ASYMPTOTICS WITH SMALL EXPONENT IN A MODEL FOR ICE-SHEET SURGING

By

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ABSTRACT

We study an equation which arises in a model describing the mechanism by which ice sheets periodically surge. The resulting ordinary differential equation looks deceptively simple, but has a complicated asymptotic character which is studied using an asymptotic approach and independent analysis. We place the emphasis on investigating the structure of the solution to a problem where a small parameter occurs as an exponent of the dependent variable.

1. Introduction

In [2], a mechanism is proposed and analysed to explain the periodic surging of ice sheets. In dimensional form, the model used reduces to:

$$\begin{aligned} h_t + (hu)_x &= a, \\ u^r &= Q^\sigma(-hh_x), \\ Q_x &= \gamma - uhh_x - \frac{\beta u}{\{\int_0^x u\,dx\}^{1/2}} - \frac{\lambda}{h}, \end{aligned} \tag{1}$$

wherein subscripts denote partial derivatives, x is a longitudinal spatial coordinate, and the variables are ice depth h, ice velocity u, and basal water flow Q. These equations represent, respectively, conservation of the ice mass, a sliding law that relates the velocity u to the basal shear stress $\tau = -hh_x$ and to the lubricating basal water flow, and a conservation law for basal water flow. The terms on the right-hand side represent respectively, geothermal heat flux, shear heating (τu), advective cooling through a basal thermal boundary layer, and conductive cooling through the ice depth.

In [2], the following are presented as being appropriate parameter values representing conditions in a large ice sheet:

$$r = 1/2, \quad \sigma = 1/6; \quad \gamma, \lambda, \beta = O(1). \tag{2}$$

Of particular interest are steady state solutions in which accumulation is ignored ($a = 0$), so that the ice flux is constant. Thus

$$uh = M, \tag{3}$$

with M being prescribed. We define

$$\xi = \int_0^x u\,dx, \tag{4}$$

so that the steady solutions of (1) satisfy, using (3),

$$\begin{aligned}
\frac{dQ}{dx} &= \gamma + \frac{M^{r+1}}{h^{r+1}Q^{\sigma}} - \frac{\beta M}{h\xi^{1/2}} - \frac{\lambda}{h}, \\
\frac{dh}{dx} &= -\frac{M^r}{h^{r+1}Q^{\sigma}}, \\
\frac{d\xi}{dx} &= \frac{M}{h}.
\end{aligned} \tag{5}$$

We simplify (5) by putting $z = \xi^{1/2}$, and using z as the independent coordinate. Thus

$$\begin{aligned}
\frac{M}{2hz}\frac{dQ}{dz} &= \gamma + \frac{M^{r+1}}{h^{r+1}Q^{\sigma}} - \frac{\beta M}{hz} - \frac{\lambda}{h}, \\
\frac{M}{2hz}\frac{dh}{dz} &= -\frac{M^r}{h^{r+1}Q^{\sigma}}.
\end{aligned} \tag{6}$$

These equations are to be solved for h, $Q > 0$ in $z > 0$, and suitable boundary conditions are to prescribe Q, $h > 0$ when $z = 0$. The solution terminates if h reaches zero (corresponding to zero ice thickness). Without loss of generality, we put $M = 1$; then if we define $S = Q + h$, we can write (6) in the form

$$\begin{aligned}
\frac{dS}{dz} &= 2(\gamma z h - \beta - \lambda z), \\
\frac{dh}{dz} &= -\frac{2z}{h^r(S-h)^{\sigma}}.
\end{aligned} \tag{7}$$

In Fig. 1 we plot the numerical solution of (7) using values $r = \frac{1}{2}$, $\sigma = \frac{1}{6}$, $\gamma = 0.2$, $\lambda = 0.36$, $\beta = 1$, together with initial values $h = 2$, $S = 2.5$ at $z = 0$. It can be seen that h decreases monotonically, tending to zero near $z = 1$, while Q initially decreases but then increases. The interesting feature is the fact that the minimum of Q is very sharp. This is associated with the asymptotic limit $\sigma \to 0$ corresponding

