Enhanced Monte Carlo Methods for Pricing and Hedging Exotic Options

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Abstract

Monte Carlo simulation is a widely used tool in finance for computing the prices of options as well as their price sensitivities, which are known as *Greeks*. The disadvantage of the Monte Carlo simulation, in its standard form, is its slow convergence rate. In the first part of this thesis, we review several methods that they have been proposed, in order to improve the convergence rate of Monte Carlo simulation. These methods find applicability in pricing exotic options such as barrier and lookback options. In the second part of this thesis, we study the applicability of Monte Carlo in estimating price sensitivities. In general, the estimation of Greeks is not as straightforward as that of option prices. Difficulties may arise by discontinuities in the option payoff function, as in the cases of barrier and digital options. The Monte Carlo methods for estimating Greeks can be divided in the following three categories: a) Finite-difference , b) Likelihood Ratio[1] and c) Pathwise methods. In this thesis, we focus on the third method, which usually gives better estimates than the other two methods, when it is applicable. A Pathwise estimator is derived by differentiating the payoff function inside the expectation operator. Thus, the interchange between differentiation and expectation is required. However, this interchange is not applicable in several cases such as the computation of delta and gamma of digital and barrier options. To overcome this obstacle we apply a smoothing technique, i.e. we approximate the discontinuous payoff through a smooth function and then we apply the Pathwise method. Although, additional error is introduced from this smoothing approximation, we can show that sufficiently good estimates of the Greeks can be obtained. Numerical results from computation of both prices and Greeks of several exotic options, are given.

Thesis Supervisor: Prof. Michael Giles

[1]In cases in which the transition density of the underlying price process is not explicitly known, ideas from Malliavin Calculus can be used to extend this method.

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Chapter 1

Introduction

Monte Carlo simulation is an essential tool in the pricing and hedging of derivative securities. A derivative security has a payoff which depends on one or more of the underlying assets. The arbitrage free price of a generic derivative security can be expressed as the discounted expected value of its payoff. Valuing derivatives thus reduces to computing expectations, namely single or multi-dimensional integrals. Valuation of a derivative security by Monte Carlo typically involves simulating paths of stochastic processes, which describe the evolution of underlying asset prices and other factors relevant to the security. Boyle first developed a Monte Carlo simulation approach for valuing options in [Boy77]. Since then, remarkable progress (see [BBG97]) has been achieved, making the Monte Carlo approach a valuable and flexible computational tool in modern finance.

The major disadvantage of the Monte Carlo method is its slow convergence rate, which is $\mathcal{O}(n^{-1/2})$ independently of the dimension of the problem, where n is the number of simulated paths. The latter makes Monte Carlo not a competitive method for pricing and hedging derivative securities when the dimension of the problem is small[1]. This means that huge computational effort is usually needed to obtain sufficiently accurate estimation of derivative prices.

Various techniques have been developed to improve the efficiency of Monte Carlo estimators. These techniques aim at computing time, variance and bias reduction, which are the major criteria for comparing alternative estimators. Various variance reduction techniques such as the control variate approach and antithetic variate method have been used to speed up the convergence rate of Monte Carlo simulation. More recently, moment matching and importance sampling methods have been used to reduce the variance of simulation estimates. Alternatives to Monte Carlo simulation, known as Quasi-Monte Carlo or low-discrepancy methods, use deterministic sequences of numbers instead of random sequences to speed up the convergence rate.

However, all the above methods can do nothing to reduce the bias in Monte Carlo estimates that results from the time-discretization of stochastic differential equations, which describe the evolution of underlying asset prices and other factors relevant to the derivative security under consideration. In the first part of this thesis, we focus on enhanced Monte Carlo methods, which reduce the bias in Monte Carlo estimates. In particular, such methods are discussed and applied to price *path-dependent* derivative securities such as *Barrier* and *Lookback* options.

In the second part of this thesis, we present enhanced Monte Carlo methods for estimating sensitivities of expectations, i.e. the partial derivatives of derivative prices, which are known as

[1] Generally, Monte Carlo simulation becomes competitive when the dimension of the problem under consideration is bigger than 3.

Greeks. Greeks are very important because they determine the trading strategy that hedges the position in the derivative security. We use a smooth function to approximate the discontinuous payoff of an option and then we apply the *pathwise* method to estimate the delta and gamma through Monte Carlo simulation. In particular, we estimate the delta and gamma of a digital call option and down-and-out barrier option. We note that without this smoothing technique the pathwise method is inapplicable in estimating the greeks of those two options. Furthermore, we carry out asymptotic analysis in order to determine the error that the smoothing introduces in our estimations and we show how we can reduce this error. Also, we study how this smoothing affects the variance of Monte Carlo estimates. Finally, we use the stratified sampling method to reduce the variance and thus to improve further the efficiency of the Monte Carlo method.

1.1 Preliminaries

1.1.1 Monte Carlo Basics

Simulation is widely used to solve problems which either do not have analytical solution or it is too difficult to obtain.

As we have already mentioned, Monte Carlo simulation is an easy way to estimate integrals in one or more dimensions. We consider for example the problem of estimating the integral of a Lebesgue integrable function, $f \in L^2(0,1)$, over the unit interval $[0,1]$. We can express this integral as an expectation

$$\mathbb{E}[f(X)] = \int_{[0,1]} f(x)\,dx \ , \tag{1.1}$$

with X uniformly distributed on $[0,1]$. This can be extended to the unit cube $[0,1]^d$ in d dimensions

$$\mathbb{E}[f(\mathbf{X})] = \int_{[0,1]^d} f(\mathbf{x})\,d\mathbf{x} \ , \tag{1.2}$$

where $\mathbf{X}$ is a vector random variable uniformly distributed on $[0,1]^d$. For simplicity, we return to the one dimensional problem. By drawing points X_i randomly, independently and uniformly from $[0,1]$, we can build the Monte Carlo estimator of the integral (1.1)

$$\hat{\mathbb{E}}[f(X)] = \frac{1}{n}\sum_{i=1}^{n} f(X_i) \ . \tag{1.3}$$

Then, by the *strong Law of Large Numbers*, we have that with probability 1

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(X_i) \longrightarrow \int_{[0,1]} f(x)\,dx \ . \tag{1.4}$$

If we set

$$\sigma_f^2 = \int_{[0,1]} \Big(f(x) - \mathbb{E}[f(x)]\Big)^2 dx \ , \tag{1.5}$$

then the error $e_n = \hat{\mathbb{E}}[f(X)] - \mathbb{E}[f(X)]$ in the Monte Carlo estimation is normally distributed, with mean 0 and standard deviation $\sigma_f/\sqrt{n}$, i.e. $e_n \approx N\big(0, \sigma_f^2/n\big)$[2]. Practically, the unknown parameter

[2] The notation $X \sim N(\mu, \sigma^2)$ abbreviates the statement that the random variable X is normally distributed with mean μ and variance σ^2

σ_f is approximated by the unbiased estimator

$$s_f = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\Big(f(X_i) - \hat{\mathbb{E}}[f(X)]\Big)^2} \ . \tag{1.6}$$

Thus, we can conclude that the standard error of the Monte Carlo method has the following form

$$e_n(f) \approx \frac{\sigma_f}{\sqrt{n}}\xi \ , \tag{1.7}$$

where $\xi \sim N(0,1)$. The latter expression shows that the error of the Monte Carlo method is $\mathcal{O}(n^{-1/2})$, independently of the dimension of the problem.

1.1.2 Risk Neutral Pricing

In this thesis, we study the problem of derivative pricing and hedging in a *risk neutral world*, where no arbitrage opportunities exist, i.e. no riskless profit can be made. We consider the Black-Scholes model, in which the price of an underlying asset S_t follows the stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \ , \tag{1.8}$$

where the growth rate μ and volatility σ of S are deterministic constants and W is standard Brownian motion. In this model the price of a risk free asset has the dynamics

$$dB(t) = rB(t)dt \quad , \tag{1.9}$$

where the r is the risk free rate, and all the assets grow at the risk free rate, i.e. $\mu = r$. Further details in risk neutral pricing of derivative securities can be found in Bjork's book [Bjo98].

In this framework, the arbitrage-free price V of a derivative with payoff $f(S(t_1), \ldots, S(t_m))$ at time $T = t_m$, is given by

$$V(t) = e^{-r(T-t)}\mathbb{E}^Q[f(S(t_0), \ldots, S(t_m))|\mathcal{F}_t] \quad , \tag{1.10}$$

where the expectation is taken under the *risk-neutral probability measure*, which is known as the *martingale measure*. $\mathcal{F}_t$ represents the history of the Brownian motion $W(t)$ up to time t, while T is the maturity time of the derivative.

To price the expectation of equation (1.10), we need to simulate the paths of the underlying asset over the time interval $[0, T]$ according to their risk-neutral dynamics (1.8). We can simulate the dynamics of the underlying asset using the Euler approximation to (1.8). The Euler approximation S_i of $S(t_i)$ on a time grid $0 = t_0 < t_1 < \ldots < t_m = T$ is defined as

$$S_{i+1} = S_i + rS_i[t_{i+1} - t_i] + \sigma S_i\sqrt{t_{i+1} - t_i}\, Z_{i+1} \ , \tag{1.11}$$

where $S_0 = S(0)$ is known and Z_i are independent standard normal random variables for $i = 0, \ldots, m-1$. With a fixed time step $\Delta t = t_{i+1} - t_i > 0$, we may write the Euler scheme as

$$S_{i+1} = S_i + rS_i\,\Delta t + \sigma S_i\sqrt{\Delta t}\, Z_{i+1} \ . \tag{1.12}$$

In summary, we use the following simple algorithm to price a derivative security in risk neutral world.

Pricing algorithm by Monte Carlo

1. Simulate the dynamics of the underlying asset using the Euler scheme.
2. Calculate the payoff of derivative security on each path.
3. Discount payoff at risk-free rate.
4. Calculate average over paths.

Table 1.1: Derivative security pricing in risk-neutral world by Monte Carlo.

1.1.3 Convergence Order

We use two categories of error of approximation to measure the quality of a discretization scheme such as Euler scheme. Thus, if $\{S_0, S_1, S_2, \ldots\}$ is any discrete-time approximation to a continuous-time process S then the weak and strong order of convergence of the discretization are defined as follows.

Definition 1.1.1. ***Strong order of Convergence***
We say that the discretization has strong order of convergence $\beta > 0$ if

$$E\big[\|S_m - S(T)\|\big] \leq c\Delta t^{\beta} \ , \tag{1.13}$$

for some constant c.

Definition 1.1.2. ***Weak order of Convergence***
We say that the discretization has weak order of convergence $\beta > 0$ if

$$\big|E[f(S_m)] - E[f(S(T))]\big| \leq c\Delta t^{\beta} \ , \tag{1.14}$$

for some constant c and for all f in a set $C^{2\beta+2}$.

The general Euler scheme, which is described by (1.12) with μ , σ not necessarily constants - they can be functions of $S(t)$ -, typically has a strong order of 1/2, while it often achieves a weak order of 1. The weak order of convergence is of greater interest in derivative pricing, because the bias of the Monte Carlo estimator $E[f(S_m)]$ of the true price $E[f(S(T))]$, where f is the discounted payoff, depends only on the distribution of $S(T)$. Thus in this thesis we concentrate on the weak order of convergence. The Euler scheme achieves a weak order of 1 under some smoothness conditions on μ, σ and f, see Kloeden and Platen [KP92]. However, in several pricing problems, these smoothness conditions are not satisfied, as a result the Euler scheme has a slower weak convergence rate. We will discuss such cases in chapter 2.

1.1.4 Sensitivities

After pricing a derivative security, it is of vital importance to hedge the risks incurred by the position that we have on this derivative security. It is possible to replicate the payoff of a derivative security by a dynamic trading strategy, so that the risk is eliminated. For example, the risk in a short position in a derivative security V is hedged by holding delta units of the underlying asset, where delta is the partial derivative of the derivative security price with respect to the current price of the underlying asset. This strategy is known as delta-hedging.

Hence, the implementation of hedging strategies, such as delta-hedging, requires the knowledge of the sensitivities of the derivative security prices, with respect to the price of the underlying asset as well as to the other model parameters. These sensitivities are known as Greeks. We introduce some standard notation for the Greeks.

Definition 1.1.3. ***Greeks***

$$\begin{aligned} \Delta &= \frac{\partial V}{\partial S} \\ \Gamma &= \frac{\partial^2 V}{\partial S^2} \\ \rho &= \frac{\partial V}{\partial r} \\ \Theta &= \frac{\partial V}{\partial t} \\ \mathcal{V} &= \frac{\partial V}{\partial \sigma} . \end{aligned}$$

An obvious approach to this numerical problem is to compute by Monte Carlo simulation the finite difference approximations of these derivatives. Let $Y(x)$ be the discounted payoff of a contingent claim, with respect to the parameter of interest x. We can simulate n independent values of $Y(x)$ and n independent values of $Y(x+\Delta x)$ for small Δx. Then, by averaging we can get the Monte Carlo estimators $\bar{Y}(x)$ and $\bar{Y}(x+\Delta x)$ of $Y(x)$and $Y(x+\Delta x)$, respectively. Thus the forward difference estimator of the derivative of Y with respect to x is

$$Y_F^{'}(x) = \frac{\bar{Y}(x+\Delta x) - \bar{Y}(x)}{\Delta x} . \tag{1.15}$$

If the simulation of the two estimators are drawn independently then it has been proved that the convergence rate is $\mathcal{O}(n^{-1/4})$. We can improve the convergence rate to $\mathcal{O}(n^{-1/3})$ by using the central difference estimator

$$Y_C^{'}(x) = \frac{\bar{Y}(x+\Delta x) - \bar{Y}(x-\Delta x)}{2\Delta x} . \tag{1.16}$$

Furthermore, by using the common random numbers for both Monte Carlo estimators, one can achieve a convergence rate of nearly $n^{-1/2}$ (see section 7.1 in [Gla04]). However, an important drawback of the finite difference method is that it usually performs very poorly, when the payoff function $Y(x)$ is not smooth enough.

In this thesis, we discuss an alternative method for estimating sensitivities which is called pathwise method, see [Gla04]. The pathwise method differentiates each simulated outcome with respect to the parameter of interest. Let again $Y(x)$ be the discounted payoff of a contingent claim with respect to the parameter of interest x. Then assuming that the payoff of the claim is almost surely continuous[3], we can estimate the derivative of the claim price $V(x) = E[Y(x)]$ with respect to parameter θ, using the pathwise derivative of Y at x

$$Y^{'}(x) = \lim_{\Delta x \to 0} \frac{Y(x+\Delta x) - Y(x)}{\Delta x} . \tag{1.17}$$

[3]Although this does not guarantee that the pathwise method is applicable, in many practical problems this requirement is sufficient. Mathematical conditions that ensure the applicability of pathwise method are discussed in page 393 in [Gla04].

We assume that this derivative exists with probability one at each x. To make (1.17) consistent, we can choose a collection of random variables $\{Y(x), x \in X\}$, which is defined on a probability space $(\Omega, \mathcal{F}, P)$. Then we can fix $\omega \in \Omega$ so that the mapping $x \mapsto Y(x, \omega)$ is a random function on X. Thus we can think $Y^{'}(x) = Y^{'}(x, \omega)$ as the derivative of the random function with respect to the x, with ω held fixed. In other words $Y^{'}(x, \omega)$ is the derivative of the simulation output with respect to x, with the random numbers held fixed.

