

The Lorenz–Krishnamurthy Slow Manifold

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ABSTRACT

The authors scale the five-mode model introduced by Lorenz and Krishnamurthy and show how explicit solutions may be obtained in the limit of small Rossby number by using the method of multiple scales. They thus obtain a characterization of the “slow manifold” of this model.

1. Introduction

In seeking to do numerical weather prediction, it is necessary to filter out the short-period gravity waves (e.g., Baer and Tribbia 1977) in favor of the longer-term behavior on the Rossby wave timescale. Lorenz (1986) and Lorenz and Krishnamurthy (1987) sought to understand the basis of this procedure by introducing a five-mode truncation of the rotating shallow-water equations. The Lorenz–Krishnamurthy model was

$$\begin{aligned} \dot{u} &= -vw + bvz - au, \\ \dot{v} &= uw - buz - av + af, \\ \dot{w} &= -uw - aw, \\ \dot{x} &= -z - ax, \\ \dot{z} &= bwv + x - az. \end{aligned} \tag{1.1}$$

In these equations, u, v, w are Rossby wave modes, while x and z are gravity wave modes. The coefficients a, b, f are positive and represent damping, coupling, and forcing, respectively. It is appropriate to take a and f as small, and Lorenz (1992) selected $a = 0.02, f = 0.2, b = 0.5$. Based on this, we define

$$a = \epsilon^2, \quad f = c\epsilon \tag{1.2}$$

and consider b, c as $O(1)$, while $\epsilon \ll 1$.

The equations (1.1) admit a steady solution, H , given by $x = z = u = v - f = w = 0$ and representative of the (unstable if $f \gg \epsilon^4$) Hadley circulation. The idea

is that solutions will generally oscillate about H . If we neglect a, f for the moment and also put $b = 0$, we see that the Rossby triad (u, v, w) and the gravity dyad (x, z) oscillate independently, the latter with a period 2π . The Rossby period depends on the amplitude, since if we put $u + iv = Ae^{i\phi/2}$, then (with $b = a = f = 0$) A is constant (and arbitrary), and ϕ satisfies $\ddot{\phi} + A^2 \sin\phi = 0$, with period (for small ϕ) $\approx 2\pi/A$.

To make sense, we need $A \ll 1$ in order that the Rossby variables evolve on a long timescale. Specifically, we now define

$$u + iv = \epsilon B e^{i\phi/2}, \quad x = \epsilon^2 \xi, \tag{1.3}$$

whence it follows that (if B and ϕ are real)

$$\begin{aligned} z &= -\epsilon^2(\dot{\xi} + \epsilon^2 \xi), \\ w &= \frac{1}{2} \dot{\phi} - \epsilon^2 \left[\frac{c}{B} \cos \frac{\phi}{2} + b(\dot{\xi} + \epsilon^2 \xi) \right], \end{aligned} \tag{1.4}$$

where the rescaling of x is also implied a posteriori by the assumption that ϕ may be slowly varying. Adopting these assumptions, we have the equations in the form

$$\begin{aligned} \ddot{\phi} + \epsilon^2 B^2 \sin\phi &= \epsilon^2 \left[2b\dot{\xi} - \left\{ \frac{c}{B} \sin \frac{\phi}{2} + 1 \right\} \dot{\phi} \right] \\ &+ \epsilon^4 \left[4b\dot{\xi} + \frac{2c}{B} \cos \frac{\phi}{2} \right] + 2b\epsilon^6 \xi, \end{aligned}$$

$$\dot{B} = \epsilon^2 \left[-B + c \sin \frac{\phi}{2} \right],$$

$$\ddot{\xi} + \xi = -\frac{b}{2} B^2 \sin\phi - \epsilon^2(\dot{\xi} + \xi) - \epsilon^4 \xi, \tag{1.5}$$

and now ξ is the gravity variable, and B and ϕ are the Rossby amplitude and phase.