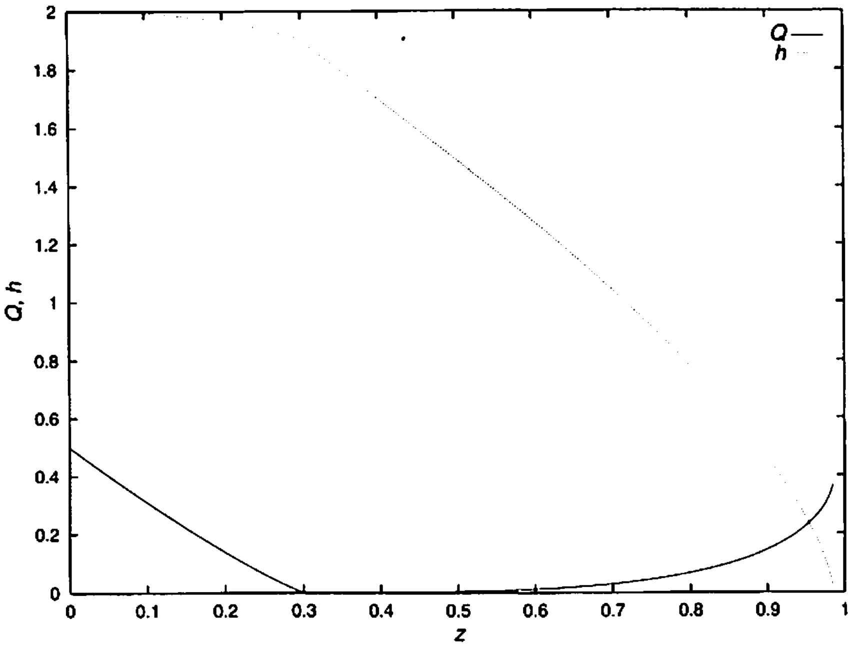

FIG. 1—Solution of (7) with $r = \frac{1}{2}$, $\sigma = \frac{1}{6}$, $\gamma = 0.2$, $\lambda = 0.36$, $\beta = 1$, with initial values $h = 2$, $S = 2.5$ at $z = 0$. Notice the very sharp transitions in the slopes of h and $Q \equiv S - h$ near $z = 0.3$.

to the actual value $\sigma = \frac{1}{6}$, because, as we shall see, the width of the transition zone is exponentially small (in $1/\sigma$).

To investigate this, we return to the original motivating example in (7), which can be written as

$$\begin{aligned} h' &= -\frac{2z}{h^r Q^\sigma}, \\ Q' &= 2[\gamma z h - \beta - \lambda z] - h'. \end{aligned} \tag{8}$$

At leading order we find:

$$\begin{aligned} h^{r+1} &= h_0^{r+1} - (r+1)z^2, \\ Q &= Q_0 + h_0 - h + \frac{\gamma}{r+2}[h_0^{r+2} - h^{r+2}] - 2\beta z - \lambda z^2, \end{aligned} \tag{9}$$

and we suppose that $Q \to 0$ at $z = z_0$, defined via

$$\begin{aligned} h^{*r+1} &= h_0^{r+1} - (r+1)z_0^2, \\ Q_0 + h_0 + \frac{\gamma}{r+2}[h_0^{r+2} - h^{*r+2}] &= h^* + 2\beta z_0 + \lambda z_0^2. \end{aligned} \tag{10}$$

The corresponding slopes are

$$h'^{*} = -\frac{2z_0}{h^{*r}},$$
$$Q'^{*} = -[h'^{*} - 2(\gamma z_0 h^{*} - \beta - \lambda z_0)], \tag{11}$$

so that

$$h \sim h^{*} + h'^{*}(z - z_0), \quad Q \sim Q^{*} + Q'^{*}(z - z_0), \quad \text{as } z \to z_0 - . \tag{12}$$

In particular, (1) is exponentially accurately approximated by

$$h' = \frac{h'^{*}}{Q^{\sigma}},$$
$$Q' = Q'^{*} + h'^{*} - h', \tag{13}$$

providing $z - z_0$, $h - h^{*}$ and $Q - Q^{*}$ are exponentially small. The equation (13) can now be written as:

$$Q' = \frac{\mu^{*}}{Q^{\sigma}} - \nu^{*}, \tag{14}$$

where

$$\mu^{*} = |h'^{*}|, \quad \nu^{*} = |Q'^{*}| + |h'^{*}|. \tag{15}$$

We now consider (14) in the particular case where the parameter values [2] are $\mu^{*} = 1$, $\nu^{*} = 2$. The problem thus reduces to a particularly innocuous form, and implies that in the transition region, $z - z_0$, $Q - Q^{*}$ and thus $h - h^{*}$ are indeed exponentially small, as assumed. Our simple reduced problem is thus a canonical simplification of (8).

In the remainder of the paper, although we attempt an asymptotic approximation in §2, the main focus will be the 'exact' approach in §3. An alternative (and improved) asymptotic approach will be reported elsewhere.

