Glaciers and ice sheets

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1 Dynamic phenomena

Glaciers are huge and slow moving rivers of ice which exist in various parts of the world: Alaska, the Rockies, the Alps, Spitsbergen, China, for example. They drain areas in which snow accumulates, much as rivers drain catchment areas where rain falls. Glaciers also flow in the same basic way that rivers do. Although glacier ice is solid, it can deform by the slow creep of dislocations within the lattice of ice crystals which form the fabric of the ice. Thus, glacier ice effectively behaves like a viscous material, with, however, a very large viscosity: a typical value of ice viscosity is 1 bar year (in the metre-bar-year system of units!). Since 1 bar = 10^5 Pa, 1 year $\approx 3 \times 10^7$ s, this is a viscosity of some 10^{12} Pa s, about 10^{15} times that of water. As a consequence of their enormous viscosity, glaciers move slowly - a typical velocity would be in the range 10-100 m y^{-1} (metres per year), certainly measurable but hardly dramatic. More awesome are the dimensions of glaciers. Depths of hundreds of metres are typical, widths of kilometres, lengths of tens of kilometres. Thus glaciers can have an important effect on the human environment in their vicinity. They are also indirect monitors of climate: for example, many lithographs of Swiss glaciers show that they have been receding since the nineteenth century, a phenomena thought to be due to the termination of the 'little ice age' in the middle ages.

Where glaciers are the glacial equivalent of rivers, i.e. channeled flow, *ice sheets* are the equivalent of droplets, but altogether on a grander scale. When an entire continent, or at least a substantial portion thereof, has a polar climate, then snow accumulates on the uplands, is compressed into ice, and flows out to cover the continent, much as a drop of fluid on a table will spread under the action of gravity. However, whereas droplets can reach a steady state through the contractile effect of surface tension, this is not relevant to large ice sheets. In them, equilibrium can be maintained through a balance between accumulation in the centre and *ablation* at the margins. This can occur either through melting of the ice in the warmer climate at the ice margin, or through calving of icebergs. (Indeed, the same balance of accumulation at higher elevations with ablation at lower elevations is responsible for the normal quasi-steady profile of valley glaciers.)

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There are two major ice sheets on the Earth, namely those in Antarctica and Greenland (the Arctic is an ocean, and its ice is sea ice, rarely more than three metres thick). They are on the order of thousands of kilometres in extent, and kilometres deep (up to four for Antarctica). They are thus, in fact, *shallow* flows, a fact which greatly facilitates the solution of mathematical models of the flow. Possibly more famous are the ice sheets which covered much of North America (the Laurentide ice sheet) and northern Europe (the Fennoscandian ice sheet) during the last ice age. Throughout the Pleistocene era (that is, the last two million years), there have been a succession of ice ages, each lasting a typical period of around 90,000 years, during which global ice sheet volume increased, interspersed with shorter (10,000 year) *interglacials*, when the ice sheets rapidly retreat. The last ice age finished some ten thousand years ago, so that we are about due for another now.

Drainage and sliding

While the motion of ice sheets and glaciers can be understood by means of viscous theory, there are some notable complications which can occur. Chief among these is that ice can reach the melting point at the glacier bed, due to frictional heating or geothermal heat input, in which case water is produced, and the ice can *slide*. Thus, unlike an ordinary viscous fluid, slip can occur at the base, and this is determined by a sliding law which relates basal shear stress τ to sliding velocity u_b and also, normally, the *effective pressure* $N = p_i - p_w$, where p_i and p_w are ice and water pressures. The determination of p_w further requires a description of the subglacial hydrology, and thus the dynamics of ice is intricately coupled to other physical processes: as we shall see, this complexity leads to some exotic phenomena.

1.2 Waves on glaciers

Just as on rivers, gravity waves will propagate on glaciers. Because the flow is very slow, they only propagate one way (downstream), and at speeds comparable to the surface speed (but slightly faster). These waves are known as surface waves, as they are evidenced by undulations of the surface: an example is shown in Fig. 1. They are examples of *kinematic* waves, driven by the dependence of ice flux on glacier depth.

A more exotic kind of wave is the 'seasonal wave'. This has no obvious counterpart in other fluid flows. It consists of (sizeable) perturbations in the surface velocity field which propagate down glacier at speeds in the order of 20-150 times the surface speed. There is no significant surface perturbation, and these waves must in fact be caused by variations of the basal sliding speed due to annual fluctuations in the basal water pressure. Although well-known and reported at the turn of the century, little attention has been paid to these

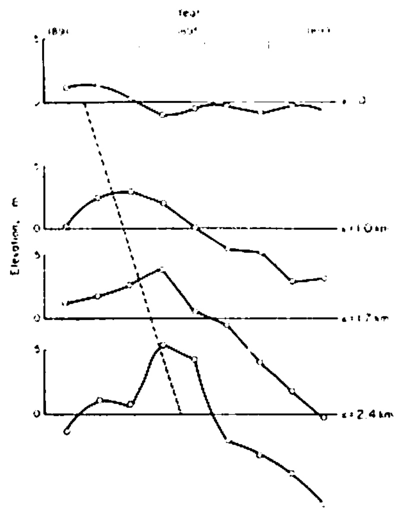

Figure 1: Changes of mean surface elevation of Mer de Glace, France, along four cross-profiles over a period of 9 years. The broken line corresponds to a wave velocity of 800 m/a. Reproduced from L. Lliboutry, IASH publ. **47**, 125-138, by permission of the International Association of Hydrological Sciences.

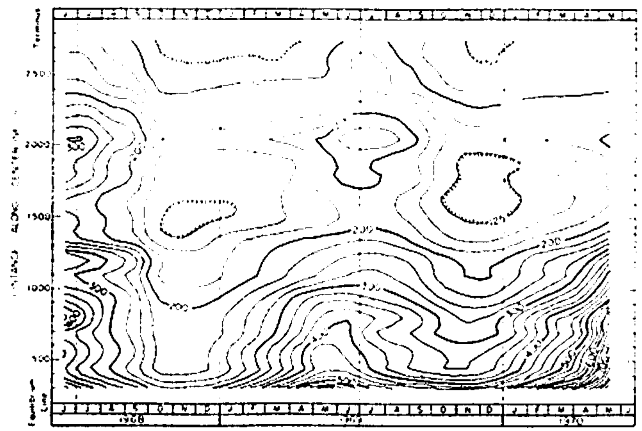

Figure 2: The measured surface speed of Nisqually Glacier, Mt.Rainier, as a function of time and distance. The contour interval is 25 mm d^{-1}. The maximum and minimum speeds occur progressively later with distance down-glacier; this represents a "seasonal wave" in the ice flow. Reproduced from Hodge (1974) by permission of the International Glaciological Society.

waves in recent years. Figure 2 shows measurements of Hodge on Nisqually Glacier which indicates the rapid passage of a velocity wave downstream.

Mention should also be made of wave ogives, although we will not deal with them in this lecture. They are bands (also known as Forbes bands) which propagate below ice-falls, and are due to the annual ablation cycle.

1.3 Surges

Perhaps the most spectacular form of wave motion is the glacier *surge*. Surges are large scale relaxation oscillations of the whole length of a glacier. They are roughly periodic, with periods on the order of 20-100 years. During a long quiescent phase, the glacier is over-extended and thin. Ice accumulation causes the glacier to thicken upstream, while the over-extended snout thins and retreats. Eventually, a critical thickness is reached, and the glacier slumps rapidly downslope again. These surges will typically last only a year or two, during which time the velocity may increase a hundred-fold. The glacier snout will then advance by several kilometres.

A typical (and much studied) example is the Variegated Glacier in Alaska. Its surge periodicity is about twenty years, while its surges last about two years. The glacier, of length twenty kilometres and depth four hundred metres, advances some six kilometres during its surge, at measured speeds of up to 65 metres per *day*. Such large velocities can only occur by basal sliding, and detailed observations during the 1982-3 surge showed that the surge was mediated by an alteration in the basal drainage system, which had the effect of raising water pressure dramatically. A dynamic model suggests, in fact, that the oscillations are caused by the competitive interaction between the basal sliding law and the hydraulics of the subglacial drainage system. When the ice is relatively thin (hence the driving shear stress is low) the drainage occurs through a network of channels incised into the ice at the glacier bed - called Röthlisberger channels. These allow effective drainage at quite low water pressures (hence high effective pressures) and thus also low velocities. At higher driving stresses, however, an instability forces the channel system to close down, and the basal water is forced into cavities which exist between the ice and bedrock protuberances (such cavities are well-known to exist). The water flow is reduced, and the sudden increase in water pressure causes a sudden increase in ice velocity - - the surge. The transition front between the linked cavity drainage system and the channel system is nucleated near the maximum depth, and propagates rapidly both upstream and downstream, at (measured) speeds on the order of hundreds of metres per hour. At the end of the surge, the channel drainage system is re-established. Figures 3 and 4 show a vertical view of Variegated Glacier in pre- and post-surge states.