Under the above assumption the expectation of $Y^{'}(x)$ is an unbiased estimator of the derivative of the price of contingent claim with respect to x. This means that we can write

$$\frac{dV}{dx} = \frac{d}{dx} E[Y(x)] = E\Big[\frac{dY(x)}{dx}\Big] \ . \tag{1.18}$$

Generally the pathwise method yields an unbiased estimate of the derivative of an option if the option's discounted payoff is almost surely continuous[4] in the parameter of differentiation. The discounted payoff is a stochastic quantity, so this rule of thumb requires continuity as the parameter varies with all random elements not depending on the parameter held fixed. However, this rule excludes digital and barrier options. To overcome this obstacle, we use a smooth function to approximate the discounted payoff function of those two options and then apply the pathwise method. We describe this method in chapter 3.

[4]A process $\{Y(x) \to x \in \mathbb{R}\}$ is almost surely continuous at x_0 if $Y(x) \to Y(x_0)$ almost surely as $x \to x_0$. In mathematics, the phrase almost surely is a subtle, precise way to say that something is certain except for cases that almost never happen, though still possible.

Chapter 2

Pricing Exotics Options

In this chapter, we discuss and apply enhanced Monte Carlo methods to price exotic options. Exotic options have more complicated payoffs than the standard European or American options. Most exotic options trade in the over-the-counter[1] market and have been designed to meet particular needs of investors. In this chapter we use Monte Carlo methods to price Barrier and Lookback options. Both of these types of options are known as path-dependent options, because their payoff depends on the path of the price of the underlying asset. In the following section we describe two enhanced Monte Carlo methods and we use them to price specific types of Barrier and Lookback options. We compare the convergence rate of those methods with that of the crude Monte Carlo simulation and we comment on the numerical results that we get from the experiments. We used the high level programming language C to implement these Monte Carlo methods.

2.1 Barrier Options

Barrier options are options where the payoff depends on whether the price of the underlying asset reaches a certain barrier level during a certain period of time. These options are attractive because they are less expensive than the corresponding standard options. Barrier options can be classified in two basic categories, the *knock-out* options and the *knock-in* option. A knock-out option is extinguished when the underlying asset price reaches a certain barrier. On the other hand, a knock-in option is activated only when the underlying asset price reaches this barrier. The simplest such options are otherwise standard calls and puts that are knocked-in or knocked-out by the price of the underlying asset. We consider the case of continuous monitoring of the barrier. Improving the estimates of the prices of the continuously monitored barrier options is of practical importance, since there are option markets such as the currency option market, where the barrier is monitored almost continuously.

First, in section 2.1.1 we describe how we can price barrier options by Monte Carlo simulation, as well as the discretization error that is introduced. In sections 2.1.2 and 2.1.3, we describe two alternatives to crude Monte Carlo simulation methods. We present and comment on numerical results in section 2.1.4.

[1] The market for securities not listed and traded on an organized exchange.

2.1.1 Discretization Error in Pricing

We consider the usual Black-Scholes model where the underlying asset price $S(t)$ follows the stochastic differential equation (1.8), while the price of a contingent claim on $S(t)$ is given by (1.10). Let now B denote the level of the barrier. Then an *up* option has $S(0) > B$ and a *down* option has $S(0) < B$. The asset price reaches the barrier for the first time at

$$\tau_B = \inf\{t > 0 : S(t) = B\} \ . \tag{2.1}$$

A *knock-in call option* with maturity time T and strike price K has payoff

$$f(S(T)) = \begin{cases} (S(T) - K)^+ & , \text{if } \tau_B \le T \\ 0 & , \text{otherwise} \end{cases} \ , \tag{2.2}$$

where $(S(T) - K)^+ = \max\{S(T) - K, 0\}$. The option's price at current time t, is

$$C_{in}(t) = e^{-r(T-t)}\mathbb{E}^Q[(S(T) - K)^+ \mathbf{1}\{\tau_B \le T\}] \ , \tag{2.3}$$

where $\mathbf{1}\{\tau_B \le T\}$ denotes the indicator function and is defined as follows

$$\mathbf{1}\{\tau_B \le T\} = \begin{cases} 1 & , \text{if } \tau_B \le T \\ 0 & , \text{otherwise} \end{cases} \ . \tag{2.4}$$

We obtain the price of a *knock-in put option* by replacing $(S(T) - K)^+$ with $(K - S(T))^+$ in (2.3). For knock-out options we replace the event $\{\tau_B \le T\}$ with its complement $\{\tau_B > T\}$. Analytic pricing formulas for all eight types of this kind of barrier options are provided by Rubinstein and Reiner in [RR91].

We can price these options using the Monte Carlo method by simulating $S(T)$ and the indicator function $\mathbf{1}\{\tau_B \le T\}$. We can do this by setting

$$\hat{\tau}_B = \inf\{i : S_i > B\} \quad , \tag{2.5}$$

for $i = 1, \dots m$ and $S_0 = S(0)$. We suppose that $B > S(0)$. Thus it is possible to approximate $(S(T), \mathbf{1}\{\tau_B \le T\})$ by $(S_m, \mathbf{1}\{\hat{\tau}_B \le m\})$ with $\Delta t = T/m$ for some discretization[2] of $S(t)$. However, even if we could simulate S exactly on the discrete grid $0, \Delta t, 2\Delta t, \dots, m\Delta t$ this would not sample $\mathbf{1}\{\tau_B \le T\}$ exactly. It is possible for a simulated path of S to cross the barrier at some time t between two grid points $i\Delta t$ and $(i+1)\Delta t$ and never be above the barrier at any of the times $0, \Delta t, 2\Delta t, \dots, m\Delta t$, see figure 2.1. This means that the knock-in call option will not be activated in this case. Thus, its payoff will be 0 instead of $(S(T) - K)^+$ and error is introduced in our estimation.

In the following two sections, we show how we can reduce the discretization error in the sampling of indicator $\mathbf{1}\{\tau_B \le T\}$ using two smart ideas before we apply the Monte Carlo simulation to price the barrier options.

2.1.2 Correction Method

Broadie, Glasserman and Kou show in [BGK97] how to adjust the pricing formula of a continuously monitored barrier option to obtain a better approximation to the price of the discretely monitored barrier option. They showed that to use the continuous price as an approximation to the discrete

[2] We denote by S_i the approximation of $S(i\Delta t)$.

Figure 2.1: Asset price path that follows Geometric Brownian Motion.

price, we should first shift the barrier B away from $S(0)$ by a factor $e^{\beta\sigma\sqrt{\Delta t}}$, if $B > S(0)$ (up options) and $e^{-\beta\sigma\sqrt{\Delta t}}$, if $B < S(0)$ (down options). In the latter expressions $\beta \approx 0.5826$ and $\Delta t = \frac{T}{m}$ is the time interval between the m monitoring points. Also, they showed that the corrected pricing formula reduces the error from $\mathcal{O}(1/\sqrt{m})$ to $o(1/\sqrt{m})$. The above are summarized in the following theorem.

Theorem 2.1.1. *Let $V_m(B)$ be the price of a discretely monitored knock-in or knock-out down call or up put with barrier H. Let $V(B)$ be the price of the corresponding continuously monitored barrier option. Then*

$$V_m(B) = V(Be^{\pm\beta\sigma\sqrt{\Delta t}}) + o(\frac{1}{\sqrt{m}}) , \tag{2.6}$$

where $+$ and $-$ applies if $B > S(0)$ and $B < S(0)$, respectively, and $\beta = -\zeta(\frac{1}{2})/\sqrt{2\pi} \approx 0.5826$ where ζ is the Riemann zeta function.

We apply similar corrections in pricing continuously monitored barrier options by Monte Carlo. In particular, we first shift the barrier by a quantity[3] know $-B\beta\sigma\sqrt{\Delta t}$ if $B > S(0)$ and $+B\beta\sigma\sqrt{\Delta t}$ if $B < S(0)$ and then we apply the regular Monte Carlo method to price the options.

2.1.3 Probabilistic Method

In this section, we present a probabilistic method, which allows us to estimate the probabilities p_i that the indicator $\mathbf{1}\{\tau_B \leq T\}$ takes the value 1 in time interval $[i\Delta t, (i+1)\Delta t)$ or for simplicity

[3]By Taylor series expansion we have that $e^{\pm\beta\sigma\sqrt{\Delta t}} \approx 1 \pm \beta\sigma\sqrt{\Delta t}$.

$[t_i, t_{i+1})$ and $i = 0, 1, \ldots, m-1$, i.e. the probabilities that the asset price crosses the barrier over this time interval.

We consider a random path of the underlying asset price S, starting from an initial level S_i at time t_i, ending to a final level S_{i+1} at time t_{i+1} where $t_{i+1} - t_i = \Delta t$. We are looking for the conditional probability that S hits the barrier during time interval $[t_i, t_{i+1})$ given its initial and final values S_i and S_{i+1}, respectively. This is equivalent to calculating the probability that the process $Z(t) = \ln S(t)$ hits the barrier B in time interval $[t_i, t_{i+1})$ given its initial and final values $Z_i = \ln S_i$ and $Z_{i+1} = \ln S_{i+1}$, respectively. By considering a Brownian bridge from Z_i to Z_{i+1} over $[t_i, t_{i+1})$, we can calculate the probability that the barrier is crossed. We can do this by deriving the probability that the maximum or minimum of the Brownian bridge is above or below the barrier, depending on whether we are pricing *up* or *down* barrier option. Results can be extended easily for double barrier options.

We consider the case in which $B > S_i$ or equivalently $\ln B = b > Z_i$. The asset price S follows the stochastic differential equation

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW(t) \tag{2.7}$$

and by Ito's Lemma, we obtain

$$dZ(t) = d\ln S(t) = (r - \frac{\sigma^2}{2})\,dt + \sigma\,dW(t)\ . \tag{2.8}$$

From the last equation, we see that the process $Z(t)$ follows a Brownian motion with drift $r - \frac{\sigma^2}{2}$ and diffusion coefficient σ^2. Hence we have that $dZ \sim N((r - \frac{\sigma^2}{2})\,dt, \sigma\sqrt{dt})$, while $dW \sim N(0, dt)$.

We consider now a random path of the Z, starting from an initial level Z_i at time t_i, ending at a final level Z_{i+1} at time t_{i+1}, see figure 2.2. Because of the symmetry with respect to barrier b of the Brownian motion starting at (t_i, Z_i) crossing the barrier at some time $t_i < \tau_b < t_{i+1}$ and ending at (t_{i+1}, Z_{i+1}), the probability of doing this is the same as the probability of going from point (t_i, Z_i) to the point $(t_{i+1}, Z^{'}_{i+1})$. This is known as the reflection principle of Brownian motion and further details can be found in pages 78-79 in [KS91]. The probability that a path of Z starts from Z_i at time t_i and ends at Z_{i+1} at time t_{i+1} is given by the transition probability density function[4]

$$p(Z_{i+1}, t_{i+1}; Z_i, t_i) = \frac{1}{\sigma\sqrt{2\pi\Delta t}} \exp\big[-\frac{(Z_{i+1} - Z_i - a\Delta t)^2}{2\sigma^2\Delta t}\big]\ ,$$

where $a = r - \frac{\sigma^2}{2}$ is the drift. Hence, we can state that

$$p\left\{Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1}\right\} = \frac{1}{\sigma\sqrt{2\pi\Delta t}} \exp\big[-\frac{(Z_{i+1} - Z_i - a\Delta t)^2}{2\sigma^2\Delta t}\big]\ . \tag{2.9}$$

Thus, the probability that such a path crosses the barrier b at some time $t_i < \tau_b < t_{i+1}$ given the initial and final values of Z, is

$$P\left\{t_i < \tau_b < t_{i+1} | Z_i, Z_{i+1}\right\} = \frac{p\left\{t_i < \tau_b < t_{i+1}, Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1}\right\}}{p\left\{Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1}\right\}}\ . \tag{2.10}$$

[4]Generally a Brownian motion $Z(t) = \mu t + \sigma W(t)$ is a Markov process, i.e. it has independent increments, and its transition probability density, namely the probability that $X(t+s) = y$ given that $X(s) = x$, is $p(y, t; x, s) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\big[-\frac{(x-y-\mu t)^2}{2\sigma^2 t}\big]$.

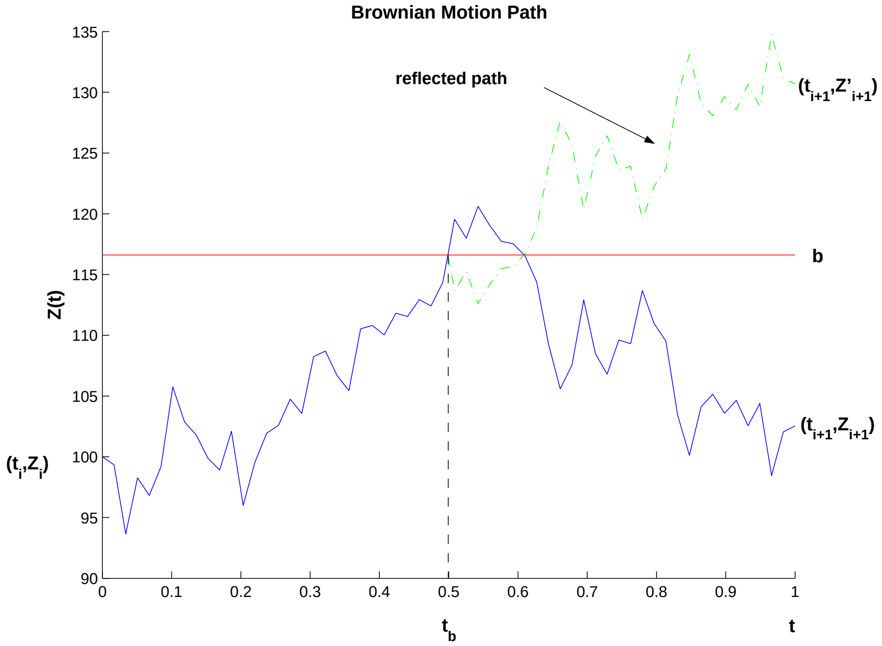

Figure 2.2: Brownian motion path and its reflected path.

We already know the denominator of (2.10), which is given by (2.9) and therefore, it remains to calculate the numerator. First note that

$$\begin{aligned} p\left\{t_i < \tau_b < t_{i+1}, Z_i, Z_{i+1}\right\} &= p\left\{t_i < \tau_b < t_{i+1}\right\} p\left\{Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1} | t_i < \tau_b < t_{i+1}\right\} \\ &= p\left\{t_i < \tau_b < t_{i+1}\right\} p\left\{Z(t_i) = Z_i, Z(t_{i+1}) = Z^{'}_{i+1} | t_i < \tau_b < t_{i+1}\right\} \\ &= p\left\{Z(t_i) = Z_i, Z(t_{i+1}) = Z^{'}_{i+1}, t_i < \tau_b < t_{i+1}\right\} \ . \end{aligned} \tag{2.11}$$

In the second line of the above equation we used the reflection principle of the Brownian motion, which states that the probability of a path of Z starting at (t_i, Z_i) crossing the barrier at some time $t_i < \tau_b < t_{i+1}$ and ending at (t_{i+1}, Z_{i+1}) is equal to the probability of a path starting from the same point and ending to the point $(t_{i+1}, Z^{'}_{i+1})$. Also from figure 2.2 is easy to see that $Z^{'}_{i+1} = 2b - Z_{i+1}$.

