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DYNAMICS OF A REDUCED MODEL OF TWO-PHASE FLOW IN A BOILING CHANNEL

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ABSTRACT

A model is presented for the flow of two-phase steam-water flow in a boiling channel. The motivation was to study the dynamics of a boiler which constituted a part of a steam engine. Particular features of the system were that the boiling tube was coiled, and that the externally supplied heat was generated by an exothermic chemical reaction. The reactant supply was controlled so as to maintain exit temperature of 540°C at an operating pressure of 50–100 bars. The inlet temperature of sub-cooled water was 10°C.

The model presented here considers the flow in the boiler to consist in general of three separate regions: sub-cooled liquid, two-phase steam and water, and superheated steam. These regions are separated by moving boundaries termed the boiling boundary and the superheat boundary. The flow is modelled in each region by one-dimensional, averaged equations of conservation of mass, momentum and enthalpy, together with an equation of state and suitable constitutive relations. In particular, the two-phase region is modelled using 'two-fluid' averaged equations, so that each phase carries its own conservation laws; the enthalpies are not necessarily those of thermodynamic equilibrium, and are related by an averaged jump equation derived from the Stefan condition. Separate constitutive relations, particularly for interfacial friction, must be prescribed for separate regions of bubbly, slug, churn and annular flow.

A much simpler model which retains the accuracy of the original can be derived using the methodology of non-dimensionalisation, scaling and asymptotic analysis. This model facilitates numerical solution, since it also filters out fast space and time scales, which manifest themselves numerically by the requirement of small space and time scales. Here we describe a simple numerical method, and illustrate its use with some typical sample calculations. Some further analysis is possible using the often realistic result that the two-phase flow speeds are much larger than the inlet flow speed. In this case the boiling boundary responds quasi-statically to fluctuations in inlet velocity, and the dynamics of the system is then controlled by the delayed feedback between the two.

1. INTRODUCTION

The motivation for this study came from a commercial contract involving design modelling of a steam engine, comprising a pump, boiler, turbine and condenser in series. The specific contractual requirement was for a fast but accurate code for the boiler. Since the exit temperature was controlled to give superheated steam, there was thought to be a significant portion of two-phase flow, much of it in the annular régime. The likelihood of widely differing velocities suggests the use of a two-fluid model, and in the final report to the company (Aldridge et al. 1991), a very general such model, due to Drew and Wood (1985) was proposed. In this paper we summarise this model, and how how it can be usefully approximated, using applied mathematical techniques, to a much simplified form which is capable of rapid and efficient numerical solution.

2. MATHEMATICAL MODEL

We consider a vertical tube $0 < z < l$ of small cross-section. Turbulent sub-cooled liquid is admitted at the base $z = 0$ and driven by a pressure gradient towards the outlet at the top. The tube is externally heated at a rate Q (per unit length per unit time), and in general there will be two moving boundaries $z = r(t)$ and $z = s(t)$ which demarcate the regions of sub-cooled flow, two-phase flow and superheated flow. We term $r(t)$ the boiling boundary, and $s(t)$ the superheat boundary.

In what follows, we describe separate models for the three regions. Appropriate equations are averaged cross-sectionally and in time, and the averaging process throws up an enormous number of terms which must be constituted in some way. In this paper, we choose the simplest path, treating the simplest boundary conditions, and ignoring certain terms in Drew and Wood's complete formulation. We must emphasise that in the original report (Aldridge et al. 1991) no such liberty was taken, and we show that the process described here can be applied to the most general two-fluid model, with exactly the same resultant simplification.