Our understanding of the Variegated surges relies on the concept of drainage switch

Figure 3: Variegated Glacier at the beginning of a surge, 29 August, 1964. Photograph by Austin Post, U.S. Geological Survey.

Figure 4: Variegated Glacier at the end of a surge, 22 August, 1965. Photograph by Austin Post, U.S. Geological Survey.

between channelised flow and linked cavities, implicitly for ice flowing over (hard) bedrock. A rather different situation appears to operate in Trapridge Glacier, another well-studied surging glacier in the Yukon. Here the glacier is cold (unlike the temperate (at the melting point) Variegated) and rests on a thick ($\sim$ 6 metres) layer of *till* — a rather sludgy mixture of water and rock particles. The till has a bimodal distribution, consisting of coarse pebbles and boulders interspersed with finer clay material. Till is produced by the erosion of brittle underlying bedrock, and is evacuated by the slow motion of the ice downstream.

The sequence of events which appear to be occurring as the Trapridge thickens is that the basal ice reaches melting point (and the till thaws). The till is then deformable, and the surge will be associated with a slump of the ice riding over the soft till; it therefore depends on the rheology of the till. While this is presently uncertain, it is clear that water pressure in unconsolidated sediments has a major effect on till rheology. As water pressure increases, the till dilates and its effective viscosity is reduced. Thus, surging here could also be explained by a sudden alteration of drainage mechanism. Quite what this could be, however, is unclear. At the moment, water is evacuated from the base through the till to a subglacial aquifer, and emerges at an outlet stream in front of the glacier. One possibility is that, as the ice thickens and more basal melt water is produced, the water pressure at the base necessary to evacuate the meltwater gradually increases towards overburden pressure, with an associated increase in basal till deformation. One might expect a runaway phenomena, as the accelerating ice flow produces yet more melt water, a process eventually relieved by the surge and the re-freezing of the base due to thinning of the ice. However, the long-awaited next surge of Trapridge has not yet occurred and such mechanisms are highly speculative.

1.4 Ice streams

Although ice sheets also flow under the horizontal pressure gradients induced by the glaciostatic pressures beneath their sloping surfaces, they rest on essentially unsloping bases, and therefore have no advective component in their dynamics. Thus ice sheets do not, at least on the large scale, exhibit wave motion: the governing equations are essentially diffusive in character. On a more local scale, however, ice sheets have interesting phenomena of their own.

Principal among these may be ice streams. Ice sheets do not tend to drain uniformly to the margin from their central accumulation zones, but rather the outflows from catchment areas are concentrated into fast-moving ice streams. Examples are the Lambert Glacier in Antarctica and Jakobshavn in Greenland, a fast-moving (8 metres per day) outlet glacier. These ice streams gain their speed by carving out deep channels through which they flow.

Indeed, there is an obvious positive feedback here. The deeper an ice flow, the larger the driving basal stress, and the warmer the basal ice (due to increased frictional heat and decreased conductive heat loss), and hence the softer the ice. Both of these effects contribute to enhanced ice flow, which explains the formation of such channels, since the erosive power of ice flow increases with the basal velocity and the basal shear stress. Indeed, flow of ice over a plane bed is subject to a lateral instability (much as overland flow of surface water is unstable to the formation of rills and gullies).

A similar kind of mechanism may operate when ice flows over deforming sediments, as in the Siple Coast of West Antarctica. Here, it is found that the flow is concentrated into five ice streams, A, B, C, D, and E, which are characterised by their heavily crevassed appearance. The flow in these ice streams is very rapid and is due to basal sliding over the underlying sediment (except for ice stream C, which appears to have 'switched off' several hundred years ago). Measurements on ice stream B indicate that the basal water pressure is high (within 0.4 bar of the overburden pressure), and that it is underlain by some eight metres of saturated till. A similar instability to that concerning ice flow over hard bedrock can explain the streaming nature of the flow. Where ice flow is larger, there is increased water production. *If* the drainage system is such that increased water production leads to increased water pressure (as one might expect, e.g. for a Darcy flow), then the higher water pressure decreases the viscosity of the till, and hence enhances the ice flow further. This is an instability mechanism, and the limiting factor is that when ice flow increases, there is increased heat loss from the base, which acts to limit the increase of melt rate. Although this mechanism is viable, it has not yet been shown that it works.

1.5 Jökulhlaups

It will be clear by now that basal water is tremendously important in determining the nature of ice flow. Equally, the basal water system can fluctuate independently of the overlying ice dynamics, most notably in the outburst floods called jökulhlaups. In Iceland, in particular, these are associated with volcanoes under ice caps, where high rates of geothermal heat flux in the confines of a caldera cause a growing subglacial lake to occur, which eventually overflows, propagates downglacier, and releases enormous floods over the southern coastal outwash plains. These floods carry enormous amounts of volcanic ash and sediments, which create vast beaches of black ash. Despite their violence, the ice flow is hardly disturbed. Jökulhlaups are essentially internal oscillations of the basal drainage system. They are initiated when the rising subglacial lake level causes leakage over a topographic rim, and the resultant water flow leads to an amplifying water flow by the following mechanism. Water flow through a channel in ice enlarges it by meltback of the walls due to frictional heating. The increased channel size allows increased flow, and thus further enlargement. The process is limited by the fact that the ice tends to close up the channel due to the excess overburden pressure over the channel water pressure, and this is accentuated when the channel is larger. In effect, the opening of the valve by the excess lake pressure is closed by the excess ice pressure. These floods occur more or less periodically, every five to ten years in the case of the best known, that of Grimsvötn under Vatnajökull in South-east Iceland.

1.6 Notes

The best source for general information about glaciers and ice sheets is the book by Paterson (1994).

An early discussion of surface waves is by Finsterwalder (1907). The modern theory is largely due to Nye (1960), who analyses linear waves; a nonlinear analysis is given by Fowler and Larson (1980b). Seasonal waves are discussed by Deeley and Parr (1914) and more recently by Hodge (1974). Wave ogives are lucidly discussed by Waddington (1986).

The surge on Variegated Glacier is discussed by Kamb et al. (1985), and theoretical descriptions are given by Kamb (1987) and Fowler (1987a). Observations of Trapridge Glacier are described by Clarke et al. (1984). The issue of the Journal of Geophysical Research in which Fowler's (1987a) article appears is a collection of articles on fast glacier flow, including both ice streams, surging glaciers, and tidewater glaciers. For a discussion of the dynamics of ice stream B in Antarctica, see Engelhardt et al. (1990). The basic theory of jökulhlaups is due to Nye (1976).

2 Mathematical models

2.1 The basic shallow ice approximation

We consider first motions of a glacier in a (linear) valley. We take the x axis in the direction of the valley axis, z upwards and transverse to the mean valley slope, and y across stream. The basic equations are those of mass and momentum conservation, which for an incompressible ice flow (neglecting inertial terms) are

$$\begin{aligned} \nabla.\mathbf{u} &= 0, \\ 0 &= -\nabla p + \nabla.\boldsymbol{\tau} + \rho\mathbf{g}, \end{aligned} \tag{2.1}$$

where $\mathbf{g}$ is the gravity vector, p is the pressure, and $\boldsymbol{\tau}$ is the deviatoric part of the stress tensor. The usual relation between stress and strain rate is

$$\tau_{ij} = \eta\dot{\varepsilon}_{ij}, \tag{2.2}$$

where η is the effective viscosity, and $\dot{\varepsilon}_{ij}$ is the strain rate

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.3}$$

The most common choice of flow law is known as Glen's law, that is

$$\dot{\varepsilon}_{ij} = \frac{1}{2}A(T)\tau^{n-1}\tau_{ij}, \tag{2.4}$$

where the second stress invariant is given by $2\tau^2 = \tau_{ij}\tau_{ij}$ (using the summation convention) and $A(T)$ is a temperature dependent rate factor which causes A to vary by about three orders of magnitude over a temperature range of 50 K: variation of A is thus significant for ice sheets (which may be subject to such a temperature range), but less so for glaciers.

If we adopt the configuration shown in Fig. 5, then $\mathbf{g} = (g\sin\alpha, 0, -g\cos\alpha)$, where α is the mean valley slope downhill.