Now, it can be shown that the probability that we want, is given by

$$p\left\{t_i < \tau_b < t_{i+1}, Z_i, Z_{i+1}\right\} = e^{\frac{2a(b-Z_i)}{\sigma^2}} \frac{1}{\sigma\sqrt{\Delta t}}\, n\left(-\frac{(2b - Z_{i+1} - Z_i + a\Delta t)}{\sigma\sqrt{\Delta t}}\right) \ , \tag{2.12}$$

where $n(x)$ is the density of the standard normal distribution. The derivation of the latter probability is given in section A.1 in Appendix A. Further details for expectation pricing of barrier options can be found in chapter 4 in [Kwo98], as well as in chapter 8 in [Jos03].

Substituting (2.9) and (2.12) in (2.10) we get

$$P\{t_i < \tau_b < t_{i+1} | Z_i, Z_{i+1}\} = \frac{p\{t_i < \tau_b < t_{i+1}, Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1}\}}{p\{Z(t_i) = Z_i, Z(t_{i+1}) = Z_{i+1}\}}$$

$$= \frac{e^{\frac{2a(b-Z_i)}{\sigma^2}} \frac{1}{\sigma\sqrt{2\pi\Delta t}} \exp\left[-\frac{(2b-Z_{i+1}-Z_i+a\Delta t)^2}{2\sigma^2\Delta t}\right]}{\frac{1}{\sigma\sqrt{2\pi\Delta t}} \exp\left[-\frac{(Z_{i+1}-Z_i-a\Delta t)^2}{2\sigma^2\Delta t}\right]}$$

and by doing some algebra we obtain

$$P\{t_i < \tau_b < t_{i+1} | Z_i, Z_{i+1}\} = \exp\left[-\frac{2(b - Z_{i+1})(b - Z_i)}{\sigma^2\Delta t}\right] . \tag{2.13}$$

We can write the above equation in terms of the underlying asset price S, by replacing $Z_i = \ln S_i$, $Z_{i+1} = \ln S_{i+1}$ and $b = \ln B$ to get

$$P\{t_i < \tau_B < t_{i+1} | S_i, S_{i+1}\} = \exp\left[-\frac{2(\ln B - \ln S_i)(\ln B - \ln S_{i+1})}{\sigma^2\Delta t}\right] . \tag{2.14}$$

Similarly, we can show that when the initial value S_i of the asset price is above the barrier B, then the probability that a random path of S crosses the barrier during time interval $[t_i, t_{i+1})$, given its initial and final values S_i and S_{i+1}, respectively, is given by

$$P\{t_i < \tau_B < t_{i+1} | S_i, S_{i+1}\} = \exp\left[-\frac{2(\ln S_i - \ln B)(\ln S_{i+1} - \ln B)}{\sigma^2\Delta t}\right] . \tag{2.15}$$

Baldi first derived these probabilities in [Bal95], and later Baldi, Caramellino and Iovino in [BCI99] extended these probabilities for the case where the barrier is not constant but depends on time. They used these probabilities to evaluate single and double barrier options using Monte Carlo simulation.

Now, we can use the probability (2.14) to calculate the probability that the asset price path crosses the barrier B at some time $\tau_B \in [0, T]$. This probability is the complement of the probability that the asset path does not cross the barrier during the time interval $[0, T]$. The latter probability can be calculated as the product of the probabilities $\hat{p}_i = 1 - p_i$, for $i = 0, 1, \ldots m - 1$, that S does not cross the barrier during the time interval $[t_i, t_{i+1})$. Of course the probability p_i is known and is given by (2.14). Thus, we have that

$$\begin{aligned} P\{t_B \leq T | S(T) = S_m\} &= 1 - P\{t_B > T | S(T) = S_m\} \\ &= 1 - \textstyle\prod_{i=0}^{m-1} \hat{p}_i \\ &= 1 - \textstyle\prod_{i=0}^{m-1} \left(1 - p_i\right) \\ &= 1 - \textstyle\prod_{i=0}^{m-1} \left(1 - \exp\left[-\frac{2(\ln B - \ln S_i)(\ln B - \ln S_{i+1})}{\sigma^2\Delta t}\right]\right)^+ . \end{aligned} \tag{2.16}$$

We return now to our initial goal, which is to evaluate a knock-in call option, which has a value that is given by (2.3). We can approximate the payoff of the option by

$$(S_m - K)^+ \prod_{i=0}^{m-1} \mathbf{1}\{t_i \leq \tau_B < t_{i+1}\} \tag{2.17}$$

and hence the conditional expectation of this expression, given the approximated discrete values of asset price S at times t_i for $i = 0, 1, \dots, m$, is

$$\begin{aligned}
&\mathbb{E}^Q\Big[(S_m - K)^+ \textstyle\prod_{i=0}^{m-1} \mathbf{1}\{t_i \le \tau_B < t_{i+1}\} \,\big|S_0, S_1 \dots S_m\Big] \\
&= (S_m - K)^+ \textstyle\prod_{i=0}^{m-1} \mathbb{E}^Q\big[\mathbf{1}\{t_i \le \tau_B < t_{i+1}\} \,\big|S_i, S_{i+1}\big] \\
&= (S_m - K)^+ P\{t_B \le T | S(T) = S_m\}\big] \ .
\end{aligned} \tag{2.18}$$

Finally by (2.16) and (2.18) we can approximate the payoff of the knock-in call option as follows

$$(S_m - K)^+ P\{t_B \le T | S(T) = S_m\} \quad , \tag{2.19}$$

where the probability in the above expression is given by (2.16), in the case in which the asset price is below the barrier initially. We can easily modify this probability using (2.15) to price down barrier options, i.e. when the asset price is below the barrier initially. Furthermore these results can be extended to double barrier options pricing.

In summary, using the reflection principle of the Brownian motion we can calculate explicitly the probability that the underlying asset price S crosses the barrier B in a time interval $[t_i, t_{i+1})$ given the initial and final values of S. This method reduces the discretization error that the Euler scheme introduces in estimation of the barrier option pricing by Monte Carlo simulation. The latter is verified by numerical experiments that we present in the next section.

2.1.4 Simulation Results

In this section, we present the numerical results from our experiments. We use the two Monte Carlo methods that we described in previous sections as well as the crude Monte Carlo simulation to price a *Down-and-Out call option*. Next, we describe briefly the implementation. Finally, we compare the convergence rate of the crude Monte Carlo simulation with that of the enhanced Monte Carlo methods and we comment on the results.

We consider the case of a continuously monitored Down-and-Out call barrier option. A Down-and-Out barrier option remains active[5], provided the price of the underlying asset S does not cross a barrier B at any time $t \in [0, T]$ during the life of the option, where $S_0 > B$. We will use the exact price of that option, which is calculated by the analytical pricing formulas provided by Reiner and Rubinstein [RR91], to compare with the prices we get by Monte Carlo simulation methods. Assuming that the barrier level is lower than or equal to the strike price, the Down-and-Out call option price is given by the following formula

$$c_{down-out} = c_{BS} - c_{down-in} \quad , \tag{2.20}$$

where c_{BS} is the value of the European call option under the Black-Scholes formula and $c_{down-in}$ is the Down-and-In call option, both with the same characteristics with Down-and-Out barrier option. The value of the Down-and-In barrier option is given by the following formula.

$$c_{down-in} = S_0 \Big(\frac{B}{S_0}\Big)^{2m} \Phi(y) - K e^{-rT} \Big(\frac{B}{S_0}\Big)^{2m-2} \Phi(y - \sigma\sqrt{T}) \ , \tag{2.21}$$

[5]The holder of the option gains the right to exercise the option.

where

$$y = \frac{\ln\left(B^2/(S_0K)\right)}{\sigma\sqrt{T}} + m\sigma\sqrt{T} \ ,$$

$$m = \frac{r-0.5\sigma^2}{\sigma^2} \ .$$

$\Phi(x)$ is the cumulative normal distribution of x, B is the barrier value, S_0 is the initial underlying asset price, K is the strike price, r is the risk-free interest rate, T is the maturity time of the option and σ is the volatility of the asset price.

We calculate the prices of a Down-and-Out call option with fixed values $S = 100$, $K = 100$, $r = 0.1$, $T = 1$ and for varying values of barrier B, time step Δt and volatility σ. We use 10^6 Monte Carlo replications in each experiment. Table 2.1 shows the prices of the Down-and-Out call option calculated by the three Monte Carlo methods, when barrier B is equal to 75, 85, 92 and 99, for constant volatility $\sigma = 0.5$. In the same table we give the absolute error of those methods, as well as the standard error which is estimated as $\frac{s_n}{\sqrt{n}}$[6]. We call *Correction 1* the method of section 2.1.2 and *Correction 2* the method of section 2.1.3. Crude or *Without Correction* Monte Carlo method is compared with the above methods. Also, we give graphs which show the convergence rate of each method in logarithmic scale. We give graphs for the cases where $\sigma = 0.3, 0.5, 0.8$.

			Price			**Absolute Error**		
Barrier	**Δt**	**$s_n/\sqrt{n}$**	**Without Cor/tion**	**Cor/tion (1)**	**Cor/tion (2)**	**Without Cor/tion**	**Cor/tion (1)**	**Cor/tion (2)**
75	0.050000	0.041610	22.13807	20.79079	20.75181	1.59926	0.25198	0.21300
	0.025000	0.041785	21.66408	20.60121	20.57608	1.12527	0.06240	0.03727
	0.012500	0.042033	21.40599	20.60248	20.59383	0.86718	0.06367	0.05503
	0.006250	0.041996	21.10827	20.52295	20.51415	0.56946	0.01586	0.02465
	0.003125	0.042025	20.98937	20.56308	20.56022	0.45057	0.02427	0.02142
85	0.050000	0.040370	18.72014	15.73397	15.50707	3.57909	0.59292	0.36602
	0.025000	0.040353	17.79299	15.44254	15.38068	2.65194	0.30148	0.23963
	0.012500	0.040028	17.02166	15.24815	15.22180	1.88061	0.10709	0.08074
	0.006250	0.039725	16.51273	15.23136	15.21365	1.37167	0.09031	0.07259
	0.003125	0.039579	16.15809	15.21323	15.20914	1.01703	0.07218	0.06808
92	0.050000	0.038067	14.88035	10.81302	9.49655	5.71225	1.64492	0.32845
	0.025000	0.037022	13.25541	9.87960	9.43394	4.08731	0.71150	0.26584
	0.012500	0.035859	12.03064	9.41718	9.31648	2.86254	0.24908	0.14838
	0.006250	0.035119	11.25417	9.30504	9.28269	2.08607	0.13694	0.11459
	0.003125	0.034394	10.63492	9.21871	9.21056	1.46682	0.05061	0.04246
99	0.050000	0.033369	10.11251	5.91648	1.23780	8.83189	4.63586	0.04283
	0.025000	0.030084	7.68624	4.37868	1.26985	6.40561	3.09806	0.01077
	0.012500	0.026712	5.82726	3.28129	1.28171	4.54664	2.00067	0.00108
	0.006250	0.023687	4.44832	2.53110	1.28511	3.16770	1.25048	0.00449
	0.003125	0.021065	3.48525	2.02422	1.29933	2.20462	0.74359	0.01871

Table 2.1: Down-and-Out Call, when $\sigma = 0.5$ and the exact price is 20.53881, 15.14105, 9.16810, 1.28062 for $B = 75, 85, 92, 99$ respectively.

[6]Standard sample deviation s_n is estimated as $s_n = \sqrt{\frac{1}{n-1}\sum_i^n (X_i - \bar{\mu})^2}$, where n is the number of Monte Carlo replications, X_i is the options price in $i - th$ simulation and $\bar{\mu} = \frac{1}{n}\sum_{i=1}^n X_i$ the mean of option price.

From the results of table 2.1 and the figures 2.3, 2.4, 2.5 we can make the following remarks :

1. No matter what is the value of the barrier, Monte Carlo methods with correction are much more accurate and have faster convergence rate than the crude Monte Carlo simulation.

2. In extreme case where $B = 99$, i.e. the barrier is very close to initial asset price $S_0 = 100$, we see that *Correction 2* gives remarkably accurate results even if the time step Δt is not so small, i.e. $\Delta t = 0.05$. On the other hand, in this extreme case *Correction 1* does not improve enough the efficiency of Monte Carlo estimates.

3. The Euler scheme has weak order of convergence 1/2 which is smaller than the optimal weak order of convergence that it can achieve, namely 1.

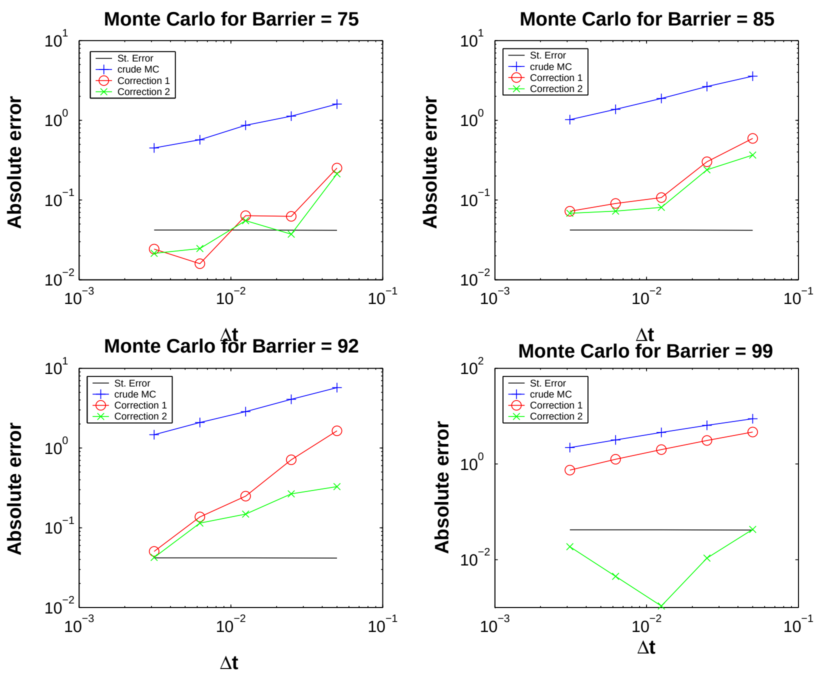

Figure 2.3: Down-and-Out Call - Convergence rate, $\sigma = 0.5$.

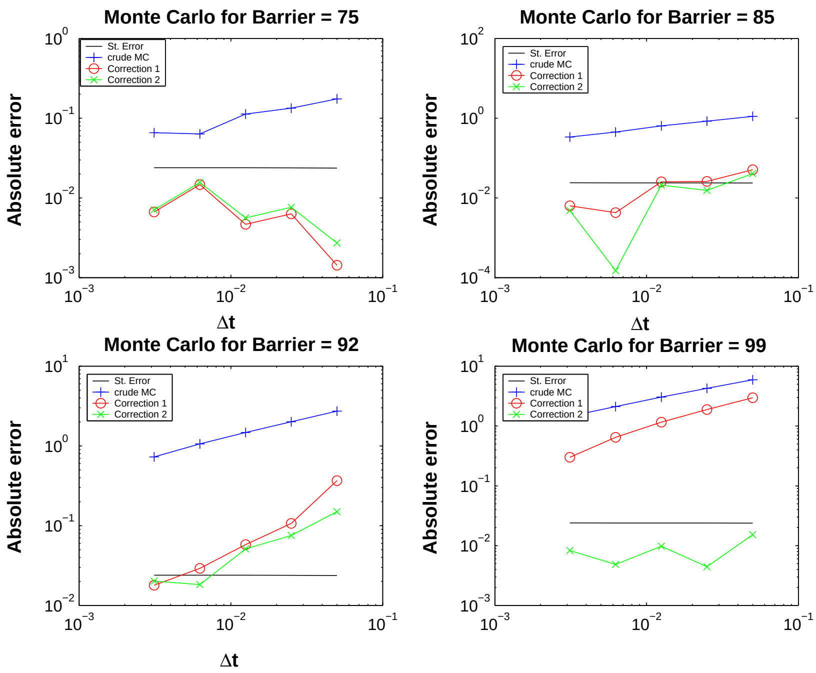

Figure 2.4: Down-and-Out Call - Convergence rate, $\sigma = 0.3$.