2.1 Sub-cooled flow

Supposing that the cross-sectional area A is constant, suitable equations in $0 < z < r$ are

$$\begin{aligned} \rho_{lt} + (\rho_l u)_z &= 0, \\ \rho_l(u_t + uu_z) &= -p_z - F_{lw}, \\ \rho_l(h_{lt} + uh_{lz}) &= \frac{Q}{A}, \end{aligned} \tag{1}$$

where ρ_l is liquid density, u is liquid velocity, p is pressure and h_l is enthalpy. The term F_{lw} represents the turbulent friction at the wall, and is normally written

$$F_{lw} = \frac{2}{d} f_{lw} \rho_l \,|u|\, u, \tag{2}$$

where d is the tube diameter, f_{lw} is the friction factor, and is a weak function of the Reynolds number. We have neglected gravity in the momentum equation. In the original problem, this was because the tube was coiled, so that the average $\bar{g}$ could be neglected. More generally, gravity will be 'small' if $\rho_l gl \ll \Delta p$, where Δp is the pressure drop. Since $\rho_l gl \sim 1$ bar if $l \sim 10$ m, this will be the case where $\Delta p > 10$ bars, a normal operating condition.

2.2 Two-phase flow

In a boiling tube, the two-phase flow will normally evolve through a sequence of different régimes as the void fraction α increases: in turn, we see bubbly flow, slug flow, churn flow and annular flow. It is only in the last of these that the gas phase can move in a relatively unimpeded way, hence the fluid velocities tend to be much larger; equivalently, the region of annular flow is longer. For both these reasons, the annular flow is the primary determinant of two-phase flow pressure drop, and hence we concentrate on this. Nevertheless, we emphasise that the difference between the models of the different flow régimes lies in the constitutive terms only. We shall come back to this point eventually.

Suitable two-fluid equations for two-phase flow are

$$(\alpha\rho_g)_t + (\alpha\rho_g v)_z = \Gamma,$$

$$(\beta\rho_l)_t + (\beta\rho_l u)_z = -\Gamma, \tag{3}$$

where α and β are gas and liquid volume fractions, and $\alpha + \beta = 1$; ρ_g is the gas density, v is the gas velocity, and Γ is the rate of change of phase with time due to boiling. The momentum equations are

$$(\beta\rho_l u)_t + (\beta\rho_l u^2)_z = -\beta p_z + F_{lw} + F_{li},$$

$$(\alpha\rho_g v)_t + (\alpha\rho_g v^2)_z = -\alpha p_z + F_{gi}. \tag{4}$$

Here, F_{lw} is the wall stress on the liquid phase, F_{li} is the interfacial stress on the liquid, $F_{gi} = -F_{li}$ being that on the gas. In writing (4), we have already assumed that $F_{gw} = 0$, as for annular flow. We have also assumed that various profile coefficients which arise in averaging the flux terms are equal to one, and that the phasic pressures are equal. This last assumption is thought to be realistic, but it does cause problems, in view of the well-known ill-posedness of the basic equations (3) and (4) when $\rho_g = \rho_l = 1$, $\Gamma = F_{lw} = F_{li} = 0$. This ill-posedness can be resolved in a number of ways, for example by differing phasic pressures, or by a profile coefficient other than one (Fowler and Lisseter 1992), but it will not be of relevance here, since in the approximations we make, the issue of well-posedness is not contentious. There are other terms in (4): Reynolds' stresses, momentum phase change, etc., which can be demonstrated to be small.

The two-fluid enthalpy equations are (approximately)

$$\beta\rho_l(h_{lt} + u h_{lz}) = \Gamma(h_l - h_{li}) + E_l + Q/A,$$

$$\alpha\rho_g(h_{gt} + v h_{gz}) = \Gamma(h_{gi} - h_g) + E_g, \tag{5}$$

where h_l and h_g are the liquid and gas enthalpies, E_l and E_g are the interfacial energy transports to the liquid and gas phases, and h_{li} and h_{gi} are the average interfacial enthalpies. Particularly for annular flow, we cannot necessarily assume that h_l and h_g are the equilibrium saturation values, and (5) must be supplemented by an averaged interfacial Stefan condition,

$$E_l + E_g + \Gamma(h_{gi} - h_{li}) = 0, \tag{6}$$

which in effect determines Γ, given the other quantities. In addition, the quantities ρ_g,ρ_l, F_{lw},F_{li},E_l,E_g,h_{li},h_{gi} need to be constituted.