A major simplification ensues by adopting what has been called the *shallow ice approximation*. It is the lubrication theory idea that the depth $d \ll$ the glacier length l, and is adopted as follows. We scale the variables by putting

$$\begin{aligned} &u \sim U;\ v, w \sim \delta U; \\ &x \sim l;\ y, z \sim d; \\ &\tau_{13}, \tau_{12} \sim [\tau]; \\ &p - p_a - (\rho g\cos\alpha)(\zeta - z) \sim \delta[\tau]; \\ &\tau_{11}, \tau_{22}, \tau_{33}, \tau_{23} \sim \delta[\tau]. \end{aligned} \tag{2.5}$$

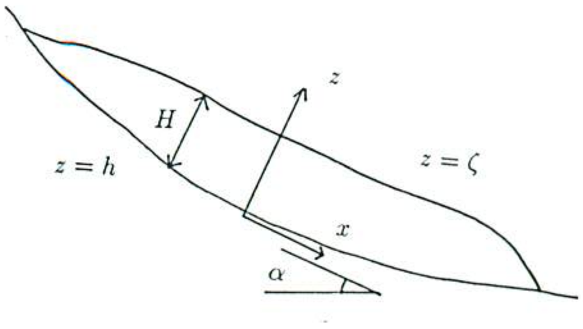

Figure 5: typical profile of a valley glacier

where $z = \zeta(x, y, t)$ is the top surface of the ice. Here

$$\delta = d/l \tag{2.6}$$

is the aspect ratio and we anticipate $\delta \ll 1$. The choice of d and $[\tau]$ has to be determined self consistently; we choose l from the given spatial variation of accumulation rate, and we choose U via $\delta U = [a]$, which is a typical accumulation rate. If we choose $[\tau] = \rho g d \sin\alpha$, and define

$$\varepsilon = \delta \cot\alpha, \tag{2.7}$$

then the scaled momentum equations are

$$\begin{aligned}
\frac{\partial\tau_{12}}{\partial y} + \frac{\partial\tau_{13}}{\partial z} &= -1 + \varepsilon\frac{\partial\zeta}{\partial x} + \delta^2\left(\frac{\partial p}{\partial x} - \frac{\partial\tau_{11}}{\partial x}\right), \\
\frac{\partial\zeta}{\partial y} &= \frac{\delta^2}{\varepsilon}\left[-\frac{\partial p}{\partial y} + \frac{\partial\tau_{12}}{\partial x} + \frac{\partial\tau_{22}}{\partial y} + \frac{\partial\tau_{23}}{\partial z}\right], \\
\frac{\partial p}{\partial z} &= \frac{\partial\tau_{13}}{\partial x} + \frac{\partial\tau_{23}}{\partial y} + \frac{\partial\tau_{33}}{\partial z}.
\end{aligned} \tag{2.8}$$

To get some idea of typical magnitudes, use values $d \sim 100$ m, $l \sim 10$ km, $\tan\alpha \sim 0.1$; then $\delta \sim 10^{-2}$, $\varepsilon \sim 10^{-1}$, so that to leading order $\zeta = \zeta(x, t)$ and

$$\frac{\partial\tau_{12}}{\partial y} + \frac{\partial\tau_{13}}{\partial z} = -1 + \varepsilon\frac{\partial\zeta}{\partial x}; \tag{2.9}$$

we retain the ε term for the moment.

The final relation to choose d (and hence also $[\tau]$) is determined by effecting a balance in the flow law. If the viscosity scale is $[\eta]$, then we choose

$$[\tau] = [\eta]U/d. \tag{2.10}$$

For example, choosing $[\eta]$ via Glen's law, we would have $[\eta] = 2/\{A[\tau]^{n-1}\}$, from which we find

$$d = \left[\frac{2[a]l}{A(\rho g \sin\alpha)^n}\right]^{1/(n+2)}, \tag{2.11}$$

which leads, with sensible choices of $A, l, [a], n$ to values of d comparable to those observed ($d \sim 100$ m).

At leading order, the important components of the flow are then

$$\tau_{13} = \eta\frac{\partial u}{\partial z}, \quad \tau_{12} = \eta\frac{\partial u}{\partial y}, \tag{2.12}$$

and if η depends on the (scaled) second invariant τ, then to leading order

$$\tau^2 = \tau_{13}^2 + \tau_{12}^2; \tag{2.13}$$

for example, Glen's flow law would be

$$\eta = [A(T)\tau^{n-1}]^{-1}, \tag{2.14}$$

where $A(T)$ would be a scaled rate factor. We see that

$$\tau = \eta|\nabla u|, \tag{2.15}$$

where $\nabla = (\partial/\partial y, \partial/\partial z)$, and for Glen's law (with $A = 1$),

$$\eta = |\nabla u|^{-(n-1)/n} \tag{2.16}$$

(note $n = 1$ for a Newtonian flow; Glen's law usually assumes $n = 3$); the determination of velocity in a glacier then reduces to the elliptic equation for u (putting $\varepsilon = 0$)

$$\nabla . [\eta\{|\nabla u|\}\nabla u] = -1 \tag{2.17}$$

with appropriate boundary conditions for no slip at the base being $u = 0$ on $z = h$, $\partial u/\partial z = 0$ on $z = \zeta$, and ζ is determined through a prescribed mass flux, $\int u\,dy\,dz = Q$ (given).

Most studies of wave motion ignore lateral variation, and in this case (with $\tau_{13} = 0$ on $z = \zeta$) (2.9) gives

$$\tau_{13} = (1 - \varepsilon\zeta_x)(\zeta - z), \tag{2.18}$$

and Glen's law is

$$\frac{\partial u}{\partial z} = A(T)\tau_{13}^n = A(T)[1 - \varepsilon\zeta_x]^n(\zeta - z)^n. \tag{2.19}$$

If $A = 1$ is constant, then two integrations of (2.19) give the ice flux $Q = \int_h^\zeta u\,dz$ as

$$Q = u_b H + (1 - \varepsilon\zeta_x)^n \frac{H^{n+2}}{n+2}, \tag{2.20}$$

where $H = \zeta - h$ is the depth, and u_b is the sliding velocity. Integration of the mass conservation equation, together with an appropriate kinematic surface boundary condition, then leads to the integral conservation law,

$$\frac{\partial H}{\partial t} + \frac{\partial Q}{\partial x} = a, \tag{2.21}$$

where a is the dimensionless accumulation rate. (2.21) is an equation of convective diffusion type, with the diffusive term being that proportional to ε.

Note that if transverse variations were to be included, we should solve

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = a, \tag{2.22}$$

where A is the cross sectional area, and Q would be given by $Q = \int_A u\,dS$, where u solves (2.17) in A, together with appropriate boundary conditions.

Ice sheets

A model for ice sheets can be derived in much the same way — typical aspect ratios are 10^{-3} — but there is no 'downslope' gravity term $\rho g \sin\alpha$ (effectively $\alpha = 0$), and the appropriate balance determines the driving shear stress at the base in terms of the surface slope. Effectively, the advection term is lost and $\varepsilon = 1$. Another difference is that $x \sim y \sim l$ ($\sim$ 3000 km) while $z \sim 3$ km is the only small position variable. An isothermal version of (2.21) is then (with $\nabla = (\partial/\partial x, \partial/\partial y)$)

$$H_t = \nabla.\left[\left\{\frac{|\nabla\zeta|^{n-1}H^{n+2}}{n+2}\nabla\zeta\right\} + H\mathbf{u}_b\right] + a, \tag{2.23}$$

and is a nonlinear diffusion type equation for H, since $\zeta = H + h$. The sliding velocity $\mathbf{u}_b$ is apparently a convective term, but in fact the sliding law usually has $\mathbf{u}_b$ in the direction of shear stress, whence $\mathbf{u}_b \propto -\nabla\zeta$, and this term also is diffusive.

Temperature

Although the isothermal models are mathematically nice, they are not quantitatively very realistic. For a glacier, probably the neglect of variation of the rate parameter $A(T)$ in the flow law is as important as the assumption of a two-dimensional flow, although the possible coupling of temperature to water production and basal sliding is also significant. For ice sheets, temperature variation is unquestionably significant, and cannot in practice be neglected.

With variables scaled as in the previous section for a shallow shear flow, the temperature equation for an ice sheet may be written approximately as

$$\frac{dT}{dt} = \alpha\tau^2/\eta + \beta T_{zz}, \tag{2.24}$$

where $T - T_m$ is scaled with ΔT (a typical surface temperature below melting point). The derivative dT/dt is a material derivative. The stress invariant τ is related to the horizontal velocity $\mathbf{u} = (u, v)$ by

$$\tau \approx \eta\left|\frac{\partial \mathbf{u}}{\partial z}\right| = (\zeta - z)|\nabla\zeta|, \tag{2.25}$$

since the horizontal stress vector $\boldsymbol{\tau} = (\tau_{13}, \tau_{23})$ satisfies

$$\boldsymbol{\tau} = \eta\frac{\partial \mathbf{u}}{\partial z} = -(\zeta - z)\nabla\zeta. \tag{2.26}$$

(For an ice sheet, this relation is derived as for (2.18), but the downslope term 1 is absent, scales are chosen so that $\varepsilon = 1$, and (2.26) represents the two (horizontal) dimensional version.)