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The Hadley state has $x = z = 0$, and in linearizing about H , we see from (1.1) that $x = z = 0$ is an invariant eigenspace, which also contains the (one dimensional) unstable eigenspace. Thus, the unstable manifold $W^u(H)$ (Guckenheimer and Holmes 1983) of H is contained in an invariant manifold, which is tangent at H to $x = z = 0$. Insofar as this implies that trajectories on $W^u(H)$ are slowly varying near H , one might conjecture that $W^u(H)$ is contained in a hypothetical three-dimensional manifold in the five-dimensional phase space, on which trajectories are slowly varying, and termed the "slow manifold" (Leith 1980). Jacobs (1991) showed that a unique three-dimensional invariant manifold does exist, which is tangent to $x = z = 0$ at H ; however, it was shown by Lorenz and Krishnamurthy (1987) and Lorenz (1992) that the trajectory on $W^u(H)$ develops fast oscillations after a finite time. According to Lorenz (1992), "the question as to just how the slow manifold ought to be defined seems to be presently unsettled."

More recently, Boyd (1994) has considered the issue. He considers an inviscid limit of (1.1) in which he puts $a = f = 0$; of course, this removes the natural small parameter from the system. It can be shown that the reduction corresponding to (1.5) is

$$\begin{aligned} \ddot{\phi} + \epsilon^2 B^2 \sin\phi &= 2b\epsilon^2 \xi, \\ \dot{B} &= 0, \\ \ddot{\xi} + \xi &= -\frac{b}{2} B^2 \sin\phi. \end{aligned} \tag{1.6}$$

The arbitrariness of ϵ is manifested by the arbitrary value of B , but to be consistent with (1.1), we select $B = O(1)$. Despite the degeneracy of this simplification, we shall see that it is entirely adequate to an understanding of the full model.

Boyd's main contribution was in analyzing a further simplification of (1.6), in which b was put to zero in (1.6), *only* [the first of the three (1.6) relations], so that the Rossby triad is uncoupled and drives the gravity dyad. By virtue of an extraordinary exact solution, he showed that periodic Rossby waves could excite gravity waves of exponentially small amplitude. In section 2 below, we examine the same question for the full Lorenz-Krishnamurthy (LK) system. Boyd (1995a) considered the nature of the slow manifold further and, in particular, related the study of this "fuzzy" manifold to the subject of superasymptotic perturbation methods (Berry 1991).

2. Perturbation analysis

In this section, we address three questions: (i) How should a slow manifold be usefully defined for the LK model? (ii) How does the unstable trajectory $W^u(H)$ from H evolve with time? (iii) Is it the case, as suggested by Boyd (1994), that exponentially small-amplitude gravity waves are always excited?

a. The slow manifold

The slow manifold is associated with solutions that vary slowly in time. It is then essential to associate this with the specific limit $\epsilon \rightarrow 0$ in (1.5). Specifically, we have the following ansatz: a slow manifold exists if there are slowly varying solutions $\phi(\tau), B(\tau), \xi(\tau)$ of (1.5), where $\tau = \epsilon t$, as $\epsilon \rightarrow 0$. If it exists, it is evidently of three dimensions.

We put $\tau = \epsilon t$, so that (1.5) becomes

$$\begin{aligned} \phi'' + B^2 \sin\phi &= -\epsilon \left\{ \frac{c}{B} \sin \frac{\phi}{2} + 1 \right\} \phi' \\ &+ \epsilon^2 \left\{ 2b\xi'' + \frac{2c}{B} \cos \frac{\phi}{2} \right\} + 4\epsilon^3 b\xi' + 2b\epsilon^4 \xi, \\ B' &= \epsilon \left[-B + c \sin \frac{\phi}{2} \right], \\ \xi &= -\frac{b}{2} B^2 \sin\phi - \epsilon^2 (\xi'' + \xi) - \epsilon^3 \xi' - \epsilon^4 \xi, \end{aligned} \tag{2.1}$$

where $\phi' = d\phi/d\tau$, etc., and we seek asymptotic expansions of the form

$$\phi \sim \phi_0 + \epsilon\phi_1 + \dots, \tag{2.2}$$

etc. Clearly, the loss of the highest derivatives of ξ allows us to prescribe $\phi(0), \phi'(0), B(0)$, and then $\xi(0)$ and $\xi'(0)$ are determined; therefore, the slow manifold is a three-dimensional subspace of the five-dimensional phase space.