2.3 Superheated flow

The relevant equations are

$$\rho_{gt} + (\rho_g v)_z = 0,$$

$$(\rho_g v)_t + (\rho_g v^2)_z = -p_z - F_{gw},$$

$$\rho_g(h_{gt} + v h_{gz}) = Q/A, \tag{7}$$

and must be supplemented by an equation of state for ρ_g, and an expression for F_{gw}.

2.4 Constitutive relations

We take $\rho_l =$ constant, but define

$$\rho_g = \frac{p}{RT_g} \tag{8}$$

where T_g is the gas temperature. F_{lw} is defined by (2), and equivalently,

$$F_{li} = \frac{2}{d} f_{li} \rho_g \left| v - \chi u \right| (v - \chi u), \tag{9}$$

where a value of $\chi = 2$ was suggested by Wallis (1969). Similarly,

$$F_{gw} = \frac{2}{d} f_{gw} \rho_g \left| v \right| v. \tag{10}$$

The friction factors f_{lw}, f_{li}, f_{gw} depend weakly on the respective Reynolds numbers of relevance: we do not pursue that.

The interfacial heat transports can be defined by

$$E_g = H_{gi}(T_{gi} - T_g)/L_s,$$

$$E_l = H_{li}(T_{li} - T_l)/L_s, \tag{11}$$

where H_{gi} and H_{li} are heat transfer coefficients, T is temperature, and L_s^{-1} is the average surface area per unit volume. Depending on flow régime, it is a function of α, and for annular flow,

$$L_s^{-1} = 4\alpha^{1/2}/d. \tag{12}$$

Lastly, we need to prescribe the enthalpies in terms of the temperatures. We assume that the interfacial temperatures T_{gi}, T_{li} are equal to T_{sat}, the local saturation value. This varies somewhat with pressure, but we ignore this here for simplicity, and thus the interfacial enthalpies are constants,

$$h_{gi} = h_{g0}, \quad h_{li} = h_{l0},$$

$$T_{gi} = T_{sat}, \quad T_{li} = T_{sat}. \tag{13}$$

Then the enthalpies are defined by

$$h_l = h_{l0} + c_{pl}(T_l - T_{sat}),$$

$$h_g = h_{g0} + c_{pg}(T_g - T_{sat}), \tag{14}$$

where c_{pl} and c_{pg} are the specific heats.

2.5 Boundary conditions

The simplest boundary conditions to prescribe are that

$$p = p_{out} \quad \text{at} \quad z = l,$$

$$p = p_{out} + \Delta p, \quad T = T_0 \quad \text{at} \quad z = 0; \tag{15}$$

an alternative to the inlet pressure condition is to prescribe the inlet mass flux

$$\rho_l u = G_l. \tag{16}$$

We shall mostly dwell on the former.

3. NON-DIMENSIONALISATION AND REDUCTION

We wish to choose scales for the variables so that the dimensionless variables are $O(1)$. Since the velocities can be expected to be large in the two-phase region, we choose to use the two-phase equations to determine the appropriate scales. Specifically, we define

$$\beta = \epsilon\beta^*, \quad v = Vv^*, \quad u = Uu^*, \quad p = p_{out} + Pp^*, \quad \Gamma = \bar{\Gamma}\gamma,$$
$$h_g = h_{g0} + c_{pg}\Delta T h_g^*, \quad h_l = h_{l0} + c_{pl}\Delta T h_l^*,$$
$$z = lz^*, \quad t = (l/U)t^*, \quad \rho_g = \bar{\rho}_g\rho_g^*, \tag{17}$$

where

$$\Delta T = T_{sat} - T_0, \tag{18}$$

and the unknown scales $\epsilon, V, U, P, \bar{\Gamma}, \bar{\rho}_g$ are yet to be chosen. We choose them specifically to balance what we infer (by examining typical orders or magnitude) are the largest terms in the equations. That is, we define