2. Asymptotics of the reduced problem

We have seen that the sharpness of the transition occurs even at quite moderate σ. We expect this fact to be associated with asymptotics based on $\sigma \to 0$, so the emphasis will be placed on determining the asymptotic structure of Q and the remainder of the paper will be devoted to the reduced equation, i.e. (14). At present, a parallel investigation is being carried out in the spirit of [1] and [6] and will be reported in due course. We consider:

$$\frac{dQ}{dx} = \frac{1}{Q^{\sigma}} - 2, \tag{16}$$

with initial condition $Q(0) = 1$. Since $\frac{dQ}{dx}$ is negative and remains so with increasing x, Q decreases and reaches the 'steady state' $Q = 2^{-1/\sigma}$ asymptotically as $x \to \infty$. Hence, for all $x > 0$, it is clear that $2^{-1/\sigma} < Q < 1$. We will consider the problem in the inverted form:

$$\frac{dx}{dQ} = -\frac{1}{2 - Q^{-\sigma}}. \tag{17}$$

We note that (16) is autonomous, and in fact, imposing the boundary condition

$$x(Q = 1) = 0 \tag{18}$$

amounts to fixing the x origin. Note that for $Q = O(1)$ we have $Q^{-\sigma} \approx 1$ for small σ, but when Q becomes very small (and specifically when $Q \sim 2^{-1/\sigma}$) then $Q^{-\sigma} \approx 2$ and is no longer approximately equal to one. With this observation it is clear that (17), when treated as a perturbation problem, must be singular as both limiting behaviours of $Q^{-\sigma}$ cannot be captured in one set of scales.

2.1. Outer solution

We can pose an outer problem for $Q = O(1)$ by using the expansion:

$$Q^{-\sigma} \sim 1 - \sigma \ln Q + \dots \tag{19}$$

and it is easily shown that a solution for x in powers of σ satisfying the boundary condition is:

$$x = 1 - Q + \sigma(Q \ln Q - Q + 1) + \dots \; . \tag{20}$$

We note that this outer solution fails to remain asymptotic when $Q = \exp[-O(1/\sigma)]$, and this verifies the need for a boundary layer. The solution above, however, remains convergent for small Q but will only converge extremely slowly, and is of little practical value.

2.2. Inner solution

The inner solution exists by virtue of the fact that $Q^{-\sigma}$ changes behaviour quite suddenly as $Q \to 2^{-1/\sigma}$. The main difficulty in obtaining a boundary layer approximation is in determining the relevant scaling for Q near $2^{-1/\sigma}$. A straightforward approach takes account of the fact that (20) is invalid when $Q = \exp[-O(1/\sigma)]$. For example, if we define

$$Q = 2^{-1/\sigma} q, \tag{21}$$

then a distinguished limit results from choosing

$$x = \frac{2^{-1/\sigma} X}{\sigma} + x_0, \tag{22}$$

whence we find that X satisfies

$$\frac{dX}{dq} \sim -\frac{1}{2 \ln q} + \dots, \tag{23}$$

with matching condition

$$X \sim \sigma q(1 - \ln 2) + \dots \tag{24}$$

as $q \to \infty$, if we choose

$$x_0 \sim 1 + \sigma + \dots . \tag{25}$$

The solution of (23) is

$$X = -\frac{1}{2}\int^{q} \frac{d\tau}{\ln \tau}, \tag{26}$$

but it is clear that this cannot match to (24), and this suggests the existence of an intermediate layer. Without going into details, it is found that pursuit of this straightforward approach is extremely problematic. An alternative approach is to consider the alternative inner scaling:

$$Q = (2 - g(\sigma)z)^{-1/\sigma}, \tag{27}$$

where $g << \sigma$ (for example, $g(\sigma) = \exp(-1/\sigma)$) which implies that $\lim_{\sigma\to 0}(2 - g(\sigma)z)^{-1/\sigma} = 2^{-1/\sigma}$. At leading order the differential equation reduces to

$$\frac{dX}{dz} = -\frac{1}{2}\frac{1}{z}$$

with solution $X = -\frac{1}{2}(\ln z + A)$, where A is an arbitrary constant of integration. We note that the outer solution is a σ power series, which can, in theory, be extended arbitrarily far. On the other hand the inner solution, at this point, has a leading order term which is $O(2^{-1/\sigma}/\sigma)$ when written in terms of x. In the absence of further information, we might assume that there are no terms in the outer solution (though this is not quite satisfactory) to influence the matching [5] to $O(2^{-1/\sigma}/\sigma)$ and we attempt to proceed as if the outer solution (20) is determined to this order [8]. The inner solution (written in terms of x rather than X) then takes the form:

$$x = 1 + \sigma + \cdots - \frac{1}{2}\frac{2^{-1/\sigma}}{\sigma}(\ln g + \ln z + A) \tag{28}$$

with $A = 0$. Note that the exponentially small terms have more than a numerical significance in the inner region [1], [7].