At leading order, we have $B = B_0$ is constant,

$$\begin{aligned} \phi_0'' + B_0^2 \sin\phi_0 &= 0, \\ \xi_0 &= -\frac{b}{2} B_0^2 \sin\phi_0, \end{aligned} \tag{2.3}$$

where ϕ_0 satisfies a nonlinear pendulum equation and is periodic, with energy

$$\frac{1}{2} \phi_0'^2 - B_0^2 \cos\phi_0 = E. \tag{2.4}$$

In particular the unstable trajectory from H has $E = B_0^2$, and

$$\phi_0 = 2 \tan^{-1} \sinh B_0 \tau. \tag{2.5}$$

Since the leading-order equation is that of a conservative nonlinear oscillator, evolution of E on a slower time scale is indicated. We put $\tilde{\tau} = \epsilon\tau$, so that $\phi_0 = \phi_0(\tau, \tilde{\tau})$. From (2.1)_{1,2},

$$\begin{aligned} \frac{dE}{d\tau} &= -\epsilon \left\{ \frac{c}{B} \sin \frac{\phi}{2} + 1 \right\} \phi'^2 + \dots \\ \frac{dB}{d\tau} &= -\epsilon \left\{ B - c \sin \frac{\phi}{2} \right\}, \end{aligned} \tag{2.6}$$

so that to leading order, the method of averaging indicates that ϕ is given by (2.4), and $B \approx B(\bar{\tau})$, $E \approx E(\bar{\tau})$, where

$$\frac{dB}{d\bar{\tau}} = -B, \quad \frac{dE}{d\bar{\tau}} = -\overline{\phi'^2}/B, \quad (2.7)$$

where $\overline{\phi'^2}$ is the average of ϕ' over a period of (2.4) (note that $\overline{\sin(\phi/2)} = \overline{\phi'^2} \sin(\phi/2) = 0$ through the symmetry of ϕ); hence, to leading order, E and B decay to zero.

Thus, it seems that an asymptotic description of trajectories on a slow manifold can be successfully developed. The slow manifold itself is defined iteratively through solving (2.1)₃ for ξ . In particular, H lies on the slow manifold.

The above analysis is purely formal (and abrupt), and some comment is in order on the nature of the expansions, and what is likely to happen at higher order, and at longer time. First, the neglect of $\epsilon^2 \xi''$ in (2.1)₂ is a singular approximation. In particular, this precludes us from satisfying initial conditions for ξ and ξ' . This often is resolved by the existence of a boundary layer; here, however, the neglected term $-\epsilon^2 \xi''$ indicates a WKB-type approximation, which simply represents the fact that, more generally, ξ will have a sinusoidal component $\alpha \cos(t + \theta)$, but as we are explicitly seeking the slow manifold in which this component is absent (to all algebraic orders of ϵ), we see that the regular expansion (2.2) is sufficient to compute this.

The right-hand sides of equations (2.1) are regular functions provided $B \neq 0$, and thus the description of the solutions above is valid until B becomes small. In other words, ϕ oscillates on the slow Rossby timescale $\tau = \epsilon t$, with slowly varying amplitude and phase described by (2.4). In fact, writing this equation as

$$\frac{1}{2} \frac{\phi'^2}{B^2} + (1 - \cos \phi) = \frac{E + B^2}{B^2}, \quad (2.8)$$

we see that the amplitude is controlled by $(E + B^2)/B^2$, with the amplitude tending to zero if $(E + B^2)/B^2 \rightarrow 0$, and becoming homoclinic [as in (2.5)] if $(E + B^2)/B^2 \rightarrow 2$. The period of the oscillation (on the slow $\tau = \epsilon t$ timescale) is $O(1/B)$. Consideration of (2.7) indicates that when B becomes small $(E + B^2)/B^2$ rapidly approaches 2. Therefore, for small B , the Rossby wave variable ϕ will approach the homoclinic trajectory (2.5). However, it is clear that if this happens the averaging procedure breaks down, and, in particular, the derivation of (2.7) from (2.6) becomes invalid.

To go beyond this, we note that a distinguished limit $B = \epsilon^{1/2} \bar{B}$, $\phi = \epsilon^{1/2} \bar{\phi}$, $\tau = \bar{\tau}/\epsilon^{1/2}$ causes a further approximate reduction to be made when B is small. This is

$$\bar{\phi}'' + \bar{B}^2 \left(\bar{\phi} - \frac{2c}{\bar{B}^3} \right) = -\epsilon^{1/2} \left[\frac{c\bar{\phi}}{2\bar{B}} + 1 \right] \bar{\phi}',$$

$$\bar{B}' = \epsilon^{1/2} \left[-\bar{B} + \frac{c\bar{\phi}}{2} \right], \quad (2.9)$$

where $\bar{B}' = d\bar{B}/d\bar{\tau}$, and we would deduce that $\bar{\phi}$ oscillates on the timescale $\bar{\tau} = O(1)$, that is, $t = O(1/\epsilon^{3/2})$, while the amplitudes of $\bar{\phi}$ and \bar{B} relax to the steady values

$$\bar{B} = c^{1/2}, \quad \bar{\phi} = 2/c^{1/2}, \quad (2.10)$$

which represent the final stable steady state of the system. This evolution is also observed numerically (see below).