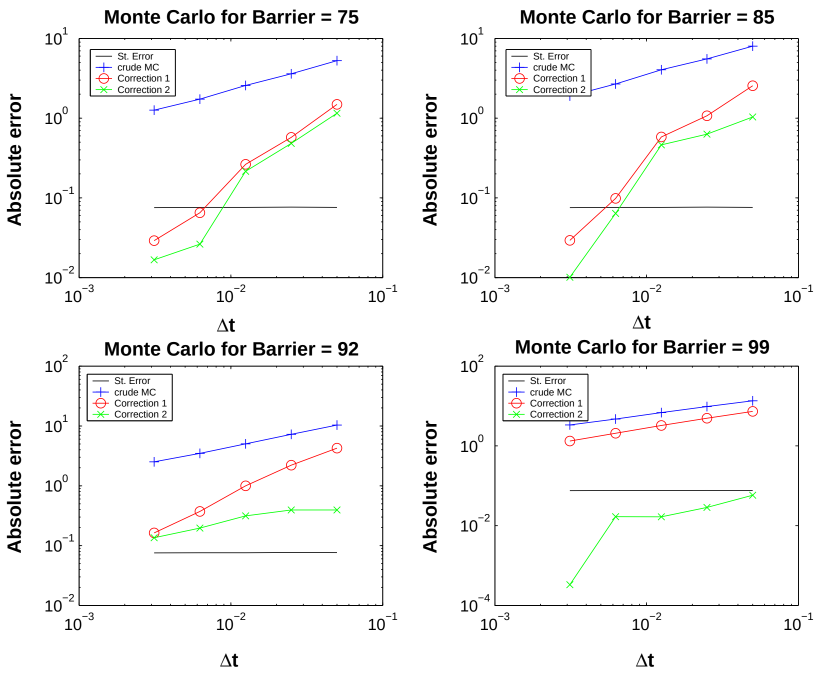

Figure 2.5: Down-and-Out Call - Convergence rate, $\sigma = 0.8$.

2.2 Lookback Options

The payoffs of Lookback options depend on the maximum or the minimum underlying asset price reached during the life of the option. There are several types of such options. The two basic types are the *floating strike* Lookback options and the *fixed strike* Lookback options.

The payoff of a floating strike Lookback call option is the difference between the minimum underlying asset price S_{min} achieved during the life of the option and the final asset price $S(T)$. Thus the payoff is

$$Float\ Call = (S(T) - S_{min})\ . \tag{2.22}$$

Similarly, the payoff of a floating strike Lookback put option is the difference between the maximum underlying asset price S_{max} achieved during the life of the option and the final asset price $S(T)$. Hence, the payoff is

$$Float\ Put = (S_{max} - S(T))\ . \tag{2.23}$$

The payoff of a fixed strike Lookback option has similar payoff to that of a standard option, with strike price K, except that the final underlying asset price $S(T)$ is replaced by the maximum and minimum asset price reached during the life of the option for a call and put, respectively. Thus, the payoffs of the fixed strike Lookback options are the following

$$Fixed\ Call = (S_{max} - K)^+ \tag{2.24}$$

$$Fixed\ Put = (K - S_{min})^+\ . \tag{2.25}$$

In this thesis, we concentrate on floating strike Lookback options. However, similar techniques can be applied to the other types of Lookback options. The floating strike Lookback options allow investors with special information on the range of the asset price to take advantage of such information, according to Goldman, Sosin and Gatto [GSG97], who introduced this type of option in 1979. A Lookback call is a way that the investor can buy the underlying asset at the lowest price during the life of the option, while a Lookback put allows an investor to sell the asset at the highest price achieved during the life of the option.

In particular, in section 2.2.1, we describe how we can price floating strike Lookback options by Monte Carlo simulation. Also, we present the discretization error that is introduced. In section 2.2.2, we describe an alternative to crude Monte Carlo simulation method that eliminates the discretization error which is introduced by the estimation of the maximum or the minimum of the underlying asset price. Finally in section 2.2.3, we give some numerical results and compare the crude and the enhanced Monte Carlo methods.

2.2.1 Discretization Error in Pricing

Again, we consider the usual Black-Scholes model where the underlying asset price S follows the stochastic differential equation (1.8), while the price of an option on S is given by (1.10).

As we have seen, a *floating strike Lookback call option*, with maturity time T, has payoff

$$f(S(T)) = (S(T) - S_{min}) \tag{2.26}$$

and thus, its price at current time t is

$$C_{float}(t) = e^{-r(T-t)}\mathbb{E}^Q[(S(T) - S_{min})]\ . \tag{2.27}$$

Similarly the price of a *floating strike Lookback put option* is given by

$$P_{float}(t) = e^{-r(T-t)}\mathbb{E}^Q[(S_{max} - S(T))] \ . \tag{2.28}$$

We can price these Lookback options using Monte Carlo method, by simulating the final asset price $S(T)$ and the maximum S_{max} or the minimum S_{min} of the asset price in time interval $[0,T]$. We concentrate on Lookback put option pricing and the estimation of S_{max}. The analysis is analogous for the call option pricing. Let $S_0, S_1, \dots, S_m$ be the Euler approximation of S over $[0, m\Delta t]$, with $\Delta t = T/m$. Then the estimate

$$\hat{S}_{max} = \max\{S_0, S_1, \dots, S_m\} \tag{2.29}$$

is the maximum of the Euler approximation to S over $[0, m\Delta t]$.

However, even if we could simulate S exactly on the discrete grid $0, \Delta t, 2\Delta t, \dots, m\Delta t$ this would not sample S_{max} exactly. It is possible that the maximum of the asset price, for a simulated path of S, to be achieved at some time t between two grid points $t_i = i\Delta t$ and $t_{i+1} = (i+1)\Delta t$. This means that the estimate $\hat{S}_{max}$ will be below the true maximum of S, see figure 2.6. Hence, the floating strike Lookback call option is underestimated in this case. On the other hand, the Euler approximation $\hat{S}_{min}$ is always bigger than the true minimum of S and therefore the floating strike Lookback call option is overestimated.

Obviously, if we can find a better way to simulate the extremes of the asset price S, then we can improve the estimates of the Lookback option prices. Fortunately, this can be done by calculating the probability distributions of the maximum S_i^{max} and minimum S_i^{min} of S, over each time interval $[t_i, t_{i+1}]$ for $i = 0, 1, \dots, m-1$. Then, we can approximate the maximum and minimum of S over $[0,T]$ as

$$\begin{aligned} \hat{S}_{max} &= \max\{S_i^{max} : i = 0, 1, \dots m\} \\ \hat{S}_{min} &= \min\{S_i^{min} : i = 0, 1, \dots m\} \ . \end{aligned}$$

We describe a method to do this in the next section.

2.2.2 Error Reduction

Andersen and Brotherton-Ratcliffe suggested a method in [ABR96], which reduces the discretization error and improves the Monte Carlo estimate for a Lookback option price. They achieved this by calculating the probability distributions of the maximum and minimum of the underlying asset price over each time interval $[t_i, t_{i+1}]$ for $i = 0, 1, \dots, m-1$, conditional upon S_i and S_{i+1}. Their calculation was based on the fact that these probability distributions are directly related to the probability distribution of the time of the first hit of a barrier, where the barrier in this case is a variable. Let τ_{max} be the first time that the asset price S reaches its maximum value S_i^{max} over $[t_i, t_{i+1}]$. Thus for any interval $[t_i, t_{i+1}]$, we have that

$$P\{\tau_{max} \in [t_i, t_{i+1}] | S_i, S_{i+1}\} = P\{\max(S(t) : t \in [t_i, t_{i+1}]) \geq S_i^{max} | S_i, S_{i+1}\} \ . \tag{2.30}$$

However, we have already calculated the above probability distributions in section 2.1.3, which is given by the equation (2.14). In this case, we rewrite this probability as follows

$$P\{\max(S(t) : t \in [t_i, t_{i+1}]) \geq S_i^{max} | S_i, S_{i+1}\} = \exp\left[-\frac{2(\ln S_i^{max} - \ln S_i)(\ln S_i^{max} - \ln S_{i+1})}{\sigma^2 \Delta t}\right] \ . \tag{2.31}$$

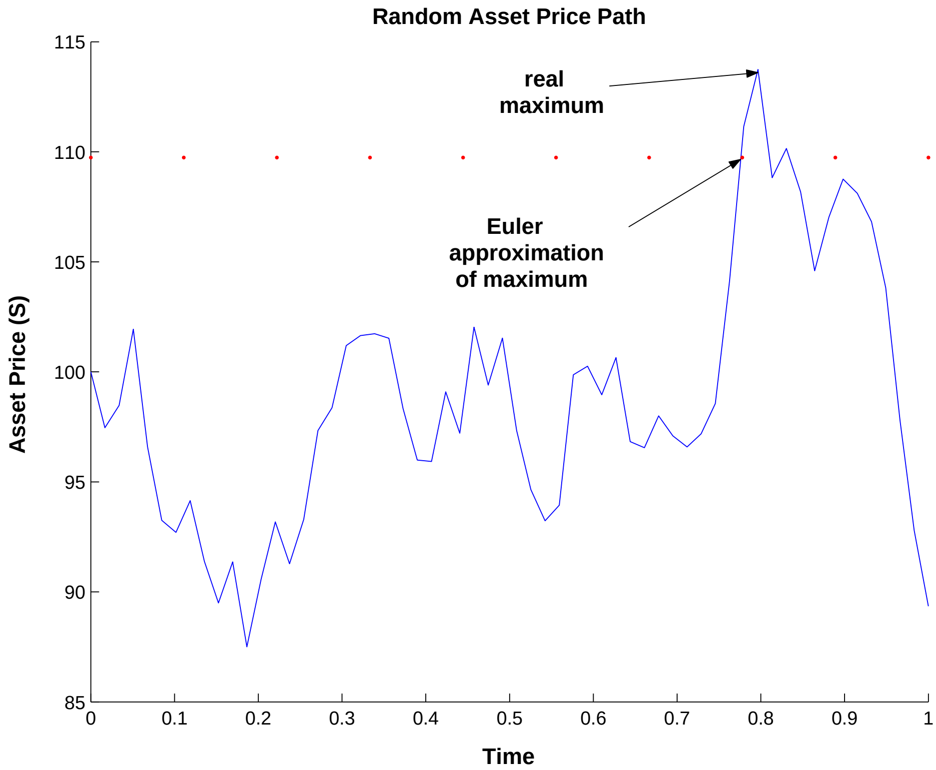

Figure 2.6: Discretization error in asset price maximum estimation.

Denoting by ξ_i the above probability and doing some algebra we can rewrite the above equation as

$$\xi_i = \exp\left(\frac{2\ln(\frac{S_i^{max}}{S_i})\ln(\frac{S_{i+1}}{S_i^{max}})}{\sigma^2\Delta t}\right) \tag{2.32}$$

and consequently we have that

$$P\left\{\max(S(t) : t \in [t_i, t_{i+1}] < S_i^{max} | S_i, S_{i+1}\right\} = 1 - \xi_i(S_i^{max}) \; . \tag{2.33}$$

Thus to simulate the maximum of the stock price over the interval $[t_i, t_{i+1}]$, given S_i and S_{i+1}, we should draw a uniform random variable $u_i \sim U(0,1)$ and set

$$u_i = \exp\left(\frac{2\ln(\frac{S_i^{max}}{S_i})\ln(\frac{S_{i+1}}{S_i^{max}})}{\sigma^2\Delta t}\right) \; . \tag{2.34}$$

Then, we have that

$$\frac{1}{2}\sigma^2\Delta t \ln u_i = \ln\left(\frac{S_i^{max}}{S_i}\right)\ln\left(\frac{S_{i+1}}{S_i^{max}}\right) \tag{2.35}$$

and by doing some algebra, we obtain the following quadratic polynomial with respect to S_i^{max}

$$\ln^2 S_i^{max} - \ln(S_{i+1}\cdot S_i)\ln S_i^{max} + \ln S_i \cdot \ln S_{i+1} + \frac{1}{2}\sigma^2\Delta t \ln u_i = 0 \; . \tag{2.36}$$

The solutions of the above quadratic give the maximum and the minimum of the price over the interval $[t_i, t_{i+1}]$, given S_i and S_{i+1}. These extremes are obtained by exponentiating the following formulas

$$\ln S_i^{max} = \frac{\ln(S_{i+1} \cdot S_i) + \sqrt{\ln\left(\frac{S_i}{S_{i+1}}\right) - 2\sigma^2 \Delta t \ln u_i}}{2} \tag{2.37}$$

and

$$\ln S_i^{min} = \frac{\ln(S_{i+1} \cdot S_i) - \sqrt{\ln\left(\frac{S_{i+1}}{S_i}\right) - 2\sigma^2 \Delta t \ln u_i}}{2} . \tag{2.38}$$

Thus, we can approximate the maximum and minimum of S over $[0, T]$ as

$$\hat{S}_{max} = \max\{S_i^{max} : i = 0, 1, \dots m\} \tag{2.39}$$

$$\hat{S}_{min} = \min\left\{S_i^{min} : i = 0, 1, \dots m\right\} \tag{2.40}$$

and therefore by simulating the asset price paths, we can estimate the prices of the floating Lookback options as

$$\hat{C}_{float}(t) = e^{-r(T-t)} E[(S(T) - \hat{S}_{min})] . \tag{2.41}$$

Similarly the price of a floating strike Lookback put option is given by

$$\hat{P}_{float}(t) = e^{-r(T-t)} E[(\hat{S}_{max} - S(T))] . \tag{2.42}$$

Beaglehole, Dydvig and Zhou suggest a similar idea in [BDZ97] for pricing Lookback options.

2.2.3 Simulation Results

In this section we present the numerical results that we obtained by applying the crude and the enhanced Monte Carlo method to price Lookback options. In particular, we consider the value of a floating strike Lookback put option, which pays the difference between the maximum stock price over the life of the option and its final price. Using the crude Monte Carlo simulation, we can estimate the price of the option along one price path as

$$\bar{P}_{float}(t) = e^{-r(T-t)} \Big(\max\{S_i : i = 0, 1 \dots, m\} - S_m \Big) . \tag{2.43}$$

Alternatively, we can estimate the Lookback put option price using the enhanced Monte Carlo method that we described in previous section. The estimate of the price using this technique is given by (2.42).

Values of the Lookback options estimated by Monte Carlo simulation are compared with the exact value of the continuous Lookback. In particular, we consider the floating strike Lookbacks which were first introduced by Goldman, Sosin and Gatto [GSG97] and can be priced within a Black-Scholes framework. A set of equations given for European floating strike Lookback puts are the following

$$p_{float} = S_0 \Big(\frac{\sigma^2}{r} N(-b_2) + N(b_2) \Big) - S_{max} e^{-rT} \Big(N(b_1) - \frac{\sigma^2}{2r} e^{x_2} N(-b_3) \Big) , \tag{2.44}$$

where

$$
\begin{aligned}
b_1 &= \frac{\ln\left(\frac{S_{max}}{S_0}\right) + (-r + \frac{\sigma^2}{2})}{\sigma\sqrt{T}} , \\
b_2 &= b_1 - \sigma\sqrt{T} , \\
b_3 &= \frac{\ln\left(\frac{S_{max}}{S_0}\right) + (r - \frac{\sigma^2}{2})}{\sigma\sqrt{T}} , \\
x_2 &= \frac{\ln\left(\frac{S_{max}}{S_0}\right) 2(r - \frac{\sigma^2}{2})}{\sigma^2} .
\end{aligned}
$$

We calculated the prices of a European Lookback put option for $S_{max} = S_0 = 100$, $r = 0.05$, $T = 1$ and various values of time step Δt and volatility σ. We used 10^6 Monte Carlo replications in each experiment. In table 2.2 we give the prices of the Lookback put option calculated by the crude Monte Carlo simulation as well as the enhanced method of section 2.2.2. Figure 2.7 shows the convergence rate of the crude and corrected Monte Carlo method for volatility $\sigma = 0.25$ (upper graph) and $\sigma = 0.5$ (down graph). Again good results are obtained with the corrected Monte Carlo method showing a substantial improvement in accuracy.

