta = 0, \tag{5.18}$$

with

$$\theta = 1, \quad \theta_z = b \text{ on } z \approx 0; \tag{5.19}$$

thus,

$$\theta = \exp\left[\frac{k}{a} - \left\{\operatorname{erf}^{-1}\left[e^{-k/a}\sqrt{\frac{2a}{\pi}}\,|z-z_m|\right]\right\}^2\right], \tag{5.20}$$

where

$$k = b^2/2, \tag{5.21}$$

and so that $\theta = 1$ at $z = 0$ and $\theta \to 0$ at $z = 1$, we require

$$\frac{k}{a} = \left\{\operatorname{erf}^{-1}\left[e^{-k/a}\sqrt{\frac{2a}{\pi}}\,z_m\right]\right\}^2,$$

$$e^{-k/a}\sqrt{\frac{2a}{\pi}}(1-z_m) = 1, \tag{5.22}$$

whence

$$\operatorname{erf}\left[(k/a)^{1/2}\right] = e^{-k/a}\sqrt{\frac{2a}{\pi}} - 1. \tag{5.23}$$

To summarize, the equations (5.6), (5.7), (5.16), (5.21), and (5.23) give nine relations for the 10 unknowns z^*, T^*, Λ, δ, N, θ^*, b, ν, a, and k. The final relation comes from solving the basal thermal boundary layer equations for ψ and θ. Thus, if $\psi = \psi_b$ at the base, then, since

$$T = -\epsilon\ln\Lambda + \mu z + \epsilon\psi, \tag{5.24}$$

and $T = \mu$ at $z = 1$, we have

$$\Lambda = e^{\psi_b}. \tag{5.25}$$

Basal Thermal Boundary Layer

Outer layer
If we define

$$W(w) = \exp\left[-\{\mathrm{erf}^{-1}(w)\}^2\right], \tag{5.26}$$

then using (5.22), we have

$$\theta = e^{k/a} W\left[\left|1 - e^{-k/a}\sqrt{\frac{2a}{\pi}}\,(1-a)\right|\right]; \tag{5.27}$$

furthermore, integration by parts of the error function as $w \to 1$ ($W \to 0$) gives

$$W\left[\frac{1}{2\{\ln(1/W)\}^{1/2}} - \frac{1}{8\{\ln(1/W)\}^{3/2}} \cdots\right] \sim \frac{\sqrt{\pi}}{2}(1-w), \tag{5.28}$$

whence we find

$$\theta \sim \sqrt{2a}\,(1-z)\left[\ln\left\{\frac{1}{1-z}\right\}\right]^{1/2} \quad \text{as} \quad z \to 1, \tag{5.29}$$

and hence

$$\psi \sim -2\ln\left[\sqrt{2a}\,(1-z)\right]. \tag{5.30}$$

The equations are invalid when $\psi_z \sim 1/\epsilon$, whence we put

$$\begin{aligned} z &= 1 + \epsilon\zeta, \\ \theta &= \sqrt{a}\,\epsilon\Theta, \\ \psi &= -2\ln\left[\sqrt{a}\,\epsilon\right] + \Psi, \end{aligned} \tag{5.31}$$

so that

$$\begin{aligned} \frac{1}{N}\left[\mu + \Psi_\zeta\right] &= 1 - \Theta^2 e^{\Psi}, \\ \Theta_{\zeta\zeta} + \frac{\Theta}{\mu}\left[\mu + \Psi_\zeta\right]e^{\Psi} &= 0, \end{aligned} \tag{5.32}$$

with suitable matching conditions at $-\infty$. (These are complicated by the logarithmic term in Θ (see (5.29)), but this is of no real consequence, in fact.)