b. Development of $W^u(H)$

How does this expansion method relate to the unstable manifold from H ? The only difference is that the averaging method must be modified when $W^u(H)$ passes close to H ; this can be done following, for example, Fowler (1984). However, as pointed out by Lorenz and Krishnamurthy (1987) and Boyd (1994), passage near a stationary point induces gravity waves of exponentially small (in ϵ) amplitude. Specifically, we have, for $\tau \sim O(1)$

$$\phi \sim 2 \tan^{-1} \sinh B_0 \tau, \quad B \sim B_0 \text{ (constant)}, \quad (2.11)$$

and thus, from (1.6),

$$\ddot{\xi} + \xi \approx -b^2 B \operatorname{sech} B \tau \tanh B \tau, \quad (2.12)$$

with solution ($\xi \rightarrow 0$ as $t \rightarrow -\infty$)

$$\xi = \frac{b^2}{\epsilon} \left[\int_{-\infty}^t \operatorname{sech} \epsilon B s \cos s ds \cos t + \int_{-\infty}^t \operatorname{sech} \epsilon B s \sin s ds \sin t \right], \quad (2.13)$$

and the difference between $t \rightarrow -\infty$ and $t \rightarrow +\infty$ is

$$\Delta \xi = \frac{b^2}{\epsilon} \int_{-\infty}^{\infty} \operatorname{sech} \epsilon B s \cos s ds \cos t \sim \frac{2\pi b^2}{B\epsilon^2} \exp\left(\frac{-\pi}{2B\epsilon}\right) \cos t. \quad (2.14)$$

In an asymptotic sense, these exponentially small gravity waves are ‘‘beyond all orders’’ of any normal perturbation expansion. Nevertheless, they can practically be quite large. If $b = 0.5$, $B = 1$, $\epsilon = 0.14$, then $\|\Delta \xi\| \sim 1.2 \times 10^{-3}$; however, if $B = 3$, then $\|\Delta \xi\| \sim 0.65$.

c. Exponential asymptotics

As indicated by Boyd (1994), this result is not constrained by the approach toward H , since the general solution of the periodic solutions of (2.3)₁ can be written as a superposition of the heteroclinic solutions. With

$$U + iV = Be^{i\phi/2}, \quad W = \phi'/2, \quad (2.15)$$

then the periodic solutions of (2.3)₁ are

$$\begin{aligned} U &= \alpha \sum_{n=-\infty}^{\infty} \operatorname{sech}[\alpha(\tau - nP)], \\ V &= -\alpha \sum_n (-1)^n \tanh[\alpha(\tau - nP)], \\ W &= -\alpha \sum_n (-1)^n \operatorname{sech}[\alpha(\tau - nP)], \end{aligned} \quad (2.16)$$

where $\alpha = \alpha(B_0)$ (and $\alpha \rightarrow B_0$ as $P \rightarrow \infty$) and $P = P(E, B_0)$ is the (Rossby) period. Since ξ satisfies, from (1.6)₃,

$$\ddot{\xi} + \xi = -bUV = R(\tau), \quad (2.17)$$

it is clear from the preceding discussion that exponentially small gravity modes are generated by the slow variation of the Rossby modes. However, as B decreases (and thus also u, v, w), the amplification with time diminishes very rapidly.

More generally, the solution of (2.17) can be written as

$$\xi = \frac{1}{\epsilon} \operatorname{Im} \int_{-\infty}^{\tau} R(U) \exp\left[\frac{i(\tau - U)}{\epsilon}\right] dU. \quad (2.18)$$