The parameters α and β are given by

$$\alpha = \frac{gd}{c_p\Delta T}, \quad \beta = \frac{\kappa}{d[a]}, \tag{2.27}$$

where d is the depth scale, c_p is specific heat, g is gravity, κ is thermal diffusivity, $[a]$ is accumulation rate. Typical sorts of value for Antarctica are $\alpha \sim 0.3$, $\beta \sim 0.12$. We see that viscous heating (the α term) is liable to be significant, while thermal conduction is small or moderate. In addition, a scaled geothermal heat flux condition at the base (cf. $T < 0$ there) is $\partial T/\partial z \approx -\Gamma$, where $\Gamma \sim 1.5$ is a typical value. Temperature variation is likely to be significant, while the rate factor in the flow law can be modelled as $A \sim \exp(\gamma T)$, with $\gamma \sim 11$ for a temperature range of 50 K.

The temperature equation for a valley glacier is the same as (2.26), although with the previous scalings, (2.25) is corrected by simply replacing $|\nabla\zeta|$ in the last expression by $(1 - \varepsilon\zeta_x)$. Although the scales are different, typical values of α and β are $\alpha \sim 0.25$, $\beta \sim 0.33$, and thus of significance. On the other hand, geothermal heat is of less importance.

2.2 Using the equations

Nonlinear diffusion

For flow over a flat base, $h = 0$, with no sliding, the isothermal ice sheet equation (2.23) is just

$$H_t = \nabla . \left[\frac{H^{n+2}|\nabla H|^{n-1}}{n+2} \nabla H \right] + a, \tag{2.28}$$

which for Glen's flow law would have $n = 3$. This is a degenerate nonlinear diffusion equation, and has singularities at ice margins ($H = 0$) or divides (where $\nabla H/|\nabla H|$ is discontinuous). In one space dimension, we have near a margin $x = x_m(t)$ where $a < 0$ (ablation),

$$\begin{aligned} H &\sim (a/\dot{x}_m)(x_m - x) \text{ if } \dot{x}_m < 0 \text{ (retreat)}, \\ H &\sim \left(\frac{2n+1}{n}\right)^{\frac{n}{2n+1}} [(n+2)\dot{x}_m]^{\frac{1}{2n+1}} (x_m - x)^{\frac{n}{2n+1}} \text{ if } \dot{x}_m > 0 \text{ (advance)}. \end{aligned} \tag{2.29}$$

This is the common pattern for such equations: margin retreat occurs with finite slope, while for an advance, the slope must be infinite. Consequently, there is a waiting time between a retreat and a subsequent advance, while the front slope grows.

Near a divide $x = x_d$, where $H_x = 0$ and $a > 0$, H is given by

$$H \sim H_0(t) - \left(\frac{n}{n+1}\right) \left[\frac{(n+2)(a - \dot{H}_0)}{H_0^{n+2}}\right]^{1/n} |x - x_d|^{(n+1)/n}, \tag{2.30}$$

and thus the curvature is infinite. Singularities of these types need to be taken into account in devising numerical methods.

Thermal runaway

One of the interesting possibilities of the thermomechanical coupling between flow and temperature fields is the possibility of thermal runaway, and it has even been suggested that this may provide an explanation for the surges of certain thermally regulated glaciers. The simplest model is that for a glacier, with exponential rate factor, thus

$$\frac{dT}{dt} = \alpha\tau^{n+1}e^{\gamma T} + \beta T_{zz}, \tag{2.31}$$

where the stress is given by

$$\tau = \zeta - z. \tag{2.32}$$

The simplest configuration is the parallel sided slab in which ζ = constant, $\mathbf{u} = (u(z), 0, 0)$, so that

$$\frac{\partial T}{\partial t} = \alpha(\zeta - z)^{n+1}e^{\gamma T} + \beta\frac{\partial^2 T}{\partial z^2}, \tag{2.33}$$

with (say)

$$T = -1 \text{ on } z = \zeta, \; T_z = -\Gamma \text{ on } z = 0. \tag{2.34}$$

For given ζ, (2.33) will exhibit thermal runaway for large enough α, and $T \to \infty$ in finite time. As the story goes, this leads to massive melting and enhanced sliding, thus 'explaining' surges. The matter is rather more complicated than this, however. For one thing, ζ would actually be determined by the criterion that the flux $\int_0^\zeta u\,dz$ is prescribed, $= s$ say, where s would be the integrated ice accumulation rate from upstream ($= \int a\,dx$).

Thus even if we accept the unrealistic parallel slab 'approximation', it would be appropriate to supplement (2.33) and (2.34) by requiring ζ to satisfy

$$\int_0^\zeta u\,dz = s. \tag{2.35}$$

Since the flow law gives

$$\frac{\partial u}{\partial z} = (\zeta - z)^n e^{\gamma T}, \tag{2.36}$$

we find, if $u = 0$ on $z = 0$, that (2.35) reduces to

$$\int_0^\zeta (\zeta - z)^{n+1} e^{\gamma T} dz = s. \tag{2.37}$$

Thermal runaway is associated with multiple steady states of (2.33), in which case we wish to solve

$$\begin{aligned} 0 &= \alpha(\zeta - z)^{n+1}e^{\gamma T} + \beta T_{zz}, \\ T &= -1 \text{ on } z = \zeta, \\ T_z &= -\Gamma \text{ on } z = 0, \\ T_z &= -[\Gamma + (\alpha s/\beta)] \text{ on } z = \zeta. \end{aligned} \tag{2.38}$$

Putting $\xi = \zeta - z$, we solve

$$\begin{aligned} T_{\xi\xi} &= -(\alpha/\beta)\xi^{n+1}e^{\gamma T}, \\ T &= -1, \; T_\xi = \Gamma + (\alpha s/\beta) \text{ on } \xi = 0, \end{aligned} \tag{2.39}$$

as an initial value problem. T_ξ is monotone decreasing with increasing ξ, and thus there is a unique value of ζ such that $T_\xi = \Gamma$ there. It follows that there is a unique solution to the free boundary problem, and in fact it is linearly stable. It then seems that thermal runaway is unlikely to occur in practice.

A slightly different perspective may allow runaway, if we admit non-steady ice fluxes. Formally, we can derive a suitable model if $\lambda = \alpha/\beta = O(1)$, $\beta \to \infty$. In this case, we can expect T to tend rapidly to equilibrium of (2.33), and then ζ reacts more slowly via mass conservation, thus

$$\zeta_t + q_x = a,$$
$$q = \frac{1}{\lambda}[T_z]_0^\zeta. \tag{2.40}$$

An x-independent version of (2.40), consistent with the previous discussion, is

$$\frac{\partial \zeta}{\partial t} = s - q(\zeta), \tag{2.41}$$

and this *will* allow relaxation oscillations if $q(\zeta)$ is multivalued as a function of ζ — which will be the case. Surging in this sense is conceivable, but the limit $\beta \to \infty$ is clearly unrealistic, and unlikely to be attained. The earlier conclusion is the more likely.

2.3 The sliding law

The sliding law relates the basal shear stress τ_b to the basal sliding velocity u_b. The classical theory, enunciated by Lliboutry, Weertman, Nye, Kamb, and others, considers ice flowing at the base of a glacier over an irregular, bumpy bedrock. The ice is lubricated at the actual interface by the mechanism of *regelation*, or melting-refreezing, which allows a thin film (microns thick) to exist at the ice-rock interface, and allows the ice to slip. The drag is then due to two processes; regelation itself, and the viscous flow of the ice over the bedrock. Regelation is dominant for small wavelength roughness, while viscous drag is dominant for large wavelengths, and early work emphasised the importance of a controlling (intermediate) wavelength (of several centimetres). More recently, the emphasis has been away from regelation and considered only the viscous flow.