Again

$$\Psi \sim -2\ln\Theta, \tag{5.33}$$

so that

$$\Theta_{\zeta\zeta} + \frac{1}{\mu\Theta}\left[\mu - \frac{2\Theta_\zeta}{\Theta}\right] = 0. \tag{5.34}$$

Inner layer
Now the solution of (5.34) tends to zero at $\zeta = \zeta_T$, say, and then

$$\Theta \sim \frac{2}{\sqrt{\mu}}(\zeta_T - \zeta)^{1/2}, \quad \Psi \sim \ln[1/(\zeta_T - \zeta)], \tag{5.35}$$

as $\zeta \to \zeta_T$. Thus, we put

$$\begin{aligned} \zeta &= \zeta_T + X/N, \\ \Theta &= \sigma/(\mu N)^{1/2}, \\ \Psi &= \ln(\mu N) + \rho, \end{aligned} \tag{5.36}$$

to find

$$\begin{aligned} \frac{\mu}{N} + \rho_x &= 1 - \sigma^2 e^{\rho}, \\ \sigma_{xx} + \sigma\left[\frac{\mu}{N} + \rho_x\right]e^{\rho} &= 0, \end{aligned} \tag{5.37}$$

and at leading order,

$$\begin{aligned} \rho_x &= 1 - \sigma^2 e^{\rho}, \\ \sigma_{xx} + \sigma\rho_x e^{\rho} &= 0, \end{aligned} \tag{5.38}$$

with

$$\sigma \sim 2(-X)^{1/2}, \quad \rho \sim -2\ln\sigma \text{ as } X \to -\infty. \tag{5.39}$$

These are equations we have seen before, and from Section 4, we can compute ρ at the value X_0 where $\sigma = 0$ as $\rho = \rho_0 = -0.365$. Thus, we have

$$\psi_b = -2\ln[\sqrt{a}\,\epsilon] + \ln[\mu N] + \rho_0, \tag{5.40}$$

so that, from (5.25)

$$\Lambda = l\mu N/a\epsilon^2, \tag{5.41}$$

where

$$l = e^{\rho_0} = 0.694. \tag{5.42}$$

(5.41) completes the determination of the 10 parameters.

Parameter Elimination

We have

$$T^* = -1 + Nz^* = \mu z^* - \epsilon \ln \Lambda, \tag{5.43}$$

so that

$$z^* = \frac{1-\epsilon\ln\Lambda}{N-\mu}. \tag{5.44}$$

Also

$$\Lambda^{1/2} = \theta^* = bz^* = \frac{b(1-\epsilon\ln\Lambda)}{N-\mu}, \tag{5.45}$$

$$\delta = \epsilon/N, \tag{5.46}$$

and

$$a = \frac{\mu R}{\Lambda} = \frac{lN\mu}{\epsilon^2\Lambda} \tag{5.47}$$

from (5.41), whence

$$N = \epsilon^2 R/l, \tag{5.48}$$

which determines the Nusselt number relationship. Note the similarity to (4.26).

From (5.45),

$$\Lambda = \frac{b^2(1-\epsilon\ln\Lambda)^2}{(1-\mu/N)^2}\left(\frac{l^2}{\epsilon^4R^2}\right), \tag{5.49}$$

so that

$$a = \frac{\mu\epsilon^4R^3(1-\mu/N)^2}{b^2l^2(1-\epsilon\ln\Lambda)^2} \approx \frac{\mu\epsilon^4R^3}{2l^2k}, \tag{5.50}$$

giving one relation between a and k. The other is then (5.23):

$$\operatorname{erf}\left[\left(\frac{k}{a}\right)^{1/2}\right] = e^{-k/a}\sqrt{\frac{2a}{\pi}} - 1. \tag{5.51}$$

We write

$$k = au^2, \tag{5.52}$$

so that

$$\operatorname{erf} u = e^{-u^2}\sqrt{\frac{2a}{\pi}} - 1 \tag{5.53}$$

and

$$a^2 \approx \frac{\mu\epsilon^4R^3}{2l^2u^2} = \frac{\mu N^2R}{2u^2}, \qquad a = \left(\frac{\mu R}{2}\right)^{1/2}\frac{N}{u}, \tag{5.54}$$

whence

$$\operatorname{erf} u = e^{-u^2}(2\mu R)^{1/4}\left(\frac{N}{\pi u}\right)^{1/2} - 1. \tag{5.55}$$