A method for extracting the exponentially small ‘‘radiative’’ component of this solution has been given by Boyd (1995b). We continue R as an analytic function in the complex plane. For example, we can do this by defining

$$F(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{R(U)dU}{U - z}, \quad (2.19)$$

and note that $F_+(z)$ ($=F, \operatorname{Im}z > 0$) and $F_-(z)$ ($=F, \operatorname{Im}z < 0$) are respectively holomorphic in $\operatorname{Im}z > 0$ and $\operatorname{Im}z < 0$ (note that $F_-(z) = -\overline{F_+(\bar{z})}$). We continue F_+ and F_- analytically into $\operatorname{Im}z < 0$ and $\operatorname{Im}z > 0$ (note that the continued functions F_+ and F_- are *not* equal); then $R(z) = (F_+(z) - F_-(z))/2$ is an analytic function that equals the prescribed $R(\tau)$ on $z = \tau \in \mathbb{R}$. For example (trivially), if $R = 1/(1 + \tau^2)$, then by direct contour integration we find $F_+ = i/(z + i)$, $F_- = i/(z - i)$, and $R = 1/(1 + z^2)$ (as we expect).

A steepest-descent path for (2.18) through $U = \tau$ is given by $U = \tau - is, s > 0$, and, providing R tends to zero at ∞ in $\operatorname{Im}z < 0$, we can deform the contour in the complex U plane, so that (2.18) becomes

$$\begin{aligned} \xi = \operatorname{Re} \left[\int_0^{\infty} R(\tau - icS)e^{-S} dS \right. \\ \left. + \frac{2\pi}{\epsilon} \sum_j R_j e^{-s_j/\epsilon} e^{i(t-t_j)} \right], \end{aligned} \quad (2.20)$$

where R has poles at $U = \tau_j - is_j, \tau_j = \epsilon t_j$, with $t_j < t$, and the corresponding residues are R_j . Application

of Laplace’s method to the integral gives the slow manifold

$$\xi \sim R(\tau) - \epsilon^2 R''(\tau) \dots, \quad (2.21)$$

and we see that when the vertical complex time contour sweeps past a (complex) singularity of $R(\tau)$ it picks up an exponentially small gravity wave. Since, if R has a singularity at z_i , it also has one at \bar{z}_i (by definition of F_{\pm}) and since the only entire function R that is zero at infinity is zero, we see that *any* R that can be analytically continued so that $R \rightarrow 0$ at ∞ will have a singularity in the lower complex half-plane and, thus, eventually produce gravity waves.

When there is no finite singularity, the situation is slightly different. For example, if $R(\tau) = e^{-\tau^2}$, then its continuation e^{-z^2} has no finite singularities but is singular at infinity. The steepest descent contour cannot proceed to $\tau - i\infty$, and one has to consider the constant phase paths of the whole integrand; thus,

$$\xi = \frac{1}{\epsilon} \operatorname{Im} \int_{-\infty}^{\tau} \exp\left[\frac{\phi(U)}{\epsilon}\right] dU, \quad (2.22)$$

where $\phi(U) = i(\tau - U) - \epsilon U^2$. The constant phase path through $U = \tau$ asymptotes to $-\infty - i/2\epsilon$ if $\tau < 0$ but to $+\infty - i/2\epsilon$ if $\tau > 0$. In the latter case, deformation of the contour requires inclusion of a steepest-descent path $U = -(i/2\epsilon) + s$ through the saddle point of ϕ at $U = -i/2\epsilon$, and as τ increases through zero, it picks up the saddle point contribution, which is $(\sqrt{\pi}/\epsilon)e^{-1/4\epsilon^2} \operatorname{sint}$. This can be compared with the effect of $R(\tau) = 1/(1 + \tau^2)$, where the sweep past $\tau = 0$ produces the term $-(\pi/\epsilon)e^{-1/\epsilon} \operatorname{sint}$.

The generation of exponentially small terms in (2.17) has been studied before, for example by Mahony (1972) and Boyd (1991), and is part of the growing subject of exponential asymptotics (Segur et al. 1991). In particular, methods have been developed for the asymptotic development beyond all orders of expressions such as (2.18) (Berry 1991; Boyd 1995b), although their direct construction from the governing differential equation is problematic.