$$\bar{\rho}_g V = l\bar{\Gamma},$$
$$\rho_l\epsilon U = l\bar{\Gamma},$$
$$f_{lw}\rho_l U^2 = f_{li}\bar{\rho}_g V^2,$$
$$P = 2lf_{lw}\rho_l U^2/d,$$
$$\bar{\rho}_g = p_{out}/RT_{sat}, \tag{19}$$

so that the flux terms balance the phase change term in (3), $F_{lw} \sim F_{li}$ in $(4)_1$, $p_z \sim F_{gi}$ in $(4)_2$, and $\rho_g^* = O(1)$ in (8). The final relation depends on the choice of whether pressure drop $(15)_2$ or inlet mass flux (16) is prescribed. We either choose

$$P = \Delta p \tag{20}$$

for prescribed pressure drop, or

$$\rho_l\epsilon U = G_l \tag{21}$$

for prescribed inlet mass flux.

3.1 Two-phase region, $r < z < s$

The dimensionless two-phase flow equations are then, from (3),(4),(5) and (6),

$$a_1[(1-\epsilon\beta)\rho_g]_t + [(1-\epsilon\beta)\rho_g v]_z = \gamma,$$
$$\beta_t + (\beta u)_z = -\gamma,$$
$$\epsilon_2[(\beta u)_t + (\beta u^2)_z] = -\epsilon\beta p_z - u^2 + S^2(v - \chi a_1 u)^2,$$
$$\epsilon_3[a_1\{(1-\epsilon\beta)\rho_g v\}_t + \{(1-\epsilon\beta)\rho_g v^2\}_z] = -(1-\epsilon\beta)p_z - S^2(v-\chi a_1 u)^2,$$
$$\gamma = q + a_4(1-\epsilon\beta)^{1/2}h_g + a_5\gamma h_l,$$
$$a_5\epsilon_6\beta[h_{lt} + uh_{lz}] = \epsilon_6 q - [(1-\epsilon\beta)^{1/2} - a_5\epsilon_6\gamma]h_l,$$
$$a_7(1-\epsilon\beta)\rho_g[a_1 h_{gt} + vh_{gz}] = -[(1-\epsilon\beta)^{1/2} + a_7\gamma]h_g, \tag{22}$$

wherein we have dropped the asterisks on the variables, and

$$S^2 = \rho_g, \quad \rho_g = \frac{1 + a_8 p}{1 + a_9 h_g}, \tag{23}$$

where

$$q = Q/\bar{\Gamma}AL, \quad L = h_{g0} - h_{l0} \tag{24}$$

is the latent heat, and the scales are defined by

$$\begin{aligned}
\epsilon &= (f_{lw}\bar{\rho}_g/f_{li}\rho_l)^{1/2}, \\
U &= G_l/\rho_l\epsilon, \\
\bar{\Gamma} &= G_l/l, \\
V &= G_l/\bar{\rho}_g, \\
P &= 2lf_{li}G_l^2/\bar{\rho}_g d.
\end{aligned} \tag{25}$$

If G_l is prescribed, these determine the scales; if Δp is prescribed, then $P = \Delta p$, and G_l can be defined by $(25)_5$. The parameters a_1–a_9 are defined by

$$\begin{aligned}
a_1 &= U/V, \\
\epsilon_2 &= \epsilon d/2f_{lw}l, \\
\epsilon_3 &= d/2lf_{li}, \\
a_4 &= 4H_{gi}\Delta T/dL\bar{\Gamma}, \\
a_5 &= c_{pl}\Delta T/L, \\
\epsilon_6 &= dl\bar{\Gamma}/4H_{li}\Delta T, \\
a_7 &= dc_{pg}\bar{\Gamma}/4H_{gi}, \\
a_8 &= P/p_{out}, \\
a_9 &= \Delta T/T_{sat}.
\end{aligned} \tag{26}$$