Since $N \gg 1$ and $R \gg 1$, we must have $u \gg 1$, so that

$$e^{u^2} \approx \frac{1}{2}(2\mu R)^{1/4}\left(\frac{N}{\pi u}\right)^{1/2}; \tag{5.56}$$

hence,

$$u \approx \frac{1}{2}\left\{\ln\left[\frac{\mu RN^2}{8\pi^2}\right]\right\}^{1/2}\left[1 - \frac{2\ln u}{\ln\left(\frac{\mu RN^2}{8\pi^2}\right)}\right]^{1/2}, \tag{5.57}$$

thus,

$$u \sim \frac{1}{2}\left\{\ln\left(\frac{\mu RN^2}{8\pi^2}\right)\right\}^{1/2}. \tag{5.58}$$

Working backwards, we find that

$$a = \left(\frac{\mu R}{2}\right)^{1/2}\frac{N}{u}, \tag{5.59}$$

$$k = \left(\frac{\mu R}{2}\right)^{1/2} Nu, \tag{5.60}$$

$$b = (2\mu R)^{1/4}(Nu)^{1/2}, \tag{5.61}$$

$$\Lambda = \mu R/a = (2\mu R)^{1/2}u/N, \tag{5.62}$$

$$z^* \sim 1/N, \tag{5.63}$$

$$\theta^* = bz^* \sim (2\mu R)^{1/4}(u/N)^{1/2}, \tag{5.64}$$

$$\nu = b\delta = \epsilon(2\mu R)^{1/4}(u/N)^{1/2}, \tag{5.65}$$

and thus

$$\nu/\theta^* \sim \epsilon \ll 1 \tag{5.66}$$

as required, and thus

$$\frac{R\theta^*\epsilon\delta}{\nu\Lambda} \sim \frac{\epsilon R}{N\Lambda} \sim \frac{1}{\epsilon\Lambda} \sim \frac{N}{\epsilon R^{1/2}u} \sim \frac{\epsilon R^{1/2}}{u}, \tag{5.67}$$

and therefore the second consistency condition after (5.9) is that $\epsilon^2 R \le 1$. This is in fact inconsistent with the requirement that $N \sim \epsilon^2 R \gg 1$. However, all is not lost. For the theory to go through as written, we only require that the change in θ_z across the asthenosphere should be small. From $(5.15)_2$, this requirement is that $a\delta \ll b$, since this means that the relative jump in θ_z across z^* is small. From (5.59) and (5.61),

$$\frac{a\delta}{b} = \frac{\epsilon(\mu R)^{1/4}}{2^{3/4}u(Nu)^{1/2}} \sim u^{-3/2}R^{-1/4}, \tag{5.68}$$

and the assumption is valid.

To conclude, we find $N \sim 1.44\,\epsilon^2 R$, just as for the temperature-dependent case, providing $N \gg 1$, i.e., $R \gg 1/\epsilon^2$, and $\epsilon \ll 1$. This is consistent with the criterion for the onset of 'convection,' which for the present instance is when

$$(1+\mu)R \ge j_{0,1}^2/2\epsilon^2, \tag{5.69}$$

where $j_{0,1} \approx 2.405$ is the first zero of the Bessel function J_0. Note that $(1+\mu)R$ is the Rayleigh number defined with the basal viscosity and the total temperature drop.