A suitable model for discussion is that of a Newtonian fluid over a rough bedrock of 'wavelength' $[x]$ and amplitude $[y]$, given, in coordinates scaled with $[x]$, by $y = \nu h(x)$, where y is now the vertical coordinate, and

$$\nu = [y]/[x] \tag{2.42}$$

is a measure of corrugation. The governing equation for slow, two-dimensional flow is the biharmonic equation

$$\nabla^4 \psi = 0 \tag{2.43}$$

for a suitably scaled stream function. Appropriate boundary conditions for no flow through the bed, and no shear stress there, are

$$\psi = 0$$
$$(1 - \nu^2 h'^2)(\psi_{yy} - \psi_{xx}) - 4\nu h' \psi_{xy} = 0, \tag{2.44}$$

on $y = \nu h$. As $y \to \infty$, the local basal flow must match to a far field flow with 'basal' velocity u_b and 'basal' stress τ_b; thus the main body of the ice flow sees the bedrock flow as a boundary layer, and u_b and τ_b are then the appropriate basal limits of the 'outer' ice flow. Specifically, we find that the correct matching condition is (in terms of correctly scaled 'outer' velocities and stresses)

$$\psi \sim u_b y + \frac{1}{2}\nu^2 \tau_b y^2 \text{ as } y \to \infty. \tag{2.45}$$

A convenient solution method can be presented if ν is small. In this case, we subtract $u_b y$ from ψ and divide by ν (so $\psi \to (\psi - u_b y)/\nu$); then to leading order in ν, the new ψ satisfies (2.43), with

$$\psi \to 0 \text{ as } y \to \infty,$$
$$\psi = -u_b h(x), \ \psi_{yy} - \psi_{xx} = 0 \text{ on } y = 0. \tag{2.46}$$

The shear stress is uncoupled from the determination of ψ, but can be determined by an integrated force balance, whence (e.g. if h is periodic with period 2π)

$$\tau_b = \frac{1}{2\pi}\int_0^{2\pi} (p + 2\psi_{xy})|_{y=0} h' dx; \tag{2.47}$$

more generally a spatial average would be used. Notice that the expression in brackets in (2.47) is simply (minus) the normal stress, and therefore is equal to the water pressure p_w in the lubricating film. We come back to this below.

A nice way to solve this problem is via complex variable theory. We define the complex variable $z = x + iy$, and then the general solution of the biharmonic equation is

$$\psi = (\bar{z} - z)f(z) - B(z) + (cc), \tag{2.48}$$

where f and B are analytic functions and (cc) denotes the complex conjugate. The zero stress condition (2.46) requires $f = -\frac{1}{2}B'$, and also $B \to 0$ as $z \to \infty$ (with $\text{Im } z > 0$), and the last condition is then

$$B + \bar{B} = u_b h \text{ on } \text{Im } z = 0. \tag{2.49}$$

If h is periodic, with a Fourier series

$$h = \sum_{-\infty}^{\infty} a_k e^{ikx}, \tag{2.50}$$

then B is simply given by

$$B = u_b \sum_{1}^{\infty} a_k e^{ikz} \tag{2.51}$$

(we can assume $a_0 = 0$, i.e. the mean of h is zero). However, it is also convenient to formulate this problem as a Hilbert problem. We define $L(z) = B''(z)$, which is analytic in $\text{Im}\, z > 0$, and then $L(z) = \overline{B''(\bar{z})}$ is analytic in $\text{Im}\, z < 0$. It then turns out that, with the usual notation,

$$\begin{aligned} L_+ + L_- &= u_b h'', \\ L_+ - L_- &= \frac{1}{2} ip, \end{aligned} \tag{2.52}$$

relate the values either side of $\text{Im}\, z = 0$; here p is ice pressure ($p = -2i(B'' - \bar{B}'')$ on $y = 0$, since $p + i\nabla^2\psi$ is analytic), and in fact $p = p_w$ on $y = 0$, since ψ_{xy} is found to be zero there. The drag (i.e. the sliding law) is then computed as (for a 2π-periodic h)

$$\tau_b = \frac{1}{i\pi} \int_0^{2\pi} (L_+ - L_-) h' dx, \tag{2.53}$$

and turns out to be

$$\tau_b = 4u_b \sum_{1}^{\infty} k^3 |a_k|^2. \tag{2.54}$$

For a linear model such as this, τ_b is necessarily proportional to u_b. For Glen's flow law, variational principles can be used to estimate

$$\tau_b \approx R u_b^{1/n}. \tag{2.55}$$

Weertman's original sliding law drew a balance between (2.55) and the linear dependence due to regelation, and the heuristic 'Weertman's law' $\tau_b \propto u_b^{1/m}$, with $m \approx (n+1)/2$ was often used.

Simplistic sliding laws such as the above have been superceded by the inclusion of cavitation. When the film pressure behind a bump decreases to a value lower than the local subglacial water pressure, a cavity must form, and indeed, such cavities are plentifully observed. An appropriate generalisation of (2.52) is then

$$\begin{aligned} L_+ + L_- &= u_b h'' \quad \text{in } C', \\ L_+ - L_- &= -\tfrac{1}{2} ip_c \quad \text{in } C, \end{aligned} \tag{2.56}$$

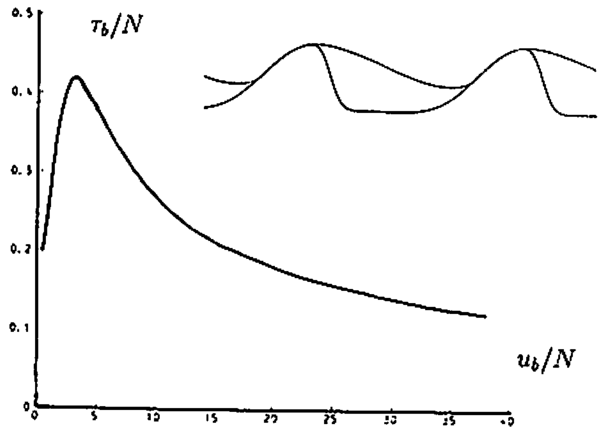

Figure 6: stress versus velocity for a bed of isolated bumps

where the bed is divided into cavities (C) where p is known ($= -p_c$), and attached regions where h is known. One can solve this problem to find the unknown cavity shapes, and for a bed consisting of isolated bumps, $\tau_b(u_b)$ increases monotonically for small u_b, reaches a maximum, and then decreases for large u_b, as shown in Fig. 6. The decreasing portion of the curve is unstable (increasing velocity decreases drag) and is caused by the roofs of the cavities from one bump reaching the next bump.

Since p in the scaled ice flow model is measured relative to ice overburden pressure, it follows that p_c in (2.56) is proportional to the effective pressure N, and in fact the sliding law has the specific form $\tau_b = N f(u_b/N)$. For a nonlinear Glen's law, the suggested generalisation is

$$\tau_b = N f(u_b/N^n). \tag{2.57}$$

The multivaluedness of $u_b(\tau_b)$ is very suggestive of surging — but is it realistic? Consideration of more realistic beds suggest that in fact $f(\cdot)$ in (2.57) will be an increasing function of its argument, since when smaller bumps start to be drowned, larger ones will take up the slack. A plausible sliding law then has $f(\xi)$ increasing as a power of ξ, or (for example)

$$\tau_b = c u_b^r N^s, \tag{2.58}$$

where we would expect $r, s > 0$. Indeed, there is some experimental and field evidence consistent with laws of this type, with $r \approx s \approx 1/3$, for example.

An apparently altogether different situation occurs when ice slides over wet, deforming till. If the till is of thickness h_T and has (effective) viscosity η_T, then an appropriate sliding law would be

$$\tau_b = \eta_T u_b / h_T. \tag{2.59}$$

In fact, till is likely to have a nonlinear rheology, and also in accordance with Terzaghi's principle of soil mechanics, one would expect η_T to depend on effective pressure N. One (measured) rheology for till gives the strain rate as

$$\dot{\varepsilon} = A_T \tau^a N^{-b}, \tag{2.60}$$

in which case the sliding law would be again of the form (2.58), with $c = (A_T h_T)^{-1/a}$, $r = 1/a$, $s = b/a$. Thus there are some good reasons to choose (2.58) as an all purpose sliding law, and this points up the necessity of a subglacial hydraulic theory to determine N.

2.4 Drainage and jökulhlaups

Subglacial water is generated both by basal melt (of significance in ice sheets) and from run-off of surface melt or rainfall through crevasses and *moulins*, which access the glacier bed. Generally the basal water pressure p_w is measured to be below the overburden ice pressure p_i, and the resulting positive effective pressure $N = p_i - p_w$ tends to cause any channels in the ice to close up (by creep of the ice). In fact, water is often seen to emerge from outlet streams which flow through large tunnels in the ice, and the theory which is thought to explain how such channels remain open asserts that the channel closure rate is balanced by melt back of the channel walls by frictional heating due to the water flow.

If we consider a single channel of cross sectional area S, through which there is a water flux Q, then conservation of mass requires

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = \frac{m}{\rho_w} + M, \tag{2.61}$$

where m is the mass of ice melted per unit length per unit time, ρ_w is water density, s is distance down channel, and M is an external source due to rainfall or surface run-off. If the flow is turbulent, then a hydraulic correlation for flow in a straight conduit is

$$\rho_w g \sin\alpha - \frac{\partial p}{\partial s} = f_1 Q^2 / S^{8/3}, \tag{2.62}$$

where α is the local bed slope, p is water pressure, and f_1 is a roughness coefficient related to the Manning friction factor. If we suppose that the frictional heat dissipated by the turbulent flow is all used to melt the walls, then

$$mL = Q[\rho_w g \sin\alpha - \frac{\partial p}{\partial s}], \tag{2.63}$$

where L is the latent heat.