We use typical numerical estimates of the parameters as follows (these were for the particular application which motivated the study):

$$\begin{gathered}
\bar{\rho}_g \sim 30\,\mathrm{kg\ m^{-3}}, \quad \rho_l \sim 760\,\mathrm{kg\ m^{-3}}, \quad \dot{m}_0 \sim .2\,\mathrm{kg\ s^{-1}}, \quad d \sim .014\,\mathrm{m}, \\
A = \pi d^2/4 \sim 1.54 \times 10^{-4}\,\mathrm{m^2}, \quad G_l = \dot{m}_0/A \sim 1.3 \times 10^3\,\mathrm{kg\ m^{-2}s^{-1}}, \quad l \sim 20\,\mathrm{m}, \\
c_{pg} \sim 4.6\,\mathrm{kJ\ kg^{-1}K^{-1}}, \quad c_{pl} \sim 5.2\,\mathrm{kJ\ kg^{-1}K^{-1}}, \quad \Delta T \sim 250\,\mathrm{K}, \\
T_{sat} \sim 270\,\mathrm{K}, \quad L \sim 1571\,\mathrm{kJ\ kg^{-1}},
\end{gathered} \tag{27}$$

and derived flow characteristics

$$\begin{gathered}
f_{lw} \sim .004, \quad f_{li} \sim .02, \\
H_{li} \sim 1.7 \times 10^5\,\mathrm{W\ m^{-2}K^{-1}}, \\
H_{gi} \sim .7 \times 10^4\,\mathrm{W\ m^{-2}K^{-1}};
\end{gathered} \tag{28}$$

hence

$$\begin{gathered}
\epsilon \sim .09, \quad a_1 \sim .44, \quad \epsilon_2 \sim .009, \quad \epsilon_3 \sim .02, \quad a_4 \sim 6, \\
a_5 \sim .76, \quad \epsilon_6 \sim .01, \quad a_7 \sim .13, \quad a_8 \sim .5 \ \ (\text{if}\, p_{out} \sim 60\,\text{bars}), \\
a_9 \sim .5.
\end{gathered} \tag{29}$$

We see that $\epsilon_2, \epsilon_3, \epsilon_6$ are very small, and ϵ and a_7 are quite small.

3.2 Subcooled region, $z < r(t)$

Here the dimensionless equations are (ρ_l being constant)

$$u_z = 0,$$
$$\epsilon_{10}[u_t + uu_z] = -p_z - u^2,$$
$$h_{lt} + uh_{lz} = \epsilon\gamma/a_5, \tag{30}$$

where

$$\epsilon_{10} = \epsilon_2/\epsilon \sim .1.$$

3.3 Superheated region, $z > s(t)$

Here we have

$$a_1\rho_{gt} + (\rho_g v)_z = 0,$$
$$\epsilon_3[a_1\rho_{gt} + (\rho_g v^2)_z] = -p_z - \rho_g v^2,$$
$$\rho_g[a_1 h_{gt} + vh_{gz}] = \gamma/a_5. \tag{31}$$

3.4 Boundary conditions

At the ends of the tube, we have

$$p = 0 \quad \text{on} \quad z = 1,$$
$$h_l = -1,$$
$$p = 1 \quad or \quad u = \epsilon \quad \text{at} \quad z = 0. \tag{32}$$

The boiling boundary is defined by

$$h_l = 0 \quad \text{on} \quad z = r, \tag{33}$$

and the superheat boundary is defined by

$$\beta = 0 \quad \text{on} \quad z = s, \tag{34}$$

and we require variables to be continuous at these.

3.5 An asymptotic reduction

3.5.1 Sub-cooled region. If significant evaporation occurs, we can expect $\gamma \sim 1$. Since $\beta = 1/\epsilon$ at $z = r$ and βu is continuous, we have that $u \sim \epsilon$ in $z < r$, thus the solution of (30) is

$$u = \epsilon u_0(t), \quad p = p_0(t), \tag{35}$$

where one or other of u_0 and p_0 can be chosen to be constant, and the enthalpy satisfies

$$h_{l\tau} + u_0 h_{lz} = \gamma/a_5, \quad \tau = \epsilon t, \tag{36}$$

whose solution is

$$z = \int_{\tau-\tau_{sc}(1+h)}^{\tau} u_0(\theta)d\theta, \tag{37}$$

where

$$\tau_{sc} = \frac{a_5}{\gamma} \tag{38}$$

is the sub-cooling delay. In particular,

$$r(\tau) = \int_{\tau-\tau_{sc}}^{\tau} u_0(\theta)d\theta. \tag{39}$$

Note that the boiling boundary will vary *slowly* with t, being a function of the slow time ϵt.