6. Discussion

Although the analysis of Section 5 followed that of Section 4, in retrospect it could have been done more directly. Beginning with θ as defined in (2.1), we have the final definition in the core, (5.20), as

$$\theta \sim \left(\frac{N}{R}\right)^{1/2}\theta^*\exp\left[\frac{k}{a} - \left\{\mathrm{erf}^{-1}\left[e^{-k/a}\sqrt{\frac{2a}{\pi}}|z - z_m|\right]\right\}^2\right], \tag{6.1}$$

and using (5.22), (5.48), and (5.52), this is

$$\theta \sim \epsilon l^{-1/2}\theta^*\exp\left[u^2 - \{\mathrm{erf}^{-1}|1-(1+\mathrm{erf}\,u)(1-z)|\}^2\right]. \tag{6.2}$$

Away from $z = 0$, we can use (5.53) and the approximation $u \gg 1$ to write (6.2) as

$$\theta \sim \epsilon l^{-1/2}\theta^*\sqrt{\frac{a}{2\pi}}\exp\left[-\{\mathrm{erf}^{-1}|2z-1|\}^2\right]. \tag{6.3}$$

In particular, we see that if we define θ as finally scaled in the core in Section 5, then $\theta \sim \epsilon l^{-1/2}\theta^*\theta_c$, and $\theta_c \sim \sqrt{a}$, as is in fact suggested by (5.18) (note that this does not affect (5.17)). Using the definitions of a and θ^* in (5.59) and (5.64), we get

$$\theta \sim \left(\frac{\mu N}{2\pi}\right)^{1/2}\exp\left[-\{\mathrm{erf}^{-1}|2z-1|\}^2\right] \tag{6.4}$$

in the convecting core.

The viscosity, η, is given in terms of ϕ and X in the asthenosphere by

$$\eta = \exp\left[-\frac{(T-\mu z)}{\epsilon}\right] = \Lambda\exp\left[-\left(\phi - \frac{\mu X}{N}\right)\right]. \tag{6.5}$$

Recall that $\eta = 1$ at the base of the cell. Thus, $\eta \sim \Lambda \sim uR^{1/2}/N$ in the asthenosphere. In the core, then,

$$\eta = \Lambda\exp[-(\phi - \mu\zeta)] = \Lambda e^{-\psi} \sim \Lambda\theta_c^2, \tag{6.6}$$

where θ_c is the core value scaled from Section 5. Thus, using (5.16),

$$\eta \sim \frac{\mu R}{2\pi} \exp\left[-2\{\mathrm{erf}^{-1}|2z-1|\}^2\right], \tag{6.7}$$

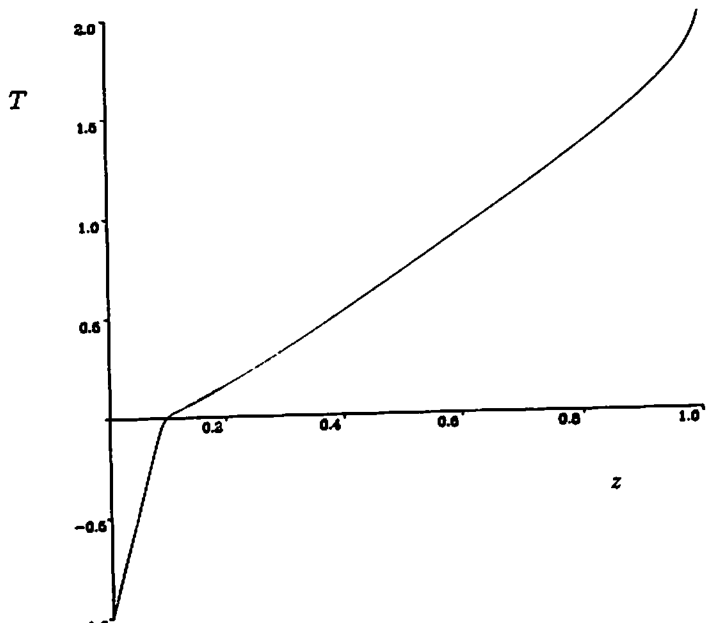
and $\eta \sim R$ in the core. In the basal (outer) thermal boundary layer, we have from (5.31) and (5.16)

$$\eta = \Lambda e^{-\phi} = \mu R \epsilon^2 e^{\Psi}, \tag{6.8}$$

so that $\eta \sim \epsilon^2 R$, and in the inner thermal boundary layer (where the heat flux is adjusted to the conductive value),