3. Discussion

Lorenz (1992) questioned whether a slow manifold actually existed for his five-mode primitive-equation model, based on the development of gravity modes on $W^u(H)$. Boyd (1994) suggested that this would generally be the case, with exponentially small gravity terms, based on a further-simplified model. What we have shown is that this conclusion applies also to the full Lorenz–Krishnamurthy model, in the limit where $\epsilon = a^{1/2} \rightarrow 0$ while $f/\epsilon = O(1)$. The use of a multiple-timescale perturbation procedure is central to this conclusion. Our analysis indicates that the Rossby wave variables, suitably scaled, are of amplitude $O(\epsilon)$ and oscillate equivalently to a pendulum on a slow time-

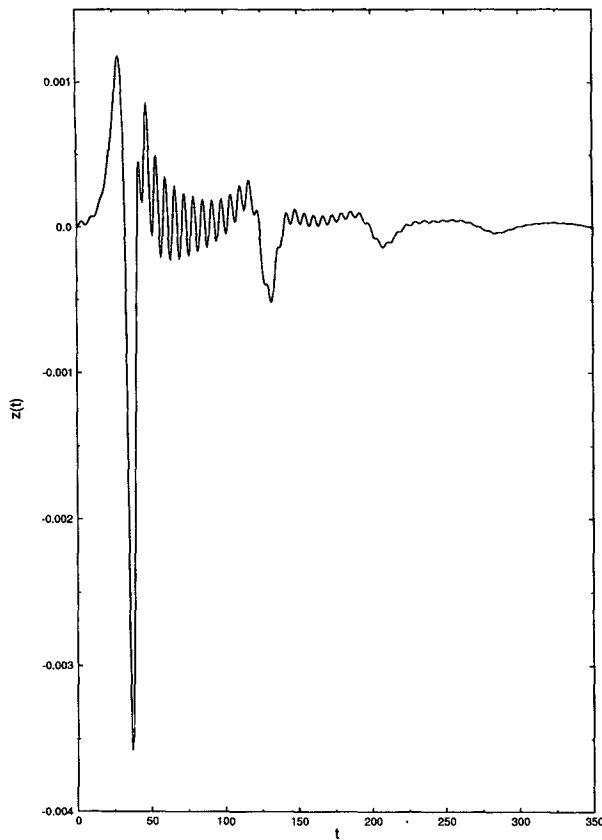


FIG. 1. Solution of (1.1) starting from $u = x = z = 0$, $v = 0.2$, $w = -0.001$, near the unstable Hadley fixed point H . The parameter values used are $b = 0.5$, $a = 0.02$, $f = 0.2$. After the first return near H , we see the generation of a gravity wave mode of amplitude (by eye) ~ 0.0004 . The theoretical amplitude [from (2.11), using (1.1)] is $\Delta z \sim (2\pi b^2/B) \exp(-\pi/2B\epsilon)$, which, with $B = c = 1.414$, is $\Delta z \sim 0.00043$. This figure shows the damping of the Rossby wave amplitude on the timescale $t \sim 1/\epsilon^2$ ($\tau \sim 1/\epsilon$). In seeking to illustrate our results, we are led to consider an initial condition close to the top of the "pendulum" [see (2.1)], i.e., $\phi \approx \pi$, $\phi' \approx 0$. This permits the switching on of the gravity waves to be easily seen when $\epsilon = 0.14$, but the near-heteroclinic nature of the trajectories causes the subsequent pulses to be much smaller, and effectively, what we see is the first gravity wave being gradually damped.

scale $t \sim 1/\epsilon$. Passage through each oscillation generates an exponentially small gravity wave of period $t = O(1)$. These oscillations are out of phase with each other, so that there is no secular buildup in time. Over a longer timescale $t \sim O(1/\epsilon^2)$, the Rossby amplitudes decrease exponentially (like $\exp[-O(\epsilon^2 t)]$), and the oscillation periods grow inversely. For times longer than this, the Rossby periods lengthen to $t = O(1/\epsilon^{3/2})$,

and the amplitudes decay to a final stable equilibrium state over a timescale $t = O(1/\epsilon^2)$. This later development is, however, not of principal concern here.

In Fig. 1 we show a representative solution that indicates the sudden acquisition of gravity modes as the nonlinear pendulum swings through an angle 2π . This figure also reveals the damping over the slower ($t \sim 1/\epsilon^2 = 50$) timescale, and we see that the strength of the fast gravity waves is damped similarly. See the figure caption for further details. Our conclusion thus follows that of Boyd (1994): a "slow" manifold can be constructed via formal power series in the "Rossby number" ϵ . In the vicinity of the fixed point H , these expansions are limited by the fact that trajectories, particularly those which approach H , "pick up" a dressing of gravity waves, whose amplitude varies exponentially with $-1/\epsilon$. While formally these gravity modes are smaller than $O(\epsilon^N)$ for any N , in practice, they can be a significant numerical contaminant.

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