The matching process above was not satisfactory and it is better to think of the inner solution as merely being a local approximation. In fact, elsewhere [3] we show that there is a double boundary layer structure within the 'inner' region. We now turn to different methods of analysis based on an exact representation of the solution, in order to elucidate some of this structure.

3. Analysis of the reduced equation

On recalling that

$$2^{-\frac{1}{\sigma}} < Q < 1 \tag{29}$$

we reconsider the differential equation (17), and look for solutions without exploiting the smallness of σ. Since, from the above inequality, we have

$$\frac{1}{2} < \frac{1}{2}Q^{-\sigma} < 1 \tag{30}$$

we can then expand the denominator as a convergent series $\sum_{n=0}^{\infty} \frac{1}{2^n} Q^{-n\sigma}$. Formally integrating with respect to Q gives

$$x = C - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \frac{Q^{1-n\sigma}}{1 - n\sigma}, \tag{31}$$

where we assume for now that $\frac{1}{\sigma} \notin \mathbb{N}$. Applying the initial conditions gives the solution:

$$x = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \left(\frac{1 - Q^{1-n\sigma}}{1 - n\sigma} \right). \tag{32}$$

This series is absolutely convergent for Q in $\left(2^{-\frac{1}{\sigma}}, 1\right]$ and the partial sums converge uniformly to x, on every interval $c \le Q \le 1$, for any c in $(2^{-\frac{1}{\sigma}}, 1)$. If $\frac{1}{\sigma} \in \mathbb{N}$, then the solution is

$$x = \sum_{n \in \mathbb{N}_0 / \{\frac{1}{\sigma}\}} \frac{1}{2^{n+1}} \left(\frac{1 - Q^{1-n\sigma}}{1 - n\sigma} \right) - \frac{1}{2^{1+\frac{1}{\sigma}}} \ln Q. \tag{33}$$

In this case, defining the positive integer $N \equiv \frac{1}{\sigma}$, the exact solution becomes

$$x = \sum_{n=0}^{N-1} \frac{N\left(1 - Q^{1-\frac{n}{N}}\right)}{2^{n+1}(N-n)} - \frac{\ln Q}{2^{N+1}} + \sum_{n=N+1}^{\infty} \frac{N\left(1 - Q^{1-\frac{n}{N}}\right)}{2^{n+1}(N-n)}.$$

Using the power series identity

$$\sum_{n=N+1}^{\infty} \frac{\alpha^n}{N-n} = \alpha^N \ln(1-\alpha)$$

for $|\alpha| < 1$, we evaluate the latter sum. This gives x as a finite sum, albeit arbitrarily long:

$$x = \sum_{n=0}^{N-1} \frac{N\left(1 - Q^{1-\frac{n}{N}}\right)}{2^{n+1}(N-n)} - \frac{\ln Q}{2^{N+1}} - \frac{N}{2^{N+1}} \ln(2 - Q^{-\frac{1}{N}}). \tag{34}$$

In particular, if $\sigma = \frac{1}{6}$, the solution is

$$\frac{13}{10} - \frac{1}{2}Q - \frac{3}{10}Q^{5/6} - \frac{3}{16}Q^{2/3} - \frac{1}{8}\sqrt{Q}$$
$$- \frac{3}{32}Q^{1/3} - \frac{3}{32}Q^{1/6} - \frac{1}{128}\ln(Q) - \frac{3}{64}\ln\left(2 - \frac{1}{Q^{1/6}}\right). \tag{35}$$

3.1. Solution behaviour away from $Q = 2^{-1/\sigma}$

We now investigate the behaviour of x as $Q \to 1^-$, that is $\ln Q \to 0^-$, by obtaining a power series expansion in σ. In fact, this will be valid for all Q in the range above, except when very close to $2^{-\frac{1}{\sigma}}$.

We first consider the case where $\frac{1}{\sigma} \notin \mathbb{N}$. Writing $Q^{1-n\sigma}$ in (32) as $e^{(1-n\sigma)\ln Q}$, and then expanding this as an exponential power series, cancelling the 1 and finally dividing out by $(1 - n\sigma)$ gives:

$$x = -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{m=1}^{\infty} \frac{(1 - n\sigma)^{m-1} (\ln Q)^m}{m!}.$$

We now shift the subscript m up by one, and interchange the sums to obtain

$$x = -\sum_{m=0}^{\infty} \frac{(\ln Q)^{m+1}}{(m+1)!} \sum_{n=0}^{\infty} \frac{(1 - n\sigma)^m}{2^{n+1}}. \tag{36}$$