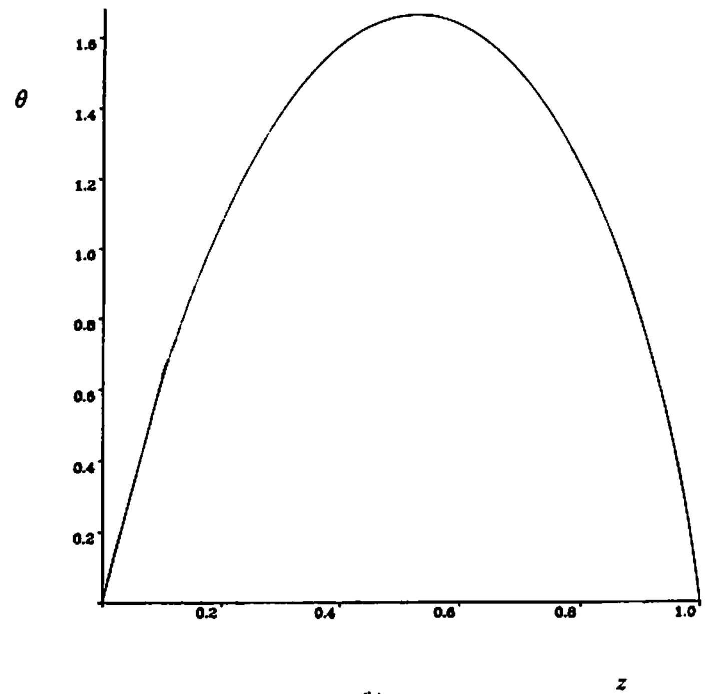
$$\eta = l e^{-\rho} \sim O(1). \tag{6.9}$$

Thus, the viscosity decreases fairly abruptly through the basal thermal boundary layer, although in a formal sense there is no boundary layer in viscosity as such. In Figure 1, we show typical computed values of T, θ, and η which are consistent with the description given above.



(a)

Figure 1. Profiles of T, θ, and η, computed for $R = 10^3$ and $\epsilon = 0.05$. A uniform mesh of size 3×10^{-4} was used and the time dependent system (2.8) was solved until a steady solution was obtained. A standard NAG algorithm, D03 PGF, was used.



(b)

Figure 1. (*Continued*)

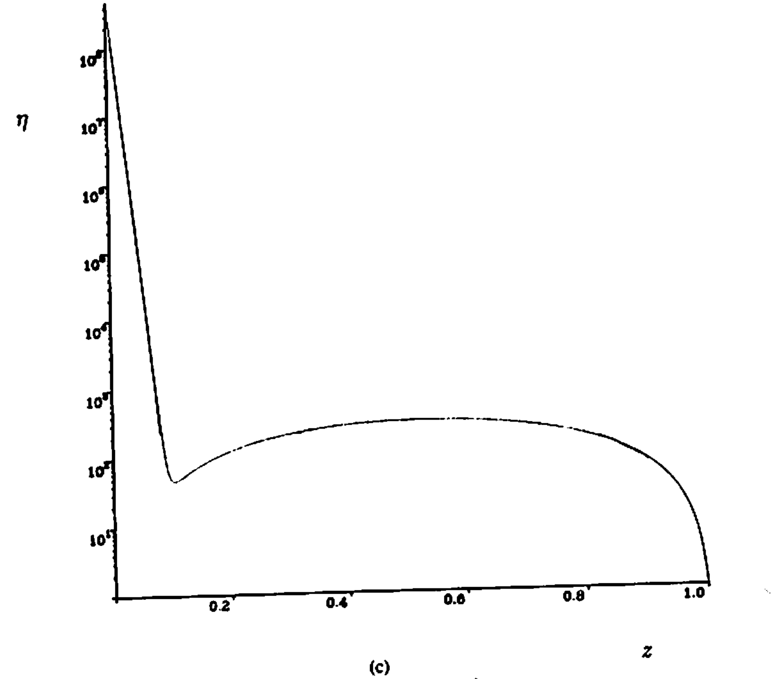
Figure 2 shows N versus $\epsilon^2 R$ for a variety of values of R and ϵ. Since the thinnest boundary layer is of order $\epsilon/N \sim 1/\epsilon R$, the resolution requirement for a uniform mesh (which we need) is that $h \le 0.3/\epsilon R$, where h is the step size. Since we also require $N \sim \epsilon^2 R \gg 1$, then