3.5.2 Two-phase region. If we neglect the small terms in $\epsilon_2, \epsilon_3, \epsilon_6$, ϵ and $a_1 a_7 \sim .06$ in (22), then an approximate model is the following:

$$\begin{aligned} a_1\rho_{gt} + (\rho_g v)_z &= \gamma, \\ \beta_t + (\beta u)_z &= -\gamma, \\ u &= \sigma v, \\ p_z &= -u^2, \\ h_l &= 0, \\ a_7\rho_g v h_{gz} &= -(1 + a_7\gamma)h_g, \\ \gamma &= q + a_4 h_g, \end{aligned} \tag{40}$$

where

$$\sigma = \frac{S}{1 + \chi a_1 S}. \tag{41}$$

The approximations involved in neglecting ϵ_2 and ϵ_3 are potentially singular, hence we may not be able to satisfy conditions of continuity of all of β, u and v at $z = r$. In this case, we might expect short regions in which the variables adjust from their sub-cooled values to the two-phase values determined by (40). Such boundary regions should not affect the large scale dynamics significantly, however, and their removal enables a direct numerical solution to be significantly easier and more efficient. The neglect of ϵ_6 causes no singular behaviour.

3.5.3 Superheat region. We neglect the term in ϵ_3, thus the reduced model is

$$\begin{aligned} a_1\rho_{gt} + (\rho_g v)_z &= 0, \\ p_z &= -\rho_g v^2, \\ \rho_g[a_1 h_{gt} + v h_{gz}] &= \gamma/a_5. \end{aligned} \tag{42}$$

3.6 Comments

The parameterisation of the two-phase flow equations implicitly assumes annular flow. In fact, near $z = r$, where $\beta = O(1/\epsilon)$ (this is *scaled*, corresponding to a liquid fraction of $O(1)$), we will have regions of bubbly, slug and churn flow. Since we can expect $u \sim v \sim \epsilon$ in these regions, the gas continuity equation suggests that $z - r \sim \epsilon$, so that the non-annular regions are relatively short. Moreover, the liquid momentum equation suggests that these regions can be adequately modelled by simply changing the definition of S in $(22)_3$. Specifically, drift flux models are consistent with choosing $S \gg 1$, but this does not affect σ significantly. Thus, it is plausible that the two-phase model given here is, in an approximate sense, a valid description for flows other than annular.

3.7 Boundary conditions

We need to specify v and βu on $z = r$. We lose the ability to specify u by losing the inertial terms from the liquid momentum equation, although the condition of equal velocities at $z = r$ ($u = v/a_1$) can be recovered by specifying $\sigma = 1/a_1$ there; we choose

$$v = \epsilon u_0/\sigma, \quad \beta u = u_0, \quad p \text{ continuous},$$

$$h_g = 0 \quad \text{at} \quad z = r. \tag{43}$$

It follows that $h_g \equiv 0$, $\gamma = q$, and only the first four of (40) are relevant.

At $z = s$, we take p, v to be continuous, and $h_g = 0$. Finally, at $z = 1$, $p = 0$.

4. NUMERICAL SOLUTION

The reduced model is of different type to the full model, being partially parabolic, and thus some care needs to be taken with the numerical method. In the sub-cooled region, we solve the enthalpy equation $(30)_3$ with a first order, explicit, upwind method. $r(t)$ is then determined by $h = 0$, interpolating between grid points. The enthalpy equation is solved similarly in $z > s(t)$, when there is a superheat region.

In $z < r$, we obtain u by quadrature; for $(30)_1$ this is trivial, though quadrature is also used if ρ_l is variable. If u is prescribed at $z = 0$, the sub-cooled region is then fully determined.