Next, we consider the case where $\frac{1}{\sigma} \in \mathbb{N}$. Defining the positive integer $N \equiv \frac{1}{\sigma}$, the exact solution becomes

$$x = \sum_{n=0}^{N-1} \frac{N\left(1 - Q^{1-\frac{n}{N}}\right)}{2^{n+1}(N-n)} - \frac{\ln Q}{2^{N+1}} + \sum_{n=N+1}^{\infty} \frac{N\left(1 - Q^{1-\frac{n}{N}}\right)}{2^{n+1}(N-n)}. \tag{37}$$

Again by writing $Q^{1-\frac{n}{N}}$ as $e^{(1-\frac{n}{N})\ln Q}$, and expanding this as an exponential power series, cancelling the 1 and finally dividing out by $(1 - \frac{n}{N})$, this equation becomes

$$x = -\sum_{n=0}^{N-1} \frac{1}{2^{n+1}} \sum_{m=1}^{\infty} \frac{\left(1 - \frac{n}{N}\right)^{m-1} (\ln Q)^m}{m!} - \frac{\ln Q}{2^{N+1}} - \sum_{n=N+1}^{\infty} \frac{1}{2^{n+1}} \sum_{m=1}^{\infty} \frac{\left(1 - \frac{n}{N}\right)^{m-1} (\ln Q)^m}{m!}.$$

Now we shift the subscript m up by one, interchange the sums and combine them to obtain

$$x = -\sum_{m=0}^{\infty} \frac{(\ln Q)^{m+1}}{(m+1)!} \left(\sum_{n=0}^{N-1} \frac{\left(1 - \frac{n}{N}\right)^m}{2^{n+1}} + \sum_{n=N+1}^{\infty} \frac{\left(1 - \frac{n}{N}\right)^m}{2^{n+1}} \right) - \frac{\ln Q}{2^{N+1}}.$$

Using the convention that $0^0 \equiv 1$, we can combine these into a single sum

$$x = -\sum_{m=0}^{\infty} \frac{(\ln Q)^{m+1}}{(m+1)!} \sum_{n=0}^{\infty} \frac{\left(1 - \frac{n}{N}\right)^m}{2^{n+1}}, \tag{38}$$

which is precisely (36) with σ replaced by $\frac{1}{N}$. So, henceforth, we can treat the two cases as one.

Next, we apply the binomial theorem to $(1 - n\sigma)^m$ and write it as $\sum_{k=0}^{m} \binom{m}{k} (-\sigma)^k$.

Interchanging the sums on n and k then gives

$$x = -\sum_{m=0}^{\infty} \frac{(\ln Q)^{m+1}}{(m+1)!} \sum_{k=0}^{m} \binom{m}{k} (-\sigma)^k \beta_k, \tag{39}$$

where $\beta_k \equiv \sum_{n=0}^{\infty} \frac{n^k}{2^{n+1}}$. The sequence β_k is known from combinatorics. The first few values are $\beta_0 = 1$, $\beta_1 = 1$, $\beta_2 = 3$, $\beta_3 = 13$, etc. It is defined recursively by

$$\beta_k = \sum_{j=0}^{k-1} \binom{k}{j} \beta_j. \tag{40}$$

The sequence may also be expressed explicitly as $\beta_k = \sum_{j=0}^{k} \left\{ {k \atop j} \right\} j!$ where $\left\{ {k \atop j} \right\}$ is a Stirling number of the second kind. Writing out the first few terms of equation (39) then gives

$$x = -\ln Q - \frac{(\ln Q)^2}{2}(1-\sigma) - \frac{(\ln Q)^3}{6}(1 - 2\sigma + 3\sigma^2)$$
$$-\frac{(\ln Q)^4}{24}(1 - 3\sigma + 9\sigma^2 - 13\sigma^3) + O(\ln Q)^4.$$

To obtain a series expansion in σ, we return to (39) and interchange the sums on m and k to get

$$x = -\sum_{k=0}^{\infty} \beta_k (-\sigma)^k \sum_{m=k}^{\infty} \frac{(\ln Q)^{m+1}}{(m+1)!} \binom{m}{k}.$$

We can represent the inner sum in terms of the *incomplete gamma function*

$$\gamma(a, \xi) \equiv \int_0^{\xi} e^{-t} t^{a-1} dt \tag{41}$$

as

$$\sum_{m=k}^{\infty} \frac{(\ln Q)^{m+1}}{(m+1)!} \binom{m}{k} = \frac{(-1)^{k+1}}{k!} \gamma(k+1, -\ln Q). \tag{42}$$

Hence, the desired outer series expression in σ is

$$x = \sum_{k=0}^{\infty} \sigma^k \frac{\beta_k}{k!} \gamma(k+1, -\ln Q). \tag{43}$$