			Price		**Absolute Error**	
σ	$\mathbf{\Delta t}$	$\mathbf{s_n/\sqrt{n}}$	**Crude Monte Carlo**	**Corrected Monte Carlo**	**Crude Monte Carlo**	**Corrected Monte Carlo**
0.25	0.050000	0.01308	15.22128	18.74419	3.50200	0.02091
	0.025000	0.01304	16.14978	18.70912	2.57349	0.01415
	0.012500	0.01303	16.87865	18.72391	1.84462	0.00063
	0.006250	0.01301	17.41949	18.74331	1.30378	0.02004
	0.003125	0.01299	17.78333	18.72772	0.93994	0.00445
0.5	0.050000	0.02609	34.77775	43.17847	8.26423	0.13649
	0.025000	0.02608	37.04758	43.17970	5.99440	0.13772
	0.012500	0.02605	38.67337	43.09348	4.36861	0.05150
	0.006250	0.02606	39.90766	43.08268	3.13432	0.04070
	0.003125	0.02605	40.77608	43.04837	2.26591	0.00639

Table 2.2: European Lookback put option for $S_{max} = S_0 = 100$, $r = 0.05$, $T = 1$. The exact price is 18.72327, 43.04198 for $\sigma = 0.25, 0.5$, respectively.

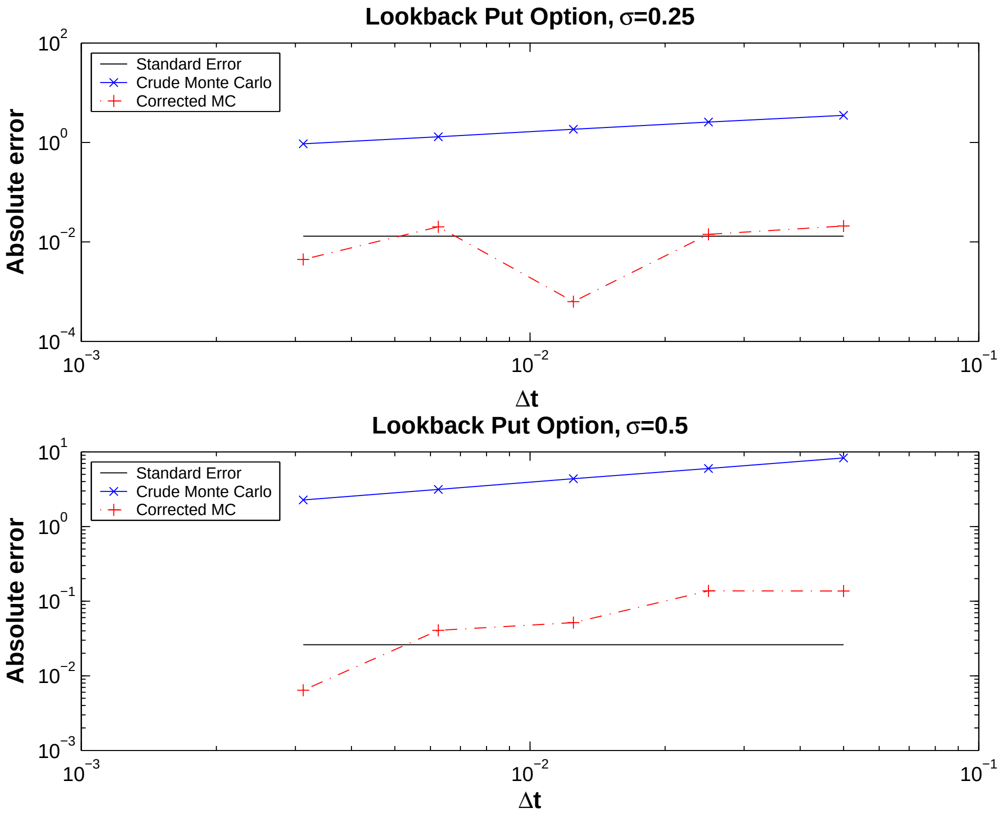

Figure 2.7: Lookback Put - Convergence rate, $\sigma = 0.5$.

Chapter 3

Estimating Sensitivities

3.1 Hedging Digital Options

Digital or binary options are options with discontinuous payoffs. The two most common types of this kind of exotic options are the *cash-or-nothing* digital options and the *asset-or-nothing* digital options. A cash-or-nothing call (put) option pays off a predetermined amount X, if the underlying asset price $S(T)$ is above (below) the strike price K at maturity time T, otherwise pays off nothing. Thus the risk neutral prices of those options are given by

$$C_{cash-or-nothing}(t) = e^{-r(T-t)}\mathbb{E}^Q[X\mathbf{1}\{S(T) > K\}] \tag{3.1}$$

$$P_{cash-or-nothing}(t) = e^{-r(T-t)}\mathbb{E}^Q[X\mathbf{1}\{S(T) < K\}] \; . \tag{3.2}$$

Similarly, asset-or-nothing call (put) option pays off the underlying asset price $S(T)$ at maturity time T, if this price is above (below) the strike price K, otherwise pays off nothing. Thus the risk neutral prices of those options are given by

$$C_{asset-or-nothing}(t) = e^{-r(T-t)}\mathbb{E}^Q[S(T)\mathbf{1}\{S(T) > K\}] \tag{3.3}$$

$$P_{asset-or-nothing}(t) = e^{-r(T-t)}\mathbb{E}^Q[S(T)\mathbf{1}\{S(T) < K\}] \; . \tag{3.4}$$

In the next section, we will show that the discontinuities in payoffs of digital options make the pathwise method inapplicable, as we noted in introduction. We can overcome this obstacle by choosing a smooth function that approximates the payoff function, and then applying the pathwise method in estimating the sensitivities of the option. Next, we apply this smoothing technique in estimating the sensitivities of a digital option.

3.1.1 Smoothing the Digital Payoff

We consider a European *cash-or-nothing call* option, which is the simplest digital (binary) call option. As usual, we assume the underlying asset price S follows the Geometric Brownian motion which is described by (1.8). Assume now that this option pays off a predetermined amount equal to 1, if the terminal underlying asset price $S(T)$ is above (below) the strike price K at maturity time T, otherwise nothing. Hence, its discounted payoff is

$$Y = e^{-rT}\mathbf{1}\left\{S(T) \geq K\right\} \ , \tag{3.5}$$

which can be written in the following form

$$Y = e^{-rT}H(S(T)-K) = \begin{cases} e^{-rT} & , \text{if } S(T)-K \geq 0 \\ \\ 0 & , \text{if } S(T)-K < 0 \end{cases} \ , \tag{3.6}$$

where $H(x)$ is the Heaviside[1] function. Thus, the price of the option at current time is

$$C_{dig} = e^{-rT}\mathbb{E}^{Q}\big[H(S(T)-K)\big] \ . \tag{3.7}$$

Obviously this payoff function is discontinuous at $S(T) = K$. The derivative of Y, with respect to $S(0)$, is 0 whenever it exists. This means that although the pathwise derivative exists with probability 1, it is useless, since

$$0 = E\Big[\frac{dY}{dS(0)}\Big] \neq \frac{d}{dS(0)}E[Y] \ . \tag{3.8}$$

The change in $E[Y]$ with a change in $S(0)$ is driven by the possibility that a change in $S(0)$ will cause $S(T)$ to cross the strike K, but this possibility is missed by the pathwise derivative. However, we can choose a smooth function of the form

$$H_{\epsilon}\left(S(T)-K\right) = h\left(\frac{S(T)-K}{\epsilon}\right) \ , \tag{3.9}$$

which approximates well the discontinuous payoff, see figure 3.1. Parameter ϵ is a small positive number, which determines the smoothness of function H_{ϵ}. Now, since the function H_{ϵ} is smooth, we can apply the pathwise method in order to estimate the Δ of the digital option, as

$$\Delta_{\epsilon} = \frac{dC_{dig}}{dS(0)} = e^{-rT}\frac{d}{dS(0)}E[H_{\epsilon}] = e^{-rT}E\Big[\frac{dH_{\epsilon}}{dS(0)}\Big] \ . \tag{3.10}$$

The last expectation in (3.10), is given by

$$E\Big[\frac{dH_{\epsilon}\left(S(T)-K\right)}{dS(0)}\Big] = \int_{0}^{\infty}\frac{dH_{\epsilon}(S-K)}{dS(0)}\,p(S)\,dS \ , \tag{3.11}$$

where $p(S)$ is the lognormal density of $S(T)$. We know that

$$S(T) = S(0)e^{(r-\frac{\sigma^2}{2})T+\sigma W} \tag{3.12}$$

where $W \sim N(0,T)$, and therefore the distribution of $S(T)$ is given by

$$\begin{aligned} P(S(T) \leq S) \ &= P\left(W \leq \frac{\log(S/S(0))-(r-\frac{\sigma^2}{2})T}{\sigma}\right) \\ &= \Phi\left(\frac{\log(S/S(0))-(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \ , \end{aligned} \tag{3.13}$$

[1]The Heaviside function is defined as $H(x) = \begin{cases} 1 & , \text{if } x \geq 0 \\ 0 & , \text{if } x < 0 \end{cases}$.

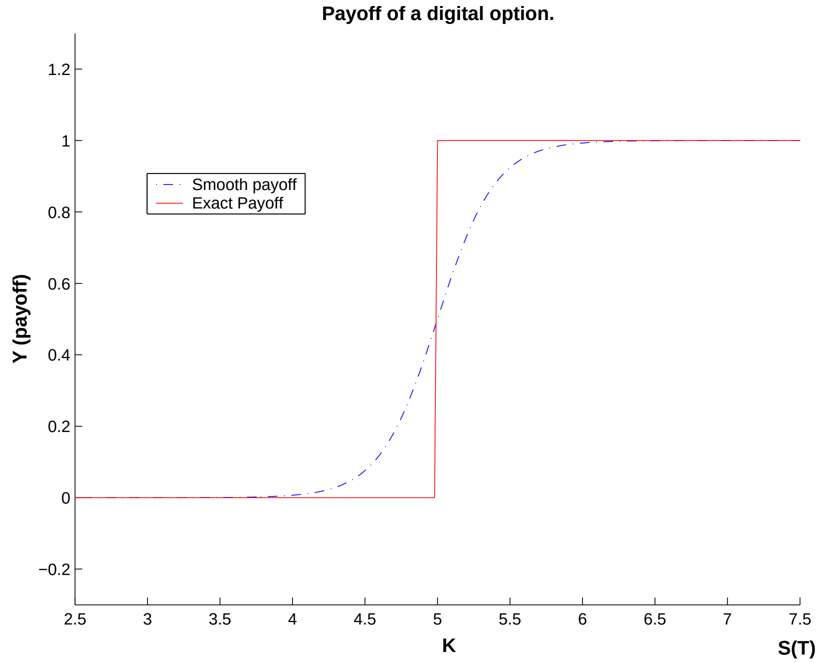

Figure 3.1: Smoothing the discontinuous payoff of a digital.

where Φ is the cumulative distribution function. Thus, by differentiating this with respect to S, we get the density of $S(T)$

$$p(S) = \frac{1}{S\sqrt{2\pi\sigma^2 T}} \exp\Big[- \frac{\big(\log(S/S(0)) - (r - \frac{\sigma^2}{2})T\big)^2}{2\sigma^2 T}\Big] . \tag{3.14}$$

Now applying the chain rule, we get

$$\frac{dH_\epsilon}{dS(0)} = \frac{dH_\epsilon}{dS(T)} \frac{dS(T)}{dS(0)} \tag{3.15}$$

and by (3.12), we have that $\frac{dS(T)}{dS(0)} = \frac{S(T)}{S(0)}$. Hence the expectation (3.11) becomes

$$E\Big[\frac{dH_\epsilon\,(S(T) - K)}{dS(0)}\Big] = \int_0^\infty \frac{dH_\epsilon\,(S - K)}{dS} \frac{S}{S(0)}\, p(S)\, dS\ . \tag{3.16}$$

Similarly, the pathwise estimator of Gamma of a digital call option is given by

$$\Gamma_\epsilon = \frac{d^2 C_{dig}}{dS^2(0)} = e^{-rT} E\Big[\frac{d^2 H_\epsilon\,(S(T) - K)}{dS^2(0)}\Big] \tag{3.17}$$

where

$$E\Big[\frac{d^2 H_\epsilon\,(S(T) - K)}{dS^2(0)}\Big] = \int_0^\infty \frac{d^2 H_\epsilon\,(S - K)}{dS^2} \Big(\frac{S}{S(0)}\Big)^2 p(S)\, dS\ . \tag{3.18}$$

3.1.2 Delta and Gamma Estimates

In this section, we study the case in which the smooth function is the following

$$H_\epsilon\left(S(T)-K\right) = h\left(\frac{S(T)-K}{\epsilon}\right) = \frac{1}{2}\left[\tanh\left(\frac{S(T)-K}{\epsilon}\right)+1\right], \tag{3.19}$$

where the function $h(x)-\frac{1}{2}=\frac{1}{2}\tanh(x)$ is an odd function, i.e. $h(x)=-h(-x)$. By differentiating (3.19) along with (3.15), we obtain

$$\frac{dH_\epsilon\left(S(T)-K\right)}{dS(0)} = \frac{1}{2\epsilon}\left[1-\tanh^2\left(\frac{S(T)-K}{\epsilon}\right)\right]\frac{S(T)}{S(0)}\ . \tag{3.20}$$

Now, the density of $S(T)$ is given by (3.14) and therefore the pathwise estimator of delta (3.16) has the following form

$$\Delta_\epsilon = e^{-rT}E\left[\frac{dH_\epsilon\left(S(T)-K\right)}{dS(0)}\right] = e^{-rT}\int_0^\infty \frac{1}{2\epsilon}\left[1-\tanh^2\left(\frac{S-K}{\epsilon}\right)\right]\frac{S}{S(0)}\,p(S)\,dS\ . \tag{3.21}$$

Similarly, we find that

$$\frac{d^2H_\epsilon\left(S(T)-K\right)}{dS^2(0)} = -\tanh\left(\frac{S(T)-K}{\epsilon}\right)\left[1-\tanh^2\left(\frac{S(T)-K}{\epsilon}\right)\right]\frac{1}{\epsilon^2}\left(\frac{S(T)}{S(0)}\right)^2 \tag{3.22}$$

and therefore the pathwise estimator of Gamma (3.18) becomes

$$\Gamma_\epsilon = e^{-rT}\int_0^\infty -\tanh\left(\frac{S(T)-K}{\epsilon}\right)\left[1-\tanh^2\left(\frac{S(T)-K}{\epsilon}\right)\right]\frac{1}{\epsilon^2}\left(\frac{S(T)}{S(0)}\right)^2 p(S)\,dS\ . \tag{3.23}$$

We can easily calculate the integrals (3.21) and (3.23) either numerically or using Monte Carlo simulation. We calculated numerically these integrals in Matlab, for a digital call option using the input parameters $K=1$, $S_0=1$, $T=1$, $\sigma=0.5$, $r=0.05$. We also estimated the above integrals by Monte Carlo simulation, using 10^6 replications. Figure 3.2 shows the values of the Delta and Gamma as parameter ϵ varies.