$$\frac{1}{\epsilon^2} \ll R \le \frac{0.3}{\epsilon h}. \tag{6.10}$$

If we allow $h = 3\times10^{-4}$ for example, then

$$\frac{1}{\epsilon^2} \ll R \le \frac{10^3}{\epsilon}. \tag{6.11}$$

Then for $\epsilon = 0.2$, $25 \ll R \le 5\times10^3$; for $\epsilon = 0.02$, $2.5\times10^3 \ll R \le 5\times10^4$. Hence, the effective range of computable R, ϵ values is not that large and becomes narrower as ϵ is reduced. There is naturally some spread in the N versus $\epsilon^2 R$ values, but the results are consistent with the prediction.



(c)
Figure 1. (*Continued*)

How does the solution manage to lead to a (relatively) isoviscous profile rather than an isothermal one? (It should be emphasized that *relatively* here means a variation by $O(R)$, but the variation in viscosity is much less than $\exp[O(1/\epsilon)]$, as one would have for an isothermal core.) Some clue is obtained by rewriting (2.8) with w and θ scaled by their core values. We put

$$\eta = R\bar{\eta}, \quad w = N^{1/2}\bar{w}, \quad \theta = N^{1/2}\bar{\theta}, \tag{6.12}$$

so that the equations can be written

$$\bar{\eta} = \bar{\theta}/\bar{w},$$

$$T = \mu z - \epsilon \ln(R\bar{\eta}),$$

$$\frac{1}{N}T_t - (\bar{\theta}\bar{w})_z = \frac{1}{N}T_{zz},$$

$$\bar{\theta}_t - \bar{w}T_z = \bar{\theta}_{zz}, \tag{6.13}$$

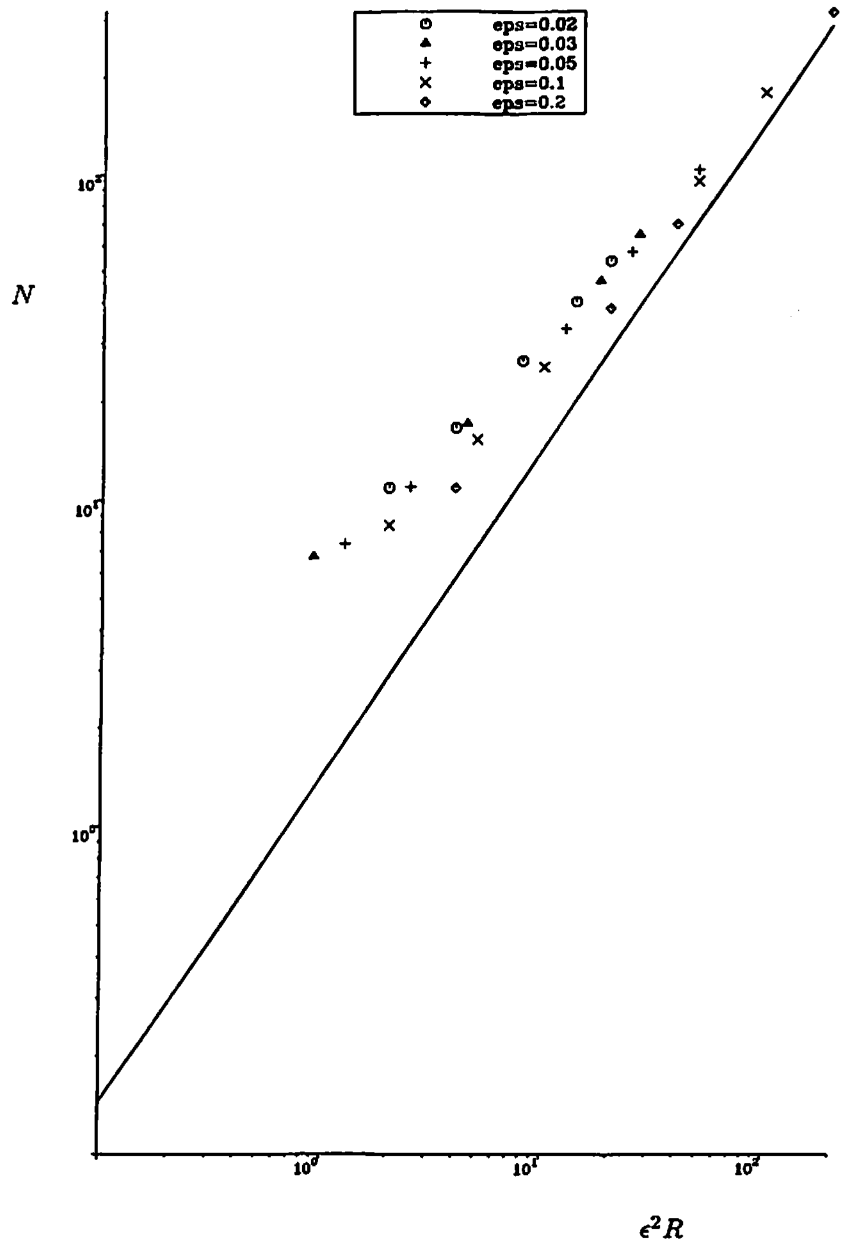


Figure 2. Values of N plotted as a function of $\epsilon^2 R$ for a variety of different values of R and ϵ in the range $10^2 < R < 10^4$, $0.02 < \epsilon < 0.2$. Also shown is the line $N = 1.44\epsilon^2 R$.

and in the core, we directly find

$$\bar{w} \sim c/\bar{\theta}, \qquad T \sim \mu z, \tag{6.14}$$

whence

$$\bar{\theta}_{zz} + \mu c/\bar{\theta} = 0, \tag{6.15}$$