The last equation to relate the four variables S, Q, p and m is essentially a kinematic boundary condition for the ice:

$$\frac{\partial S}{\partial t} = \frac{m}{\rho_i} - KS(p_i - p)^n; \tag{2.64}$$

here m/ρ_i is the rate of enlargement due to melt back, while the second term on the right hand side represents closure to Glen's law viscosity of the ice.

Steady state drainage can occur. One can show that (for glaciers) $M \gg m/\rho_w$, and with M prescribed, effectively the water flux Q is a prescribed function of s. It is also found that typically $\partial p/\partial s \ll \rho_w g \sin\alpha$ (in fact, we expect $\partial p/\partial s \sim \rho_w g d/l$, so that in the notation of (2.7), the ratio of these terms is of $O(\varepsilon)$); the neglect of the $\partial p/\partial s$ term in (2.62) and (2.63) is singular, and causes a boundary layer of size $O(\varepsilon)$ to exist near the terminus in order that p decrease to atmospheric pressure. Away from the snout, then

$$S \approx \left[\frac{f_1 Q^2}{\rho_w g \sin\alpha}\right]^{3/8}, \quad KSN^n \approx \frac{Q\rho_w g \sin\alpha}{\rho_i L}, \tag{2.65}$$

where N is the sought for effective pressure. Thus

$$N \approx \beta Q^{1/4n}, \tag{2.66}$$

where β is a material parameter which depends (inversely) in roughness. Typical values give $N = 30$ bars when $Q = 10\ \mathrm{m^3\,s^{-1}}$. Since $p_i = 9$ bars for a 100 metre deep glacier, it is clear that the computed N may exceed p_i. In this case, p must be atmospheric and there will be open channel flow. It is likely that seasonal variations are important in adjusting the hydraulic régime.

Jökulhlaups

These equations can also describe an outburst flood. In this case, M is irrelevant, and a suitable scaling shows that $Q \approx Q(t)$, and a dimensionless model is

$$\frac{\partial S}{\partial t} = \Phi^{3/2} S^{4/3} - SN^3, \tag{2.67}$$

where Φ is the (scaled) hydraulic potential gradient. The model is supplemented by a boundary condition which prescribes the water pressure at the lake outlet to be hydrostatic. As the jökulhlaup proceeds, lake level falls, thus N increases, and the rate of increase is related to the water flux. A suitable dimensionless model is then

$$\frac{\partial N}{\partial t} = \nu S^{4/3}. \tag{2.68}$$

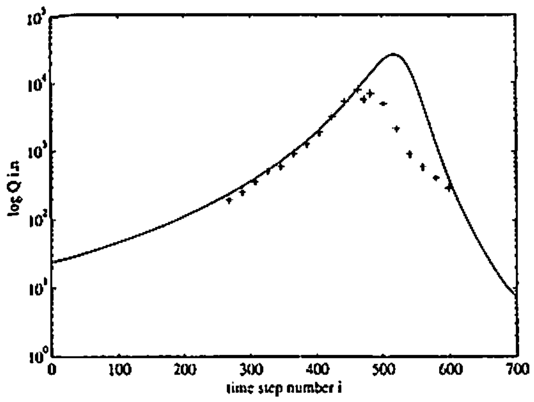

Figure 7: numerical simulation of a jökulhlaup and the hydrograph of the 1972 jökulhlaup from Grimsvötn

where N and S are measured at the lake outlet; the parameter ν is small. These two equations can simulate a flood. For simplicity, take $\Phi = 1$. In the initial stages, N is negligible, so that $\dot{S} = S^{4/3}$, and $S \sim 27(-t)^{-3}$ (with $t < 0$). Also $N \sim \nu S$, so that $SN^3 \sim \nu^3 S^4$, and the closure suddenly switches on when $S \sim \nu^{-9/8}$: this causes the jökulhlaup to self-terminate. A numerical example is shown in Fig. 7.

In order to model the periodicity of jökulhlaups, a regeneration term $-\mu$ must be added to (2.68) ($\mu \ll 1$), which corresponds to the slow refilling of the lake; in addition a trigger must be set. Grimsvötn appears to 'switch on' when N decreases to 6 bars, for reasons which are opaque, but are in any case outside the scope of these equations.

2.5 Notes

The basic scaling in the shallow ice approximation is due to Fowler and Larson (1978): it is elaborated in the book by Hutter (1983). For ice sheets, similar derivations have been done by Morland (1984), Hutter et al. (1986) and Fowler (1992), of whom we follow the latter. The concept of thermally induced instability was enunciated by Robin (1955) and taken up by Clarke et al. (1977) and Yuen and Schubert (1979), but more or less scotched by Fowler and Larson (1980a).

The theory of basal sliding over hard beds stems from Weertman (1957) and Lliboutry (1968). Two reviews of progress by the end of the seventies are in Lliboutry (1979) and Weertman (1979). The linear theory is primarily due to Nye (1969, 1970) and Kamb (1970), while the material presented here is based largely on Fowler (1986, 1987b). Till rheology is discussed by Boulton and Hindmarsh (1987).

The classical theory of drainage is due to Röthlisberger (1972), while the development for jökulhlaups follows Nye (1976).

3 Large scale fluctuations

3.1 Waves on glaciers

Waves on glaciers are mostly easily understood by considering an isothermal, two-dimensional model. We suppose the base is flat ($h = 0$), so that equations (2.20) and (2.21) give

$$H_t + \left[\{1 - \varepsilon H_x\}^n \frac{H^{n+2}}{n+2} + u_b H\right]_x = s'(x), \tag{3.1}$$

where $s'(x)$ is the accumulation rate, and $\varepsilon \sim 0.1$. If we firstly put $\varepsilon = 0$ and also $u_b = 0$, then

$$H_t + H^{n+1} H_x = s'(x), \tag{3.2}$$

which has the steady state

$$\frac{H_0^{n+2}}{n+2} = s(x). \tag{3.3}$$

With $s' > 0$ in $x < 0$ (say) and $s' < 0$ in $x > 0$ ($x = 0$ is then the *firn line*) (3.3) defines a concave profile like that in figure 5. (3.2) is clearly hyperbolic, and admits wave like disturbances which travel at a speed H^{n+1}, which is in fact $(n + 1)$ (≈ 4) times the surface speed. For an arbitrary initial condition $H = A(x)$ at $t = 0$, the solution by characteristics is

$$\begin{aligned} \frac{H^{n+2}}{n+2} &= s(x) - s_1(\sigma), \\ t &= \int_\sigma^x \frac{dx}{[(n+2)\{s(x) - s_1(\sigma)\}]^{(n+1)/(n+2)}}, \end{aligned} \tag{3.4}$$

where s_1 is defined by

$$\frac{A^{n+2}(\sigma)}{n+2} = s(\sigma) - s_1(\sigma). \tag{3.5}$$

Thus, for small perturbations, s_1 is small.

The characteristics of (3.2) propagate downstream and reach the snout (where $H = 0$) in finite time. If we wish to approximate the characteristic solution where s_1 is small, straightforward linearisation is invalid near the snout where $H_0 = 0$; rather, a uniformly valid approximation can be obtained by linearising the characteristics:

$$H_t + H_0^{n+1} H_x \approx s'(x) \tag{3.6}$$

for $H \approx H_0$, where the general solution is

$$H = H_0(x) + \phi(\xi - t), \tag{3.7}$$

where

$$\xi = \int_0^x \frac{dx}{H_0^{n-1}(x)} \tag{3.8}$$

is a characteristic spatial coordinate (note ξ is finite at the snout). (3.7) clearly reveals the travelling wave characteristic of the solution.

If H is increased locally (e.g. due to the surge of a tributary glacier) then a shock travels forward. The rôle of the term in ε is then to diffuse such shocks. A shock at $x = x_s$ will propagate at a rate

$$\dot{x}_s = \frac{[H^{n+1}]_-^+}{(n+2)[H]_-^+}. \tag{3.9}$$

where $[\ ]_-^+$ denotes the jump across x_s. When the shock reaches the snout, it then propagates at a speed $H_-^{n+1}/(n+2)$, which is *slower* than the surface speed.

In the neighbourhood of a shock (with $u_b = 0$), we put

$$x = x_s + \nu X, \tag{3.10}$$

so that

$$\frac{\partial H}{\partial t} - \frac{\dot{x}_s}{\nu}\frac{\partial H}{\partial X} + \frac{1}{\nu}\left[\left\{1 - \frac{\varepsilon}{\nu}H_X\right\}^n \frac{H^{n+2}}{n+2}\right]_X = s'(x_s + \nu X); \tag{3.11}$$

if ν is small, the profile rapidly relaxes to the steady travelling wave described by

$$\dot{x}_0 H_X = \left[\{1 - H_X\}^n \frac{H^{n+2}}{n+2}\right]_X, \tag{3.12}$$

providing we choose $\nu = \varepsilon$, which thus gives the width of the shock structure. (3.12) can be solved by quadrature.