The integer-indexed incomplete gamma function has a finite series representation

$$\gamma(k+1, \xi) = k! \left(1 - e^{-\xi} \sum_{j=0}^{k} \frac{\xi^j}{j!} \right) \tag{44}$$

and so we obtain the final form of the outer expansion:

$$x=\sum_{k=0}^{\infty}\sigma^k\beta_k\left(1-Q\sum_{j=0}^{k}\frac{(-\ln Q)^j}{j!}\right). \tag{45}$$

This is an exact (if impractical) solution of the differential equation (17), of which the first few terms were already derived in (20) via a different route. Expanding a few terms further, we find that:

$$x=(1-Q)+\sigma(1-Q(1-\ln Q))+3\sigma^2\left(1-Q\left(1-\ln Q+\frac{(\ln Q)^2}{2}\right)\right)$$
$$+13\sigma^3\left(1-Q\left(1-\ln Q+\frac{(\ln Q)^2}{2}-\frac{(\ln Q)^3}{6}\right)\right)+O(\sigma^4). \tag{46}$$

3.2. Solution near $Q=2^{-1/\sigma}$

To examine the behaviour of the solution in this region we write Q in the form: $Q=2^{-\frac{1}{\sigma}}e^{\frac{\eta}{\sigma}}$. The inner limit is now $\eta=0$ and the outer boundary condition is $x(\eta=\ln 2)=0$. The differential equation (17) in terms of η becomes

$$\frac{dx}{d\eta}=-\frac{1}{\sigma 2^{1+\frac{1}{\sigma}}}\frac{e^{(1+\frac{1}{\sigma})\eta}}{e^{\eta}-1}. \tag{47}$$

This (times η) is just the exponential generating function of the Bernoulli polynomials. Thus the differential equation is

$$\frac{dx}{d\eta}=-\frac{1}{\sigma 2^{1+\frac{1}{\sigma}}}\sum_{m=0}^{\infty}B_m\left(1+\frac{1}{\sigma}\right)\frac{\eta^{m-1}}{m!}, \tag{48}$$

where $B_m(t)$ is the mth Bernoulli polynomial. Now integrating with respect to η and applying the initial condition $x=0$ when $\eta=\ln 2$ we obtain the series solution

$$x=-\frac{\ln(\frac{\eta}{\ln 2})}{\sigma 2^{1+\frac{1}{\sigma}}}+\frac{1}{\sigma 2^{1+\frac{1}{\sigma}}}\sum_{m=1}^{\infty}B_m\left(1+\frac{1}{\sigma}\right)\frac{(\ln 2)^m-\eta^m}{m(m!)}. \tag{49}$$

We can change the argument of the B_m term using the Bernoulli polynomial identity $B_m(1+t)=B_m(t)+mt^{m-1}$, for $m\geq 1$, and do the factorial sum to write this as

$$x=-\frac{\ln(\frac{\eta}{\ln 2})}{\sigma 2^{1+\frac{1}{\sigma}}}+\frac{1}{2}-\frac{e^{\frac{\eta}{\sigma}}}{2^{1+\frac{1}{\sigma}}}+\frac{1}{\sigma 2^{1+\frac{1}{\sigma}}}\sum_{m=1}^{\infty}B_m\left(\frac{1}{\sigma}\right)\frac{(\ln 2)^m-\eta^m}{m(m!)}. \tag{50}$$

As this last series is not asymptotic, we make a renewed attack on the problem by resorting to the variables of §2. First, we use the change of variables $X\equiv\sigma 2^{\frac{1}{\sigma}}x$ and $Z\equiv 2-Q^{-\sigma}$. The differential equation, in terms of these variables, is

$$\frac{dX}{dZ}=-\frac{1}{2Z}\left(1-\frac{Z}{2}\right)^{-\frac{1}{\sigma}-1} \tag{51}$$

with condition $X(Z = 1) = 0$ at the outer boundary. We can formally integrate this to obtain the solution

$$X(Z) = \int_{\frac{Z}{2}}^{\frac{1}{2}} \frac{(1-t)^{-\frac{1}{\sigma}-1}}{2t}\, dt. \tag{52}$$

We can represent this as a series by expanding $(1-t)^{-\frac{1}{\sigma}-1}$ using the binomial theorem and integrating term by term. One thus finds the series representation

$$X(Z) = -\frac{1}{2}\ln Z + \sum_{n=1}^{\infty} \binom{-\frac{1}{\sigma}-1}{n} \left(-\frac{1}{2}\right)^{n+1} \left(\frac{Z^n}{n}\right) + C(\sigma), \tag{53}$$

where the constant of integration is

$$C(\sigma) \equiv -\sum_{n=1}^{\infty} \frac{1}{n} \binom{-\frac{1}{\sigma}-1}{n} \left(-\frac{1}{2}\right)^{n+1}. \tag{54}$$