From the graphs, we can see that the smaller the value of the parameter ϵ the better estimates of Greeks are obtained by numerical integration. Also, we can see that as the ϵ increases, the Monte Carlo estimates converge to those, which are obtained by numerical integration. However, as ϵ tends to zero, the Monte Carlo simulation gives very poor estimates. The above observations become more apparent in the case of Gamma estimation. We observe that Monte Carlo method gives poor results, even if the value of ϵ is not so small.

3.1.3 Asymptotic Analysis

In this section, we carry out the asymptotic analysis for our problem. We have seen that the value of the option, which is calculated as the expectation of its discounted payoff, is given by

$$V = e^{-rT}\int_0^\infty p(S)H(S-K)\,dS\ , \tag{3.24}$$

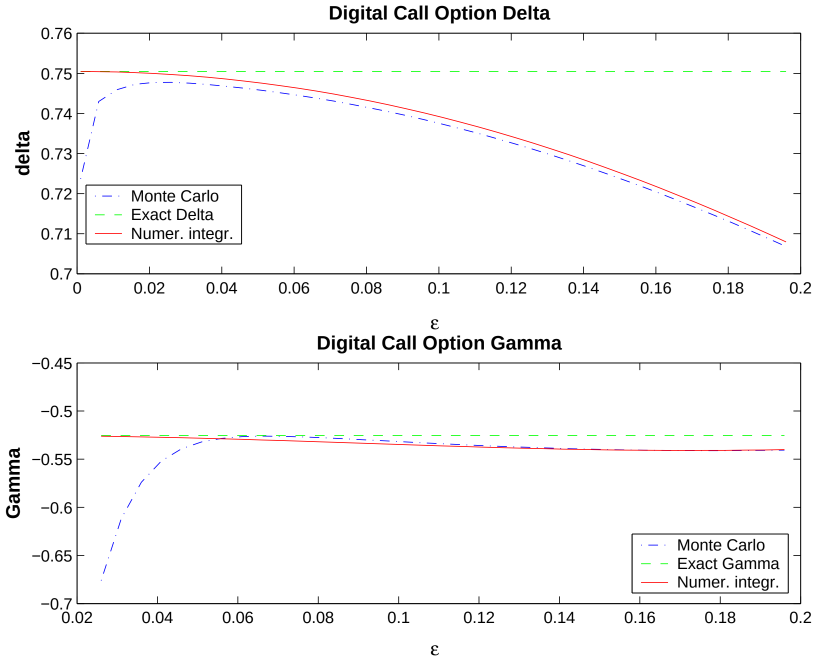

Figure 3.2: Digital Call - Delta and Gamma estimates vs ϵ. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

while the value of the option, which is estimated through the smooth approximation $H_\epsilon(S(T) - K)$ of its payoff, is

$$V_\epsilon = e^{-rT} \int_0^\infty p(S) H_\epsilon(S - K)\, dS \ . \tag{3.25}$$

Thus, we have that

$$V - V_\epsilon = e^{-rT} \int_0^\infty p(S) \Big[H(S - K) - H_\epsilon\left(S(T) - K\right) \Big]\, dS \tag{3.26}$$

and by changing variable $S = K + \epsilon x$, we get

$$V - V_\epsilon \approx e^{-rT} \int_{-\infty}^\infty p(K + \epsilon x) \Big[H(\epsilon x) - h(x) \Big]\, \epsilon\, dx \ , \tag{3.27}$$

since $H_\epsilon(\epsilon x) = h(x)$. Now $H(\epsilon x) = H(x)$, and therefore (3.27) becomes

$$V - V_\epsilon \approx e^{-rT} \int_{-\infty}^\infty p(K + \epsilon x) \Big[H(x) - h(x) \Big] \epsilon\, dx \ . \tag{3.28}$$

By Taylor series expansion of $p(K + \epsilon x)$, we have

$$p(K + \epsilon x) = p(K) + \epsilon x \frac{dp}{dS} + \frac{(\epsilon x)^2}{2!} \frac{d^2 p}{dS^2} + \frac{(\epsilon x)^3}{3!} \frac{d^3 p}{dS^3} + \frac{(\epsilon x)^4}{4!} \frac{d^4 p}{dS^4} + \dots \tag{3.29}$$

Hence inserting this expansion in (3.28), we have

$$\begin{aligned} V - V_\epsilon & \approx \epsilon\, e^{-rT} \int_{-\infty}^{\infty} p(K) \Big[H(x) - h(x)\Big] dx \\ & + \epsilon^2\, e^{-rT} \int_{-\infty}^{\infty} \frac{dp}{dS} x \Big[H(x) - h(x)\Big] dx \\ & + \epsilon^3\, e^{-rT} \int_{-\infty}^{\infty} \frac{1}{2} \frac{d^2 p}{dS^2} x^2 \Big[H(x) - h(x)\Big] dx \\ & + \epsilon^4\, e^{-rT} \int_{-\infty}^{\infty} \frac{1}{6} \frac{d^3 p}{dS^3} x^3 \Big[H(x) - h(x)\Big] dx \\ & + \mathcal{O}(\epsilon^6) \end{aligned} \tag{3.30}$$

But the odd power terms of the above series vanish, since $H(x) - h(x)$ is an odd function. Therefore, we can write (3.30) as follows

$$V - V_\epsilon = c_2(S_0)\, \epsilon^2 + c_4(S_0)\, \epsilon^4 + \mathcal{O}(\epsilon^6) \tag{3.31}$$

with

$$c_i(S_0) = \begin{cases} e^{-rT} \int_{-\infty}^{\infty} \frac{1}{(i-1)!} \frac{d^{(i-1)} p}{dS^{(i-1)}} x^{(i-1)} \Big[H(x) - h(x)\Big] dx & \text{, if } i \text{ is even} \\ 0 & \text{, if } i \text{ is odd} \end{cases} \tag{3.32}$$

for $i = 1, 2, \ldots$.

However, we are interested in the error of the Delta approximation, which is

$$\Delta - \Delta_\epsilon = \frac{d}{dS_0}[V - V_\epsilon] = \frac{dc_2(S_0)}{dS_0} \epsilon^2 + \frac{dc_4(S_0)}{dS_0} \epsilon^4 + \mathcal{O}(\epsilon^6) \tag{3.33}$$

and therefore we need to calculate the coefficients of the above series, which have the following form

$$\frac{dc_i(S_0)}{dS_0} = \begin{cases} \int_{-\infty}^{\infty} \frac{1}{(i-1)!} \frac{d}{dS_0} \left[\frac{d^{(i-1)} p}{dS^{(i-1)}}\right] x^{(i-1)} \Big[H(x) - h(x)\Big] dx & \text{, if } i \text{ is even} \\ 0 & \text{, if } i \text{ is odd} \end{cases} \tag{3.34}$$

for $i = 1, 2, \ldots$. Thus the coefficient of the first term in series (3.33), is

$$\frac{dc_2(S_0)}{dS_0} = \int_{-\infty}^{\infty} \frac{d}{dS_0} \left[\frac{dp}{dS}\right] x \Big[H(x) - h(x)\Big]\, dx\ . \tag{3.35}$$

We calculated numerically this coefficient for the smooth function H_ϵ of (3.19) and we found that it is about 1.1956. This means that the error of the estimation of Delta using numerical integration, as we explain at the previous section, must be

$$\Delta - \Delta_\epsilon \approx 1.1956 \epsilon^2 + \mathcal{O}(\epsilon^4)\ . \tag{3.36}$$

Using the results from the previous section, we can see that the approximation error of Delta through numerical integration has a similar behavior to the theoretical error, which is given by (3.36). In particular, figure 3.3 shows the theoretical error matches well with the error in calculation of Delta using numerical integration, as parameter ϵ varies. Figure 3.4 shows the error of estimates as well as the error of numerical integration. In the same figure, we plot the upper and the lower bound of Monte Carlo error[2].

[2] For example in the case of Delta estimation these bounds define the interval $\Big[\Delta - \Delta_\epsilon - 2\frac{s_n}{\sqrt{n}}, \Delta - \Delta_\epsilon + 2\frac{s_n}{\sqrt{n}}\Big]$.

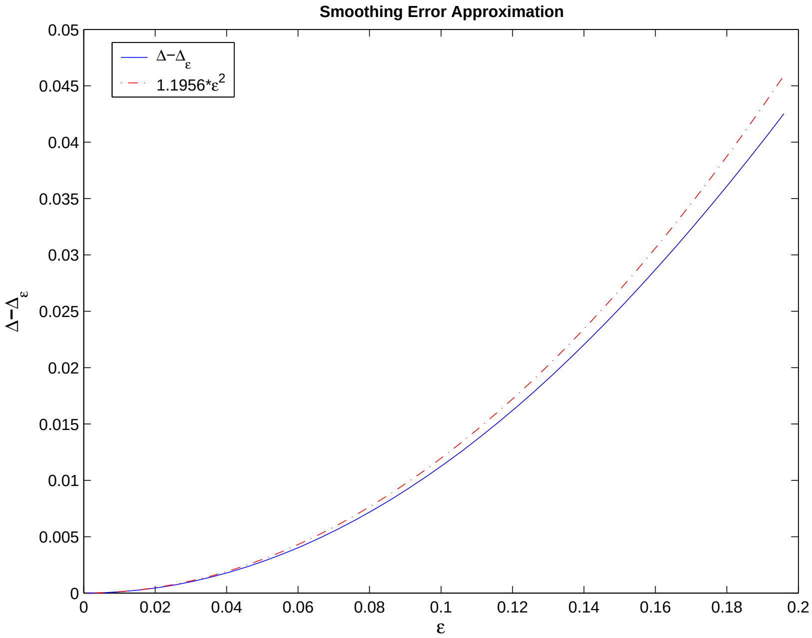

Figure 3.3: Digital Call - Theoretical and Numerical error vs ϵ. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

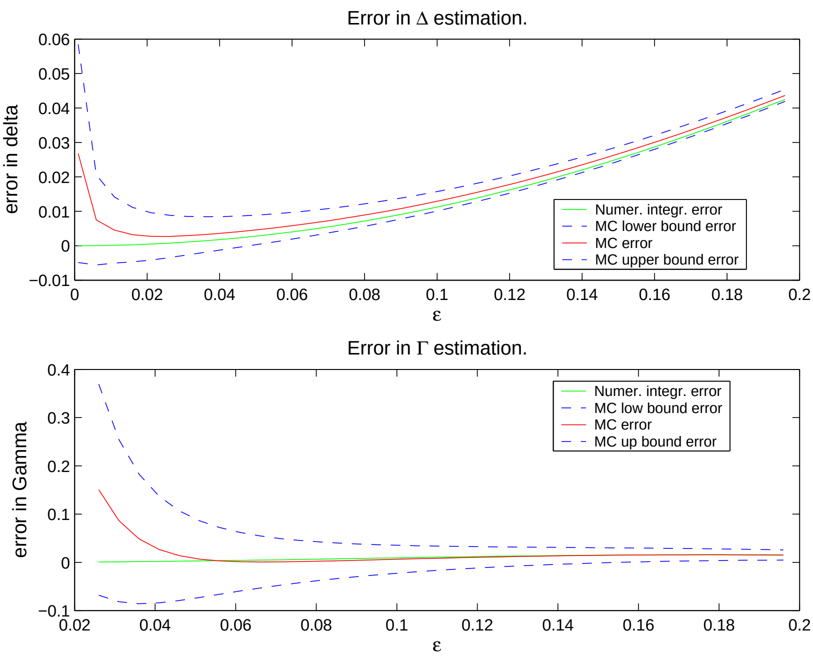

Figure 3.4: Digital Call - Monte Carlo and numerical integration error vs ϵ. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

Similarly, we can show that

$$\Gamma - \Gamma_\epsilon \sim \mathcal{O}(\epsilon^2)\ . \tag{3.37}$$

Now differentiating the asymptotic error expansion (3.31) with respect to ϵ, we have

$$-\frac{d}{d\epsilon}V_\epsilon = 2\,c_2(S_0)\,\epsilon + 4\,c_4(S_0)\,\epsilon^3 + \mathcal{O}(\epsilon^5)\ . \tag{3.38}$$

The above equation can be written as follows

$$-\frac{1}{2}\,\epsilon\,\frac{d}{d\epsilon}V_\epsilon = c_2(S_0)\,\epsilon^2 + 2\,c_4(S_0)\,\epsilon^4 + \mathcal{O}(\epsilon^6) \tag{3.39}$$

and thus by subtracting this from (3.31), we can reduce the leading order term from $\mathcal{O}(\epsilon^2)$ to $\mathcal{O}(\epsilon^4)$, i.e. we obtain

$$V - \left(V_\epsilon - \frac{1}{2}\,\epsilon\,\frac{d}{d\epsilon}V_\epsilon\right) = -c_4(S_0)\,\epsilon^4 + \mathcal{O}(\epsilon^6)\ . \tag{3.40}$$

Thus by approximating the value of the option using the estimator

$$V_\epsilon - \frac{1}{2}\,\epsilon\,\frac{d}{d\epsilon}V_\epsilon\ , \tag{3.41}$$

we obtain an approximation with error of order $\mathcal{O}(\epsilon^4)$. The latter quantity is calculated as follows

$$V_\epsilon - \frac{1}{2}\,\epsilon\,\frac{d}{d\epsilon}V_\epsilon = e^{-rT}\int_0^\infty p(S)\left[H_\epsilon(S-K) - \frac{1}{2}\,\epsilon\,\frac{d}{d\epsilon}H_\epsilon(S-K)\right]dS\ . \tag{3.42}$$

We can repeat the same procedure in order to reduce further the leading order term of the error.

Similarly, we can apply the above arguments to reduce the error of Delta and Gamma estimation. In particular, we can show that

$$\Delta - \left(\Delta_\epsilon - \frac{1}{2}\,\epsilon\,\frac{d}{d\epsilon}\Delta_\epsilon\right) \sim \mathcal{O}(\epsilon^4) \tag{3.43}$$

and

$$\Gamma - \left(\Gamma_\epsilon - \frac{1}{2}\,\epsilon\,\frac{d}{d\epsilon}\,\Gamma_\epsilon\right) \sim \mathcal{O}(\epsilon^4)\ . \tag{3.44}$$

We calculate again the value of Delta and Gamma of the digital option of section 3.1.2 by numerical integration, using the corrected approximations of Delta and Gamma. Figures 3.5 and 3.6 verify the above theoretical results. For example, we can see from the lower graphs of those figures that the error after the correction is of order $\mathcal{O}(\epsilon^4)$, while without correction it is $\mathcal{O}(\epsilon^2)$.