Seasonal waves

There is no explanation of seasonal waves available. On the face of it, we might seek waves of amplitude of velocity of $O(1)$ propagating at a speed $O(1/\mu)$, where μ is the ratio of one year to the convective time scale, so $\mu \lesssim 0.05$. Apparently we should associate the variations in u with variations in water supply, so that a natural model would involve only sliding, so

$$H_t + (H u_b)_x = s'(x). \tag{3.13}$$

and if $u_b = \phi(t/\mu)H^m/(m+1)$, where $\phi(t/\mu)$ represents the seasonal variation of water supply and hence of N, then

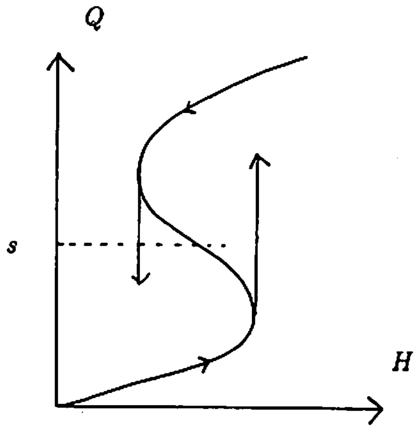

Figure 8: A multi-valued flux-depth relation can cause oscillatory surges.

$$H_t + \phi(t/\mu)H^m H_x = s'(x). \tag{3.14}$$

Unfortunately, while the surface speed will indeed oscillate seasonally in this model, the kinematic wave equation propagates waves at $(m+1)$ times the surface speed, so there seems to be no mechanism for the rapid propagation which has been observed. Another possibility is then that the variations in N force a wave passage in the hydraulic system itself, but this has not been explored.

3.2 Surges

It has long been suggested that the fast velocities during surges could only be caused by rapid sliding. Therefore it is sufficient to analyse the mass conservation equation in the form

$$H_t + (Hu)_x = s'(x), \tag{3.15}$$

where u is the sliding velocity. Also, it has been thought that if the sliding velocity were a multi-valued function of basal stress τ_b (i.e. $\tau_b(u)$ has a decreasing portion)) then since $\tau_b = H(1 - \varepsilon\zeta_x) \approx H$, this would cause the ice flux $Q = uH$ to be multi-valued as shown in fig. 8, in which case we might expect relaxation oscillations to occur for values of s intermediate between the two noses of $Q(H)$. Two fundamental questions arise. Firstly, is there any genuine reason why $\tau_b(u_b)$ should be non-monotone, and secondly, how would such a relaxation oscillator work in the spatially dependent case?

The discussion in section 2 suggested the possibility of non-monotone $\tau_b(u)$ for flow over a periodic bedrock. However, more realistic bedrocks probably do not have this feature, and τ increases with both u and N. What observations of the 1982-3 surge of Variegated Glacier showed, however, was that there is a switch in drainage pattern during a surge. There are two possible modes of drainage: the Röthlisberger channel described in section 2 with the value of N determined by the water flow, N_R, say; however, if no channels are present, then water will fill cavities at the bed and leak from one to another. This is called the linked cavity régime and operates at a higher water pressure and thus lower effective pressure, N_c, than in the channel drainage. The crucial factor which enables surges to take place is the switch mechanism, and this depends on the ice flow over the cavities. If the sliding law is, as discussed in section 2, of the form $\tau_b = Nf(u/N^n)$, then in fact the stresses in the ice are actually determined by u/N^n, and in particular the water stored by cavities depends on this parameter.

It turns out that a simple model of combined water flow through both cavities and a channel system exhibits instability (the channels close down) if the cavity storage volume is large enough, and thus the instability occurs at a critical value of $\Lambda = u/N^n$, denoted Λ_c. It follows from this that a combined model of the drainage system is

$$\begin{aligned} N &= N_R, \quad u/N^n < \Lambda_c; \\ N &= N_c, \quad u/N^n > \Lambda_c; \end{aligned} \tag{3.16}$$

and if this is written as a function $N(u)$, it is multi-valued, as shown in Fig. 9. As a consequence of this, the sliding law is indeed multi-valued, and hence $Q(H)$ has the form shown in Fig. 10.

There are two critical values of Q in fig. 10, denoted Q_+, Q_-: these are the values at the noses of the curve (where also $H = H_+, H_-$). If $s(x) < Q_+$, then an equilibrium glacier profile exists in which $Q = s(x)$. However, if the maximum value of s, s_{max}, is greater than Q_+, then such a stable equilibrium cannot occur, and the glacier surges.

The sequence of events in a surge is then as follows. The glacier grows from a quiescent state in which $Q < Q_+$ on the lower (slow) branch everywhere. When the maximum depth reaches H_+, there is a *reservoir* zone where $H > H_-$. The ice flux at H_+ jumps to the upper (fast) branch by switching drainage pattern, and this switch propagates upstream and downstream to where $H = H_-$. These *activation waves* propagate at rates of hundreds of metres per *hour* (and in effect have been observed). Once the activation waves have propagated to the boundaries of the reservoir zone, it is in the fast mode on the upper branch, and the activated reservoir zone propagates rapidly downstream, overriding the stagnant snout and propagating forwards as a front. In terms of fig. 10, the surge terminates when H reaches H_- everywhere, and deactivation waves propagate inwards from the boundaries of the exhausted reservoir zone to re-establish the channel

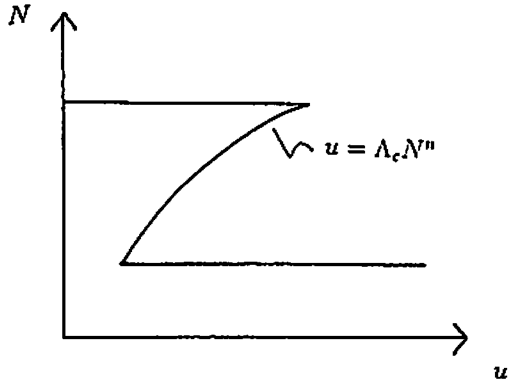

Figure 9: N is a multi-valued function of u

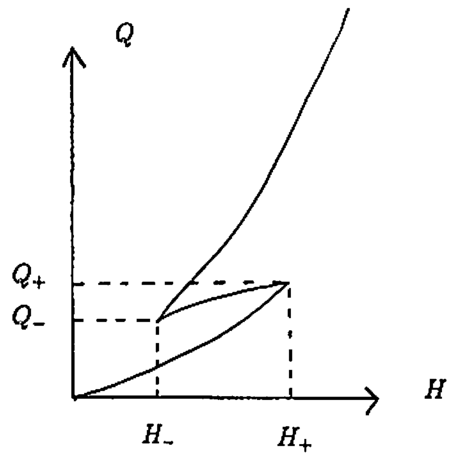

Figure 10: Q is a multi-valued function of H

drainage system. There then follows another quiescent phase where the maximum value of H increases from H_- to H_+ before the next surge is initiated.

3.3 Sliding and ice streams

It is not known why the ice flow on the Siple Coast of Antarctica, which flows out to the floating Ross ice shelf, segregates itself into the five distinct ice streams A to E. The picture which one has of this region is of a gently sloping (slope $\alpha \sim 10^{-3}$) kilometer thick ice sheet which flows in the ice streams at typical rates of 500 m y^{-1}. Such rapid velocity can only be due to basal sliding, and the seismic evidence indicates that the ice is underlain by several metres of wet till. One can expect that a sliding law of the form advocated previously is appropriate, that is

$$\tau_b = c u_b^r N^s, \tag{3.17}$$

with r and s positive. The issue then arises as to how to prescribe N. Recall from section 2 that for drainage through Röthlisberger channels, an appropriate law is $N = \beta Q_w^{1/4n}$, where Q_w is water flux. When ice flows over till, an alternative flow route is possible, that is, that water excavates 'canals' in the subglacial till. A theoretical description of this drainage system suggests that it is more likely for gently sloping ice flow, and also that the relation between N and Q_w is of the opposite sense, that is, that $\partial N/\partial Q_w < 0$.

In this case an interesting feedback exists. In Antarctic ice streams, there is little, if any, surface melt reaching the bed, and the basal water flow is due to melting there. The quantity of meltwater produced per unit area per unit time is given by the melt velocity

$$v_m = \frac{G + \tau_b u_b - g}{\rho_w L}, \tag{3.18}$$

where ρ_w is water density, L is latent heat, G is geothermal heat flux, and g is the basal heat flux into the ice. This assumes the base is at the melting point. Thus we expect the basal water flux $Q_w \sim G + \tau_b u_b - g$, and so Q_w increases with u_b (the dependence of g on u_b is likely to be weaker — boundary layer theory would suggest $g \sim u_b^{1/2}$). If also N decreases with Q_w, then N decreases as u_b increases. But this causes further increase of u_b via the sliding law. This positive feedback can lead to a runaway phenomenon which we may call hydraulic runaway.