Note that the series in Z is absolutely convergent for $|Z| < 2$, and since the constant corresponds to $Z = 1$, its series is also convergent. One easily shows that the terms of the Z series are decreasing in magnitude for $n \geq \frac{Z}{\sigma}$. This means that if $Z \gg \sigma$ or $Z = O(\sigma)$, one may have to add up a large number of terms before they start to decrease, but if $Z = o(\sigma)$, then the terms decrease rapidly from the start. The integration constant $C(\sigma)$, however, always requires $O(\frac{1}{\sigma})$ terms, and so we now find an alternative asymptotic representation of this as $\sigma \to 0$. First we observe that

$$C(\sigma) = \int_0^{\frac{1}{2}} \frac{(1-t)^{-\frac{1}{\sigma}-1} - 1}{2t}\, dt, \tag{55}$$

which upon making the substitution $e^v = 2(1-t)$ becomes

$$C(\sigma) = \int_0^{\ln 2} \frac{1}{2} \left(\frac{2^{\frac{1}{\sigma}+1} e^{-\frac{v}{\sigma}} - e^v}{2 - e^v} \right) dv. \tag{56}$$

An asymptotic expression for $C(\sigma)$ follows from applying Laplace's method to the first part of the integrand in this integral. We use the series expansion

$$\frac{1}{2 - e^v} = \sum_{k=0}^{\infty} \frac{\beta_k v^k}{k!}, \tag{57}$$

which follows from the recursion equation (40) and the usual method for determining the exponential generator of a sequence (in this case β_k). Interchanging sums then gives the divergent asymptotic series

$$C(\sigma) \sim 2^{\frac{1}{\sigma}} \sum_{k=0}^{\infty} \frac{\beta_k}{k!} \int_0^{\ln 2} v^k e^{-\frac{v}{\sigma}}\, dv. \tag{58}$$

Next, we change variable again via $v = w\sigma$ to get

$$C(\sigma) \sim \sigma 2^{\frac{1}{\sigma}} \sum_{k=0}^{\infty} \frac{\beta_k \sigma^k}{k!} \int_0^{\frac{\ln 2}{\sigma}} w^k e^{-w}\, \mathrm{d}w. \tag{59}$$

Finally, the integral can be approximated, introducing only exponentially subdominant terms, by

$$\int_0^{\infty} w^k e^{-w}\, \mathrm{d}w = k!. \tag{60}$$

So the desired asymptotic representation is

$$C(\sigma) \sim \sigma 2^{\frac{1}{\sigma}} \sum_{k=0}^{\infty} \beta_k \sigma^k = \sigma 2^{\frac{1}{\sigma}}(1 + \sigma + 3\sigma^2 + \cdots). \tag{61}$$

To see that this is divergent we obtain the behaviour of β_k for large k. First observe that it can be shown that $\frac{1}{2-e^v}$ has a simple pole at $v = \ln 2$ with residue $-\frac{1}{2}$ there. So in the neighbourhood of $v = -\frac{1}{2}$,

$$\sum_{k=0}^{\infty} \frac{\beta_k v^k}{k!} = \frac{1}{2 - e^v} \sim \frac{-\frac{1}{2}}{v - \ln 2} = \sum_{k=0}^{\infty} \frac{v^k}{2(\ln 2)^{k+1}}. \tag{62}$$

Hence, we obtain the asymptotic form, for large k,

$$\beta_k \sim \frac{k!}{2(\ln 2)^{k+1}}. \tag{63}$$

From this we see that the terms of the series above for $C(\sigma)$ behave as $k!\frac{\sigma^k}{2(\ln 2)^{k+1}}$, so the series is in fact divergent for all $\sigma > 0$. However, the terms decrease rapidly at first until $k = K \equiv \frac{\ln 2}{\sigma}$. By terminating the summation at this point, we are neglecting terms of order $\frac{K!}{K^K}$ (i.e $O(e^{-k})$ via Stirling's formula), which are thus exponentially subdominant in σ.