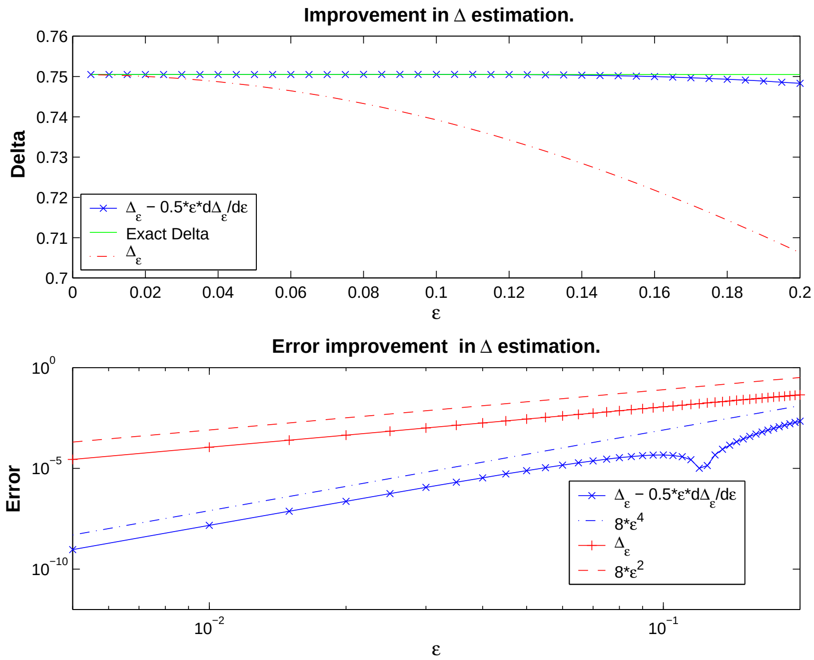

Figure 3.5: Digital Call - Delta correction. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

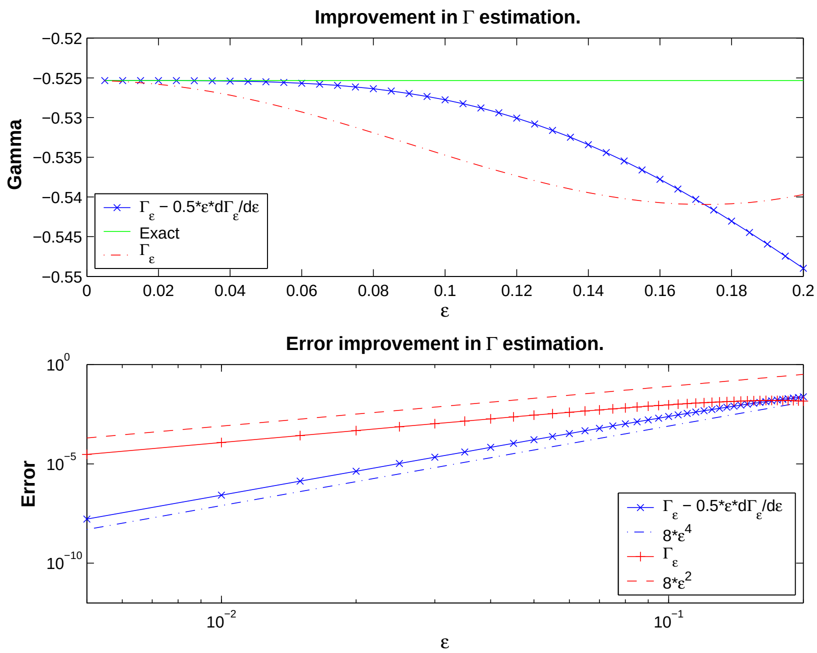

Figure 3.6: Digital Call - Gamma correction. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

3.1.4 Monte Carlo Variance

In this section, we study the variance in Monte Carlo estimates. In general it holds

$$\mathrm{Var}[f(x)] = E[f^2(x)] - E[f(x)]^2 \ . \tag{3.45}$$

First, we consider the variance of delta. In this case, we need to calculate the variance of the function $\frac{dH_\epsilon(S(T)-K)}{dS(0)}$, with

$$H_\epsilon\left(S(T) - K\right) = \frac{1}{2}\left[\tanh\left(\frac{S(T)-K}{\epsilon}\right) + 1\right] . \tag{3.46}$$

By differentiating the above function with respect to $S(0)$, we get

$$\frac{dH_\epsilon(S(T)-K)}{dS(0)} = \frac{1}{2\,\epsilon}\left[1 - \tanh^2\left(\frac{S(T)-K}{\epsilon}\right)\right]\frac{S(T)}{S(0)} \ . \tag{3.47}$$

As before, we have used the chain rule and the fact that $\frac{dS(T)}{dS(0)} = \frac{S(T)}{S(0)}$. Thus the variance of delta is given by the following expression

$$\mathrm{Var}\left[e^{-rT}\frac{dH_\epsilon}{dS(T)}\frac{S(T)}{S(0)}\right] = e^{-2rT}E\left[\left(\frac{dH_\epsilon}{dS(T)}\frac{S(T)}{S(0)}\right)^2\right] - e^{-2rT}E\left[\frac{dH_\epsilon}{dS(T)}\frac{S(T)}{S(0)}\right]^2 . \tag{3.48}$$

The first expectation of the right hand side of (3.48) is the following

$$E\left[\left(\frac{dH_\epsilon}{dS(T)}\frac{S(T)}{S(0)}\right)^2\right] = \int_0^\infty \left\{\frac{1}{2\epsilon}\left[1 - \tanh^2\left(\frac{S-K}{\epsilon}\right)\right]\frac{S}{S(0)}\right\}^2 p(S)\,dS \ , \tag{3.49}$$

where $p(S)$ is the probability density of $S(T)$ and is given by (3.14). By setting $S = K + \epsilon x$ the above expectation becomes

$$E\left[\left(\frac{dH_\epsilon}{S(T)}\frac{S(T)}{S(0)}\right)^2\right] \approx \int_{-\infty}^\infty \frac{1}{\epsilon^2}\left(\frac{1}{2}\left[1 - \tanh^2(x)\right]\frac{K+\epsilon x}{S(0)}\right)^2 p(K+\epsilon x)\,\epsilon\,dx \tag{3.50}$$

and by Taylor expansion of $p(K + \epsilon x)$, we get

$$E\left[\left(\frac{dH_\epsilon}{dS(T)}\frac{S(T)}{S(0)}\right)^2\right] \approx \underbrace{\frac{1}{\epsilon}\int_{-\infty}^\infty \left(\frac{1}{2}\left[1 - \tanh^2(x)\right]\frac{K+\epsilon x}{S(0)}\right)^2 p(K)dx}_{\mathcal{O}(\epsilon^{-1})} + \mathcal{O}(1) \ . \tag{3.51}$$

Thus the first expectation in (3.48) is of order $\mathcal{O}(\epsilon^{-1})$. Now the second expectation in (3.48) is given by

$$E\left[\frac{dH_\epsilon}{dS(T)}\frac{S(T)}{S(0)}\right] = \int_0^\infty \frac{1}{2\,\epsilon}\left[1 - \tanh^2(x)\right]\frac{S}{S(0)}\,p(S)\,dS \ . \tag{3.52}$$

So by setting $S = K + \epsilon x$ as previously, we have

$$E\left[\frac{dH_\epsilon}{dS(T)}\frac{S(T)}{S(0)}\right] \approx \int_{-\infty}^\infty \frac{1}{2\,\epsilon}\left[1 - \tanh^2(x)\right]\frac{K+\epsilon x}{S(0)}p(K+\epsilon x)\,\epsilon\,dx \ . \tag{3.53}$$

and therefore

$$E\left[\frac{dH_\epsilon}{dS(T)}\frac{S(T)}{S(0)}\right] \approx \underbrace{\int_{-\infty}^{\infty}\frac{1}{2}\left[1-\tanh^2(x)\right]\frac{K+\epsilon x}{S(0)}\,p(K)\,dx}_{\mathcal{O}(1)} + \mathcal{O}(\epsilon)\ . \tag{3.54}$$

Hence, the second expectation in (3.48) is of order $\mathcal{O}(1)$ and consequently we have that

$$\mathrm{Var}\left[e^{-rT}\frac{dH_\epsilon}{dS(T)}\frac{dS(T)}{dS(0)}\right] = \mathcal{O}(\epsilon^{-1}) + \mathcal{O}(1) \sim \mathcal{O}(\epsilon^{-1})\ . \tag{3.55}$$

We have shown that the variance of Monte Carlo estimate of Delta is of order $\mathcal{O}(\epsilon^{-1})$. This means that as $\epsilon \to 0$ then the variance tends to infinity. The latter explains the poor Monte Carlo estimates of Delta when ϵ takes small values, as we saw in the previous section.

In the case of Gamma estimation, we need to calculate the variance of the second derivative of $H_\epsilon(S(T)-K)$ with respect to $S(0)$. By differentiating (3.47) with respect to $S(0)$, we get

$$\frac{d^2H_\epsilon(S(T)-K)}{dS^2(0)} = -\frac{1}{\epsilon^2}\tanh\left(\frac{S(T)-K}{\epsilon}\right)\left[1-\tanh^2\left(\frac{S(T)-K}{\epsilon}\right)\right]\left(\frac{S(T)}{S(0)}\right)^2\ . \tag{3.56}$$

Thus we have to calculate the variance

$$\mathrm{Var}\left[e^{-rT}\,\frac{d^2H_\epsilon(S(T)-K)}{dS^2(0)}\right] = E\left[\left(e^{-rT}\,\frac{d^2H_\epsilon(S(T)-K)}{dS^2(0)}\right)^2\right] - E\left[e^{-rT}\,\frac{d^2H_\epsilon(S(T)-K)}{dS^2(0)}\right]^2\ . \tag{3.57}$$

Similarly as above, we can show that the first expectation in (3.57) is of order $\mathcal{O}(\epsilon^{-3})$, while the second expectation is of order $\mathcal{O}(\epsilon^{-2})$ and consequently

$$\mathrm{Var}\left[e^{-rT}\,\frac{d^2H_\epsilon}{dS^2(0)}\right] \sim \mathcal{O}(\epsilon^{-3})\ . \tag{3.58}$$

This means that the variance of Monte Carlo estimate of Gamma grows extremely fast as ϵ decreases. This explains the error behavior of Monte Carlo simulation and the inaccuracy of Gamma estimator even when ϵ takes relatively large values.

Figure 3.7 shows the variance (in logarithmic scale) of Delta and Gamma Monte Carlo estimates, for the call digital option of the previous section. It is obvious that, as we expected, the variance in Delta behaves like ϵ^{-1}, while the variance in Gamma behaves like ϵ^{-3}.

3.1.5 Variance Reduction through Stratified Sampling

Stratified sampling, as it is defined in [Gla04], refers broadly to any sampling mechanism that constrains the fraction of observations drawn from specific subsets (or *strata*) of the sample space. Suppose, for example, that our goal is to estimate $E[X]$ with X real-valued, and let $A_1, A_2, \ldots, A_I$ be disjoint subsets of the real line for which $P(X \in \cup_{i=1}^{I} A_i) = 1$. Then

$$E[X] = \sum_{i=1}^{I} P(X \in A_i)E[X|X \in A_i] = \sum_{i=1}^{I} p_i\mu_i\ , \tag{3.59}$$

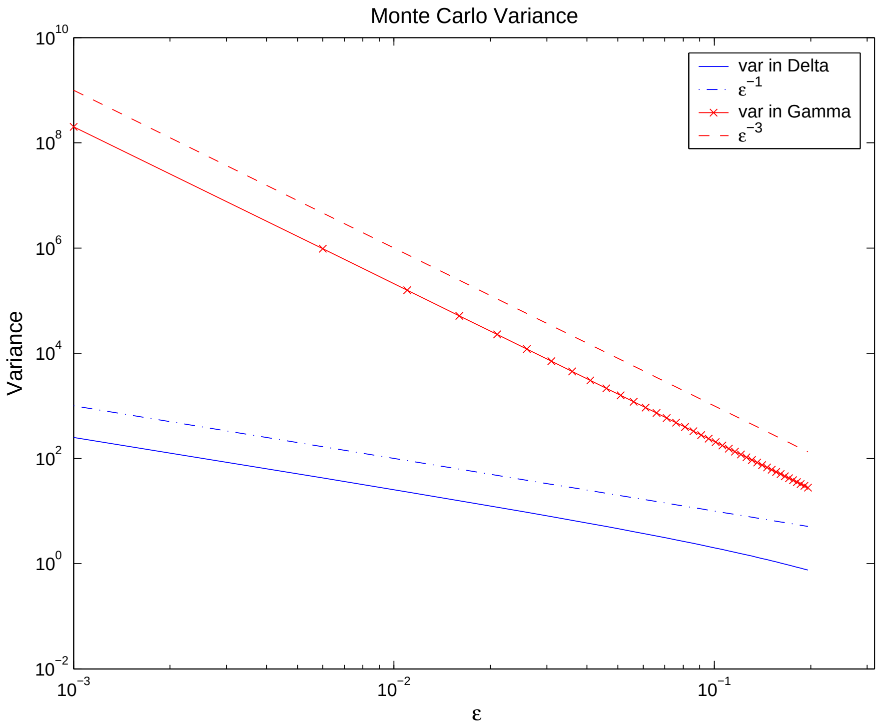

Figure 3.7: Variance of Δ and Γ Monte Carlo estimates. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

where $p_i = P(X \in A_i)$ and $\mu_i = E[X|X \in A_i]$. In stratified sampling, we decide in advance what fraction of samples should be drawn from each stratum A_i. Also each observation drawn from A_i is constrained to have the distribution of X conditional on $X \in A_i$.

The simplest case of stratified sampling is that with proportional allocation of sampling. According to this technique, we draw $n_i = np_i$ samples from the stratum A_i, where n is the total sample size and assume that n_i is rounded such that it is always integer. Let now X_{ij}, with $j = 1, \ldots, n_i$ for each $i = 1, \ldots, I$, be independent draws from the conditional distribution of X, given that $X \in A_i$. Then an unbiased estimator of $E[X|X \in A_i]$ is provided by the sample mean of the observations from the i-th stratum. It follows from (3.59) that the unbiased estimator of $E[X]$ is given by

$$\hat{X} = \sum_{i=1}^{I} p_i \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij} = \frac{1}{n} \sum_{i=1}^{I} \sum_{j=1}^{n_i} X_{ij} \; . \tag{3.60}$$

If now we allow the stratum allocations $n_1, \ldots, n_I$ to be arbitrary rather than proportional to probabilities $p_1, \ldots, p_I$ and we assume that $q_i = n_i/n$ is the fraction of observations drawn from the i-th stratum, then the above estimator becomes

$$\hat{X} = \sum_{i=1}^{I} p_i \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij} = \frac{1}{n} \sum_{i=1}^{I} \frac{p_i}{q_i} \sum_{j=1}^{n_i} X_{ij} \; . \tag{3.61}$$

The stratified estimators of (3.60), (3.61) should be contrasted with the crude Monte Carlo estimator

$$\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n} \ . \tag{3.62}$$

Compared with $\bar{X}$, the stratified estimator $\hat{X}$ eliminates sampling variability across strata, without affecting sampling variability within strata. Stratified sampling with a proportional allocation can only decrease the variance in our estimation, while by optimizing the allocation we can achieve further variance reduction. Next, we study the variance of this method in comparison with the variance of the standard Monte Carlo estimator.