as obtained previously. It seems that the rheology chooses the mean temperature $(6.13)_2$, the energy equation chooses a velocity field $\bar{w}$ consistent with this, and the role of the 'momentum' equation $(6.13)_1$ is to choose the viscosity. This was the recipe suggested by Fowler [8] for a much simpler model, and it is here borne out in a more physical context. It remains to be seen whether the convective style here can be used to study the full convection equations, but there is no obvious reason why it should not. In particular, both mean-field and full two-dimensional calculations give viscosity depth profiles that are akin to that in Figure 1 (Quareni *et al.* [12]).

7. Conclusions

The primary conclusion that must be considered is that the effect of a strongly temperature-*and*-pressure-dependent viscosity drives the internal temperature away from isothermal towards an isoviscous profile. *If* this result extends to a full Boussinesq model of convection, then it should allow us to reconcile a *relatively* constant mantle viscosity with a vigorously convecting mantle. In addition, there is no problem with having a core mantle boundary temperature of 3500, 4000, or 5000K, as this simply determines the viscosity of the convecting flow, a value which is hardly constrained by mineral properties at the CMB.

There will remain the issue of subduction. It has already been found that with temperature-dependent viscosity, stresses of order 10 kilobars can easily be generated in the surface layer. Such stresses are sufficient to cause plastic yielding, and if (as is likely) they also exist for the temperature-and-pressure-dependent case, then it is to be expected that subduction will naturally occur if the temperature-and-pressure-dependent rheology is made viscoplastic. Nevertheless, the real earth is still some distance away, due to the question of the existence and effects of phase changes and chemical discontinuities at various depths. These and other issues await future study.

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