To get a crude idea of how this works, we denote the ice thickness as h and slope $\sin\alpha$. If the velocity is u, then the ice flux is

$$Q = hu, \tag{3.19}$$

the basal shear stress is

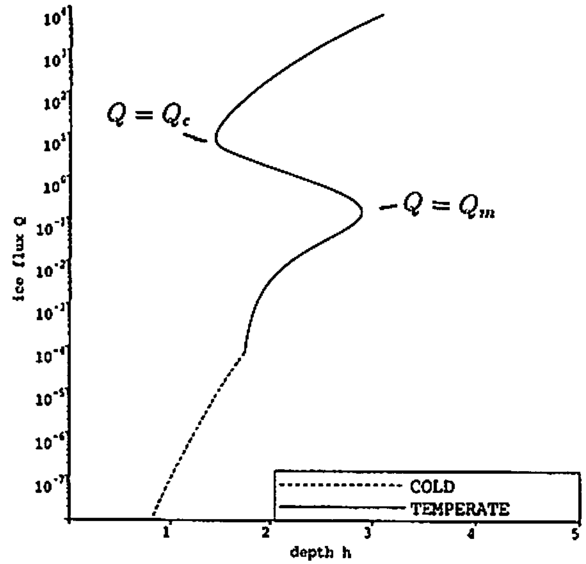

Figure 11: thermal feedback causes a multi-valued ice flux.

$$\tau = \rho g h \sin \alpha = c u^r N^s, \tag{3.20}$$

we suppose

$$N = \gamma Q_w^{-p}, \tag{3.21}$$

and that

$$Q_w = b[G + \tau u - g], \tag{3.22}$$

with

$$g = a u^{1/2}. \tag{3.23}$$

Consequently

$$h = \frac{f u^r}{[G + (\rho g \sin \alpha) h u - a u^{1/2}]^m}, \tag{3.24}$$

where

$$m = ps, \; f = \frac{c \gamma^s}{(\rho g \sin \alpha) b^m}. \tag{3.25}$$

It is not difficult to see from (3.24) if f is low enough (equivalently, the friction coefficient c is low enough), that u and hence the ice flux Q will be a multivalued function of h, as shown in fig. 11. In fact, application of realistic parameter values suggests that such multi-valued flux laws are normal.

What then happens in a region such as the Ross ice shelf area? We suppose that the ice flux is determined by conditions upstream, so that if the ice flux per unit width is q, and the width of the discharge region is W, then

$$Wq = s. \tag{3.26}$$

where s is the volume flux of ice discharged. Now if $s/W < Q_c$ (see fig. 11), then a uniform slow moving ice flow is possible. Similarly, if $s/W > Q_m$, a uniform fast moving ice stream is possible. What if $Q_m < s/W < Q_c$? A uniform stream is now unstable, and we may expect an instability to occur, whereby ice streams spontaneously occur, as observed. Such an instability would be mediated by transitions in water pressure, since basal water will flow from fast streams at high water pressure to slower ice at low water pressure. This generates a lateral enthalpy flux, and in a steady state this can be balanced by a heat flux in the ice in the opposite direction, since cooling (g) is less effective at lower u, therefore the slow ice is warmer near the base than the ice streams. No analysis of this idea has yet been done, though it is an intriguing mathematical problem.

3.4 Heinrich events and the Hudson strait mega- surge

What if the drainage channel of an ice sheet over deforming till is relatively narrow? By analogy of the pattern formation mechanism in reaction diffusion equations, one would expect that a multivalued flux-depth relation would not allow separate streams to form if the channel width is too small, and in this case we would expect periodic surges to occur down the channel, if the prescribed mass flux lies on the unstable position of fig. 11.

A situation of this type appears to have occurred during the last ice age. The Laurentide ice sheet which existed in North America drained the ice dome which lay over Hudson Bay out through the Hudson Strait, a 200 km wide trough which discharged the ice (as icebergs) into the Labrador sea and thence to the North Atlantic.

Hudson Bay is underlain by soft carbonate rocks, mudstones, which can be mobilised when wet. It is suggested that the presence of these deformable sediments, together with the confined drainage channel, led to the occurrence of semi-periodic surges of the Hudson Strait ice stream. The evolution of events is then as follows. When ice is thin over Hudson Bay, the mudstones are frozen at the base, there is no sliding and very little ice flow. Consequently, the ice thickens and eventually the basal ice warms. The basal muds thaw, and sliding is initiated. If the friction is sufficiently low, then the multi-valued sliding law of fig. 11 is appropriate, and if the accumulation rate is large enough, cyclic surging will occur. During a surge, the flow velocity increases dramatically (calculations suggest a velocity of 2 centimetres a *second*!), and there results a massive iceberg flux into the North Atlantic. On the lower branch of fig. 11, water production is virtually absent. Q_w is low in (3.22) since the flow is slow and the geothermal and viscous heat at the base can be conducted away by the ice. The low value of Q_w gives high N, consistent with low u. On the upper branch, however, viscous heat dominates, and Q_w is large, N is small, also consistent with a high u.

At the end of a surge, the rapid ice drawdown causes the water production to drop, and the rapid velocities switch off. This may or may not also be associated with re-freezing of the basal mudstones.

When water saturated soils freeze, frost heave occurs by sucking up water to the freezing front via capillary action, and this excess water freezes (at least for fine grained clays and silts) in a sequence of discrete ice lenses. Heaving can occur at a typical rate of perhaps a metre per year, though less for fine grained soils, and the rate of heave is suppressed by large surface loads. Calculations suggest a surge period of perhaps a hundred years, with a drawdown of a thousand metres, and a recovery period on the order of 5,000–10,000 years. During the surge, the rapidly deforming basal muds will dilate (in the deforming horizon, likely to be only a metre or so thick). At the termination of a surge, this layer re-consolidates, and we can expect the total heave to be a certain (small) fraction of the frost penetration depth. In effect, the ice lenses freeze the muds into the ice stream, so that when the next surge phase is initiated, some of this frozen-in basal sediment will be transported downstream, and thence rafted out into the North Atlantic in iceberg discharge.

In fact, there is evidence that this rather glamorous sequence of events actually occurs. Heinrich events are layers of ice-rafted debris in deep-sea sediment cores from the North Atlantic which indicate (or are consistent with) massive iceberg discharges every 7000 years or so. In addition, oxygen isotope concentrations in ice cores from Greenland indicate that severe cooling cycles occurred during the last ice age. One theory has it that such cooling events can be caused by a switch-off of North Atlantic deep water (NADW) circulation — effectively switching off the convective heat transport from equatorial latitudes and thus cooling the atmosphere. It seems that bunches of these cooling cycles are terminated by Heinrich events, in the sense that following Heinrich events the climate warms suddenly. There are two reasons why this should be so. On the one hand, the sudden reduction in ice thickness should warm the air above, and also it can be expected that a massive iceberg flux to the North Atlantic acts as a source of negative thermal buoyancy, which can re-initiate an otherwise stagnant circulation. Rather than being lumbering beasts, glaciers and ice sheets show every sign of being dynamically active agents in shaping the climate and the earth's topography.

3.5 Notes

The theory of surface waves on glaciers was really worked out by several early authors, and is discussed by Lliboutry (1965) in his voluminous treatise. The modern linear theory is worked out by Nye (1960), and is expanded on by Fowler and Larson (1980b). Fowler (1982) obtained a theory of seasonal waves on the rather dubious basis of a sliding law with $\partial u_b/\partial \tau_b \gg 1$, but this is unlikely to be correct.

Surges are discussed by Kamb et al. (1985) and Clarke et al. (1984). The present discussion is based on work by Fowler (1987a), the mathematical details of which are worked out in Fowler (1989).

The dynamics of ice streams are reviewed by Bentley (1987), while the theory of Hudson Strait mega-surges is due to MacAyeal (1993). Heinrich events are discussed by Bond et al. (1992), while the present discussion is based on a paper by Fowler and Johnson (1995).

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Some free boundary problems in theorical glaciology.

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Plan

1 – Mathematical Models and the Shallow-Ice Approximation

- **1.1.** Introduction
- **1.2.** Shallow-ice modelling
- **1.3.** Kinematic description of the free surface
- **1.4.** A simplified ice-water model

2 – Mathematical Analysis of the Glacier Kinematics

- **2.1.** The steady-state case
- **2.2.** The evolutionary problem
- **2.3.** The asymptotic behaviour in time

3 – Analysis of the Shallow-Ice Temperature with Phase Change

- **3.1.** The parabolic steady-state case
- **3.2.** The ultraparabolic Stefan problem
- **3.3.** The asymptotic stabilization in time

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