4.1 Pressure/velocity equations

In $z > r$, ρ_g depends on p (and h_g). Focussing on the p dependence, the gas mass and gas momentum equations have the structure of the set

$$p_t + f_z = 0,$$

$$p_z = -f, \tag{44}$$

where $f = \rho_g v$, and we have omitted irrelevant coefficients. This is a *diffusion* equation for p, and thus we require boundary data at both ends. If u_0 is prescribed, then $p = 0$ at $z = 1$ and v is prescribed at $z = r$; but if p is prescribed at $z = 0$, then we have it at $z = r$ also, and u_0 is then determined from (43) by finding v. And then, in fact, we can integrate u backwards through the sub-cooled region. We use a fully implicit method for (44), which can be got by using forward differences (in z) in one equation, and backward differences in the other.

The remaining equation is that for the liquid volume fraction β. We assume that this equation remains hyperbolic, despite the dependence of u on v and hence p. An explicit method can then be used, and the superheat boundary is determined (by interpolation) when $\beta = 0$.

The method described above has been implemented in FORTRAN and runs at Oxford on a VAX 11/780. It is relatively quick, though there appear to be some regions of numerical instability. This may be due to a change of equation *type* with certain parameters.

Detailed results will be presented elsewhere.

5. ANALYSIS

Suppose, for simplicity, that there is no superheat region. Evidently, r varies on the slow time scale $\tau = \epsilon t$, while disturbances propagate through the two-phase region on a *timescale* of $O(1)$. It follows that, barring instability in the two-phase region, the equations in $z > r$ can be solved quasi-statically. Specifically

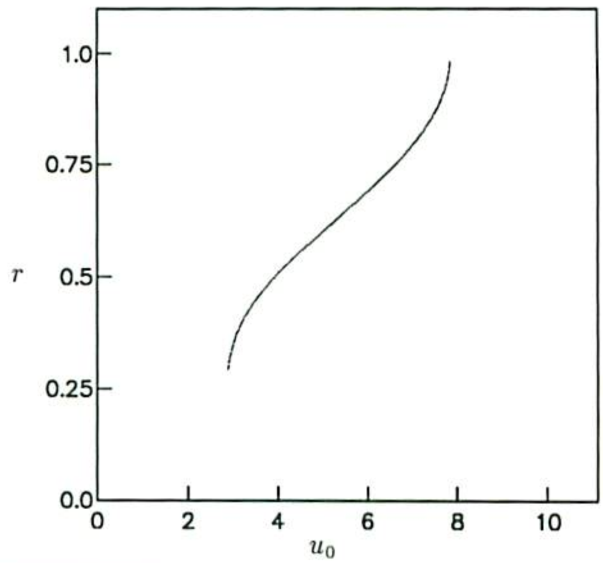

Figure 1. Boiling boundary r *vs.* inlet velocity u_0 for the quasi-static case, with $\gamma = 4$, $p_0 = 0.5$.

$$\begin{aligned} \rho_g v &\approx \gamma(z - r), \\ \beta u &= u_0 - \gamma(z - r), \\ u &= \sigma v, \\ p_z &= -\sigma^2 v^2. \end{aligned} \tag{45}$$

Now $\sigma = S/(1 + \chi a_1 S)$, and Aldridge et al. (1991) show that Reynolds number dependence leads to

$$S \approx \left[\frac{\rho_g(\beta + b)(\beta u)^{1/4}}{(\rho_g v)^{1/4}}\right]^{1/2}, \quad b \approx 0.14; \tag{46}$$

in particular σ depends on β, $\partial\sigma/\partial\beta > 0$. Evidently p can be determined from quadrature. Since (including the subcooled pressure drop, which becomes significant as u_0 increases) $\epsilon^2 u_0^2 r - \int_r^1 p_z dz = p_0$, and σ increases with u_0, it follows that this quadrature determines r as an increasing function of u_0, for example as shown in Figure 1.