So the solution in this region has asymptotic form

$$X(Z) \sim -\frac{1}{2}\ln Z + \sum_{n=1}^{\infty} \binom{-\frac{1}{\sigma} - 1}{n} \left(-\frac{1}{2}\right)^{n+1} \left(\frac{Z^n}{n}\right) + \sigma 2^{\frac{1}{\sigma}}(1 + \sigma + 3\sigma^2 + \cdots). \tag{64}$$

Using the fact that $\binom{-1}{n} = (-1)^n$, we can approximate this as

$$X(Z) \sim -\frac{1}{2}\ln Z - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(Z/2)^n}{n} + \sigma 2^{\frac{1}{\sigma}}(1 + \sigma + 3\sigma^2 + \cdots) \tag{65}$$

or more compactly as

$$X(Z) \sim -\frac{1}{2}\ln Z + \frac{1}{2}\ln\left(1 - \frac{Z}{2}\right) + \sigma 2^{\frac{1}{\sigma}}(1 + \sigma + 3\sigma^2 + \cdots). \tag{66}$$

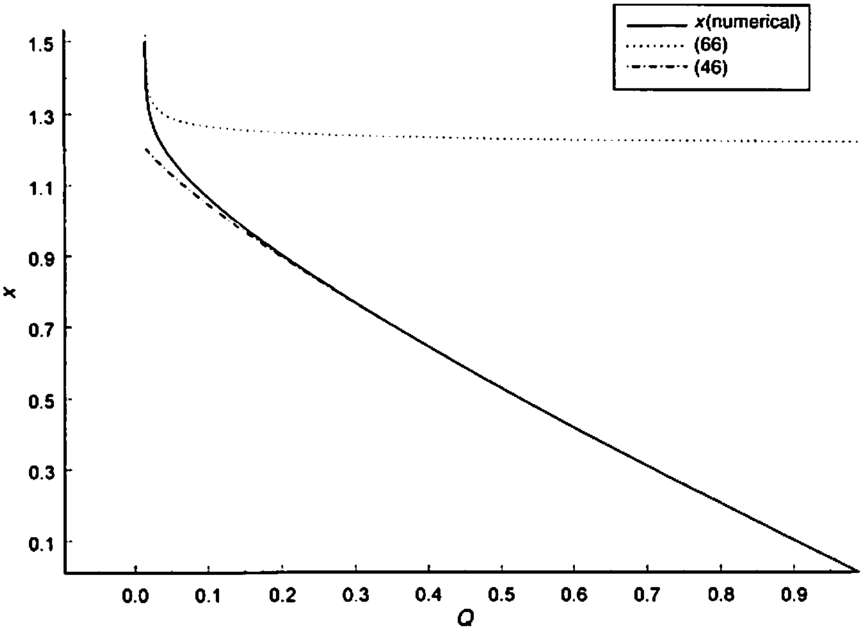

FIG. 2—Numerical and approximate solutions [(46) and (66)] of equation (17) with $\sigma = \frac{1}{6}$, and an initial condition of $Q = 1$ at $x = 0$.

This expansion in fact gives an appropriate representation of the 'inner' solution when Q is exponentially small. The exponentially large constant term simply represents an origin shift, and we see that the variables defined in §2.2 are the 'right' inner variables. However, it is not easy to see how to derive (66) from (51)—a multiple scale procedure with fast and slow variables Z and $(2-Z)^{-1/\sigma}$ is probably necessary (but is not pursued here). Note that when $Z \ll \sigma \ll 1$, $\ln(1-\frac{Z}{2})$ is always much less than $\sigma 2^{\frac{1}{\sigma}}$. The term $-\frac{1}{2}\ln Z$ begins to dominate when $-\frac{1}{2}\ln Z = \sigma 2^{\frac{1}{\sigma}}$, that is when $Z \leq Z_1 \equiv \exp(-\sigma 2^{1+\frac{1}{\sigma}})$. To illustrate the scale involved we look at some values:

σ	$\frac{1}{6}$	0.1	0.01
Z_1	5.4×10^{-10}	1.13×10^{-88}	$10^{-1.1\times 10^{30}}$

which very quickly moves one into the realm of the googolplex! To compare with the Q values, note that if $Z \ll 1$, then $Q \sim 2^{-\frac{1}{\sigma}} e^{\frac{Z}{2\sigma}}$, and if furthermore $Z \ll \sigma \ll 1$, then $Q \sim 2^{-\frac{1}{\sigma}}(1 + \frac{Z}{2\sigma})$. Thus, near the boundary layer $Q - 2^{-\frac{1}{\sigma}} \sim \frac{Z}{2\sigma}$.

4. Discussion

The presence of a small term occurring in an exponent gives rise to surprisingly complicated asymptotic behaviour. Our initial attack on the simplified problem (17) using an asymptotic approach reveals the necessity for matched *asymptotic* expansions in both inner and outer regions, but direct perturbation methods are confounded by the intricacy of the solution behaviour. As an alternative, we have used various series and integral representations of the exact solution to obtain inner (66) and outer (46) approximations to the solution which give a reasonable result from a practical point of view, when plotted against the 'exact' numerical solution in Fig. 2.

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