The whole dynamics of the system is then described by the equation

$$r(u_0) = \int_{\tau - \tau_{sc}}^{\tau} u_0(\theta) d\theta : \tag{47}$$

this represents a decided simplification of the original model! And yet, no serious quantitative errors have been made.

5.1 Steady state

These are simply solutions of

$$r(u_0) = \tau_{sc} u_0, \tag{48}$$

and hysteresis is possible for the sigmoidal curve illustrated in Fig. 1. This is associated with the *Ledinegg instability* (Ledinegg 1938) which occurs on regions of the Δp versus u_0 curve which have negative slope. The transition to instability occurs at the values of u_0 where $r' = \tau_{sc}$.

5.2 Stability

Perturbations to a steady solution exist proportional to $\exp(\lambda\tau)$, provided

$$r'(u)\lambda = 1 - e^{-\lambda\tau_{sc}}. \tag{49}$$

For large τ_{sc}, $\lambda \approx 1/r'$ or Re $\lambda < 0$, and Re $\lambda = 0$ only if $r' = \tau_{sc}$. In particular, this reduction does not admit oscillatory solutions. It follows that oscillatory instability must be associated with low values of τ_{sc}. Putting

$$\tau_{sc} = \epsilon\tau_0, \tag{50}$$

then

$$r(t) = \epsilon \int_{t-\tau_0}^{t} u_0(s)ds, \tag{51}$$

and we essentially ignore the sub-cooled region. Instability is then determined by the solution of the two-phase equations, that is

$$\begin{aligned}
a_1\rho_t + (\rho v)_z &= \gamma,\\
\beta_t + (\beta u)_z &= -\gamma,\\
u &= \sigma v,\\
p_z &= -u^2,\\
\rho &= 1 + a_8 p,
\end{aligned} \tag{52}$$

together with

$$\begin{aligned}
p = 1, \quad \beta u = u_0, \quad \rho v \approx 0 \quad &\text{at} \quad z \approx 0,\\
p = 0 \quad &\text{at} \quad z = 1.
\end{aligned} \tag{53}$$

We must choose u_0, in order that the third condition holds. If σ is independent of β, the equation for β uncouples, and that for ρ and v can be reduced to (with appropriate rescaling)

$$\rho_t = 1 - \frac{\partial}{\partial z}\left[\rho\left\{-\frac{\partial\rho}{\partial z}\right\}^{1/2}\right], \tag{54}$$

with ρ being prescribed at $z = 0$ and $z = 1$ ($\rho_0 > \rho_1$). A linear stability analysis shows that this nonlinear diffusion-type equation is (as we might expect) stable.

If σ depends on β (as in fact it does), then ρ, β, v are all coupled: in such circumstances, instability may be more likely.

A final possibility occurs when heating is low, and the annular flow régime is not reached. We rescale the variables as

$$t \sim 1/\epsilon, \quad u \sim \epsilon, \quad v \sim \epsilon a_1, \quad \gamma \sim \epsilon a_1, \quad \beta \sim 1/\epsilon, \quad p \sim \epsilon^2, \tag{55}$$

so that (in rescaled terms)

$$\begin{aligned}
r &= \int_{t-\tau_{sc}}^{t} u_0(\theta)d\theta,\\
[\rho(1-\beta)]_t + [\rho(1-\beta)v]_z &= \gamma,\\
\beta_t + (\beta u)_z &= 0,\\
p_z &= -u^2,\\
u &= \sigma a_1 v,\\
\rho &= 1 + a_8 p,
\end{aligned} \tag{56}$$

in the two-phase region, and the pressure drop condition is

$$1 = u_0^2 r + \int_r^1 u^2 dz. \tag{57}$$

This is almost the homogeneous model (and is if $\sigma a_1 = 1$), which is known to have oscillatory solutions (Davies and Potter 1967, Fowler 1978).

CONCLUSIONS

Through the use of non-dimensionalisation and scaling procedures, complicated two-phase flow problems can be reduced to much simpler, yet no less accurate, forms, which greatly facilitate both numerical computations and the ability to carry out analysis.

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