The variance of the estimate $\hat{X}$ is calculated as follows

$$\begin{aligned}
\mathrm{Var}(\hat{X}) \ &= \mathrm{Var}\Big(\textstyle\sum_{i=1}^{I} p_i \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij} \Big) \\
&= \textstyle\sum_{i=1}^{I} \mathrm{Var}\Big(p_i \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij} \Big) \\
&= \textstyle\sum_{i=1}^{I} p_i^2 \, \mathrm{Var}\Big(\frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij} \Big) \\
&= \textstyle\sum_{i=1}^{I} p_i^2 \frac{\sigma_i^2}{n_i}
\end{aligned} \tag{3.63}$$

and thus in the case of proportional allocation the latter expression reduces to

$$\mathrm{Var}(\hat{X}) = \frac{1}{n} \sum_{i=1}^{I} p_i \, \sigma_i^2 \ . \tag{3.64}$$

On the other hand, the standard Monte Carlo estimator has variance σ^2/n, where $\sigma^2 = \mathrm{Var}(X)$. Therefore, if we show that $\sum_{i=1}^{I} p_i \sigma_i^2 < \sigma^2$, we will have proved that the proportional stratified estimator has a lower variance than the usual Monte Carlo estimator. The proof is based on the *conditional variance formula*, which states

$$\mathrm{Var}(X) = E\big[\mathrm{Var}(X|X \in A_i)\big] + \mathrm{Var}\big(E[X|X \in A_i]\big) \tag{3.65}$$

but we have that

$$E\big[\mathrm{Var}(X|X \in A_i)\big] = \sum_{i=1}^{I} p_i \, \sigma_i^2 \tag{3.66}$$

and therefore

$$\sigma^2 = \mathrm{Var}(X) \geq E\big[\mathrm{Var}(X|X \in A_i)\big] = \sum_{i=1}^{I} p_i \, \sigma_i^2 \ , \tag{3.67}$$

since by Jensen's inequality[3]

$$\mathrm{Var}\big(E[X|X \in A_i]\big) = \sum_{i=1}^{I} p_i \, \mu_i^2 - \Big(\sum_{i=1}^{I} p_i \, \mu_i \Big)^2 \geq 0 \ , \tag{3.68}$$

[3]If f is a convex function on the interval $[a, b]$, then

$$\sum_{i=1}^{n} p_i \, f(\mu_i) \geq f\left(\sum_{i=1}^{n} p_i \, \mu_i \right) \ ,$$

where $0 \leq p_i \leq 1$, $p_1 + p_2 + \ldots + p_n = 1$ and each $\mu_i \in [a, b]$.

with strict inequality unless all μ_i are all equal. In general and particularly in our problem, μ_i are not equal. Thus, we have proved that in case of stratified sampling with a proportional allocation, it always holds that

$$\sum_{i=1}^{I} p_i \sigma_i^2 \leq \sigma^2 \ . \tag{3.69}$$

The latter inequality means that the variance of stratified estimator is always equal or less than that of standard Monte Carlo estimator.

Back to our problem, in order to estimate the values of Delta and Gamma, which are given by (3.21) and (3.23), respectively, we need to simulate the terminal value $S(T)$ of the underlying asset's price. We remember that the latter is given by

$$S(T) = S(0)e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z} \ , \tag{3.70}$$

where $Z \sim N(0,1)$, and we choose Z to be our stratification variable.

Initially, we consider the case of I equiprobable strata and a proportional allocation of the total sample size, i.e. $p_i = 1/I$ and $n_i = n/I$, for $i = 1, \ldots, I$. Note that we implicitly assume that n_i is always integer. Then, we can easily draw n_i independent identically distributed random variables $Z \sim N(0,1)$ from each stratum, by drawing equal number of independent uniform random variables $U_1, \ldots, U_I$ over $[0,1]$ and then setting

$$V_i = \frac{i-1}{I} + \frac{U_i}{I} \ , \quad i = 1, \ldots, I \ . \tag{3.71}$$

Finally, $\Phi^{-1}(V_i)$ is a stratified sample from the standard normal distribution. We denote with Φ^{-1} the inverse cumulative standard normal distribution function[4]. We summarize this procedure in the algorithm of table 3.1.

Stratified Sampling Algorithm

```
Inputs : I = #strata, n= total sample size
for i = 1,...,I
    set n_i = n/I
    for j = 1,...,n_i
        1. Generate U ~ Unif[0,1]
        2. Set V = (i-1+U)/I, such that V ~ Unif[(i-1)/I, i/I]
        3. Calculate Z_{i,j} = Φ^{-1}(V)
    end for
end for
```

Table 3.1: Generation of stratification variables.

Now, using the stratified samples $Z_{i,j}$, we can first draw a sample of terminal asset price $S(T)$ and then compute the Monte Carlo estimator of Delta and Gamma. The algorithm of table 3.2 describes the steps of this procedure.

[4]An algorithm inverse for calculating a value of this function, as well as a C routine implementation, can be found in Peter J. Acklam's web page http://www.math.uio.no/̃jacklam .

Monte Carlo with Proportional Stratified Sampling - Algorithm

Inputs : $I = \#$strata, $n=$ total sample size, vector p with probabilities p_i
Use *Stratified Sampling Algorithm* to draw a sample of $Z_{i,j}$
for $i = 1, \ldots, I$
  set $n_i = I/n$
  for $j = 1, \ldots, n_i$
    1. Use $Z_{i,j}$ to calculate $S_{i,j}(T)$ by (3.70)
    2. Calculate the value of Delta as
       $\Delta_{i,j} = e^{-rT}\frac{1}{2\epsilon}\left[1 - \tanh^2\left(\frac{S_{i,j}(T)-I}{\epsilon}\right)\right]\frac{S_{i,j}(T)}{S(0)}$
    3. Calculate the value of Gamma as
       $\Gamma_{i,j} = -e^{-rT}\tanh\left(\frac{S_{i,j}(T)-I}{\epsilon}\right)\left[1 - \tanh^2\left(\frac{S_{i,j}(T)-I}{\epsilon}\right)\right]\frac{1}{\epsilon^2}\left(\frac{S_{i,j}(T)}{S(0)}\right)^2$
  end for
  Calculate the means $\hat{\Delta}_i = \frac{1}{n_i}\sum_{i=1}^{n_i}\Delta_{i,j}$ and $\hat{\Gamma}_i = \frac{1}{n_i}\sum_{i=1}^{n_i}\Gamma_{i,j}$
  Calculate the variances $\sigma^2_{i,\Delta}$ and $\sigma^2_{i,\Gamma}$ of the $i-th$ stratum
end for
Calculate the stratified estimates $\hat{\Delta}_\epsilon = \sum_{i=1}^{I}\hat{\Delta}_i p_i$ and $\hat{\Gamma}_\epsilon = \sum_{i=1}^{I}\hat{\Gamma}_i p_i$

Calculate the total variances $\mathrm{Var}(\hat{\Delta}_\epsilon) = \sum_{i=1}^{I} p_i^2\frac{\sigma^2_{i,\Delta}}{n_i}$ and $\mathrm{Var}(\hat{\Gamma}_\epsilon) = \sum_{i=1}^{I} p_i^2\frac{\sigma^2_{i,\Gamma}}{n_i}$

Table 3.2: Estimation of Greeks through stratified Monte Carlo with proportional allocation.

Using the algorithm of table 3.2, we estimated the Delta and Gamma for the Digital options of the previous sections. We used $I = 10000$ strata and $n_i = 20$ samples per strata. Figure 3.8 shows the values of both Greeks for different values of parameter ϵ. In the graphs of this figure we plot the values of Delta and Gamma, which have been estimated by numerical integration and Monte Carlo with proportional allocated stratified sampling. We compare these values with the exact values of Greeks. The latter are given by the following formulas

$$\Delta = e^{-rT}\frac{n(d_2)}{S_0\sigma\sqrt{T}} ,$$

$$\Gamma = e^{-rT}\frac{n(d_2)d_1}{S_0^2\sigma^2 T} ,$$

where $n(x)$ is the density of the standard univariate normal distribution, with

$$d_2 = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_1 = d_2 + \sigma\sqrt{T} .$$

From the upper graph of figure 3.8, we can see that the Monte Carlo estimator of Delta is almost identical to the Delta value which is computed by numerical integration, even if the ϵ is very small.

Also, from the lower graph, we observe again that the Monte Carlo simulation and the numerical integration give almost the same results, at least when $\epsilon > 0.02$. The latter observations show how stratified sampling can improve the efficiency of Monte Carlo estimator compared to the standard Monte Carlo estimator. Figure 3.9 shows the error in the estimates as well as the error of the numerical integration. In the same figure, although it is not visible, we plot the upper and the lower bound of the Monte Carlo error.

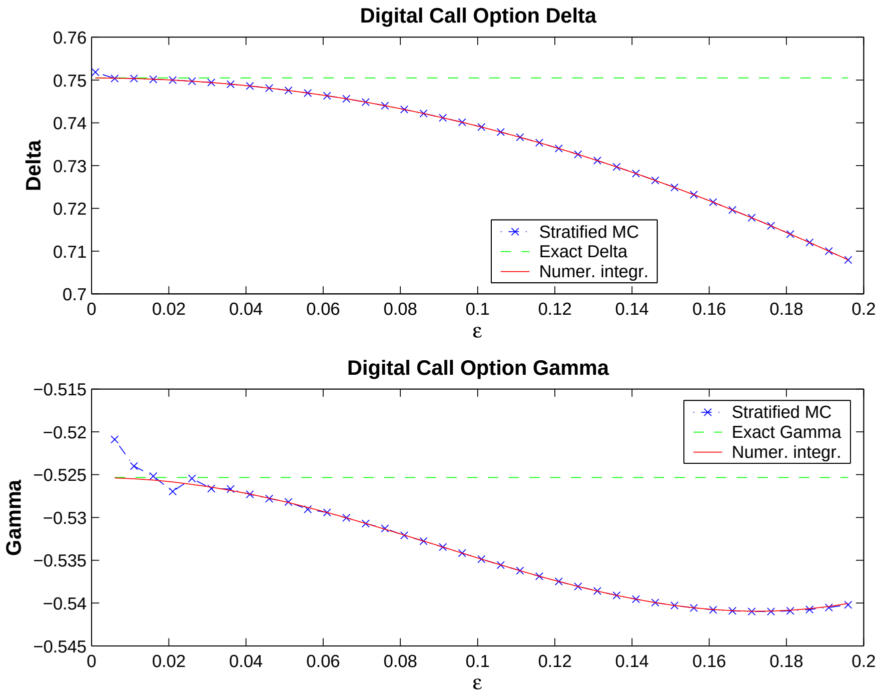

Figure 3.8: Δ and Γ of a digital call option. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

As we have noticed, further variance reduction can be achieved by using optimal allocation instead of proportional allocation. In particular, we can minimize the variance

$$\mathrm{Var}(\hat{X}) = \sum_{i=1}^{I} p_i^2 \frac{\sigma_i^2}{n_i} \,, \tag{3.72}$$

by solving the following constrained optimization problem

$$\begin{aligned} \mathbf{min}_i \ \mathrm{Var}(\hat{X}) \quad &= \textstyle\sum_{i=1}^{I} p_i^2 \frac{\sigma_i^2}{n_i} \\ \textbf{subject to} \quad &n_1 + \ldots + n_I = n \end{aligned} \tag{3.73}$$

Solving this problem using Langrange multipliers, we get the following optimal solution

$$n_i^* = \left(\frac{p_i \, \sigma_i}{\sum_{l=1}^{I} p_l \, \sigma_l} \right) n \quad , \quad \text{i=1,\ldots, I} \,. \tag{3.74}$$

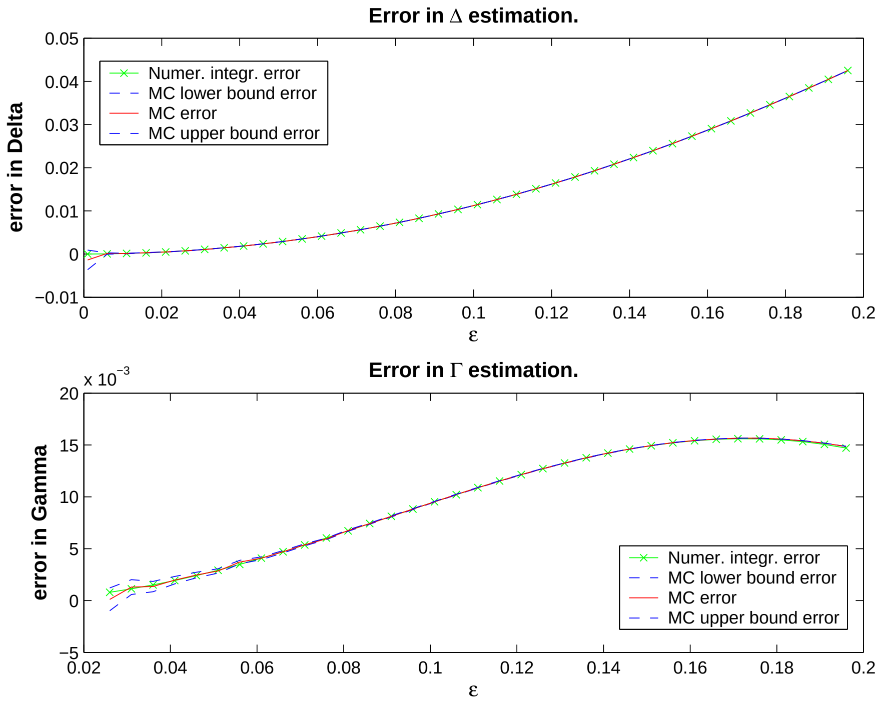

Figure 3.9: Error in digital call option estimation, for $I = 10000$, $n_i = 20$. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

Consequently the optimized variance is given by

$$\mathrm{Var}(\hat{X}_{n_i^*}) = \frac{\left(\sum_{i=1}^{I} p_i\sigma_i\right)^2}{n} \quad . \tag{3.75}$$

By 3.74, we can see that the optimal allocation for each stratum is proportional to the product of the stratum probability and the stratum standard deviation. This makes intuitive sense and it can be interpreted as follows. If p_i is large and other things being equal, then it makes sense to put more effort simulating from the $i-$th stratum. Similarly if σ_i is large and other things being equal, it make sense to draw more often from the $i-$th stratum, so as to get a more accurate estimate from this stratum. A drawback of using the optimal allocation is that σ_i are unknowns and thus we may firstly run a pilot simulation to estimate σ_i, and then run the original Monte Carlo simulation. A subject of discussion is how much computational effort we should put into pilot simulations. In general, we should put such an effort so as to obtain a reasonably good estimate of the σ_i's but without increasing too much the total computational effort. For example, when we have a large number of strata, then these pilot simulations can be computationally very expensive and thus it may not worthwhile using optimal allocation at all.

Using a similar algorithm to that of table 3.2, except that we now use the optimal allocation[5], instead of the proportional one, we estimate the values of Delta and Gamma again for the digital call option of the previous sections. However, in this case we cannot estimate in the same internal loop

[5]The optimal allocation is predetermined through a pilot simulations, as we have described.

both Greeks, since in general the number of draws per stratum n_i is different for these two Greeks. Thus, we should have two internal loops something which adds more computational effort in our calculations. In particular, we use the algorithm of table 3.3 as a pilot simulation to estimate the variances in each stratum. Then, we estimate the optimal number of samples per stratum through the expression (3.74). Next we use a loop, in which the Monte Carlo estimates of Delta and Gamma are estimated in two separate loops, for each stratum. Finally, we combine all the estimates from each stratum and we calculate the total estimates of Delta and Gamma. The algorithm for this procedure is given in table 3.3.

Figure 3.10 shows the variance of standard Monte Carlo (solid line) in contrast with that of Monte Carlo with proportional (dash-dot line) and optimal allocation (dash line) of stratified sampling. We see that by using Monte Carlo with proportional stratified sampling a significant reduction of variance is achieved, while further reduction is achieved using optimal allocation. Figure 3.11 shows the error[6] of each method against the parameter ϵ. From the graphs, we observe that the stratified sampling with optimal allocation allows us to get good estimates even when the crude Monte Carlo and the stratified sampling with proportional allocation fail to give an accurate estimate, i.e. when the ϵ is very small.

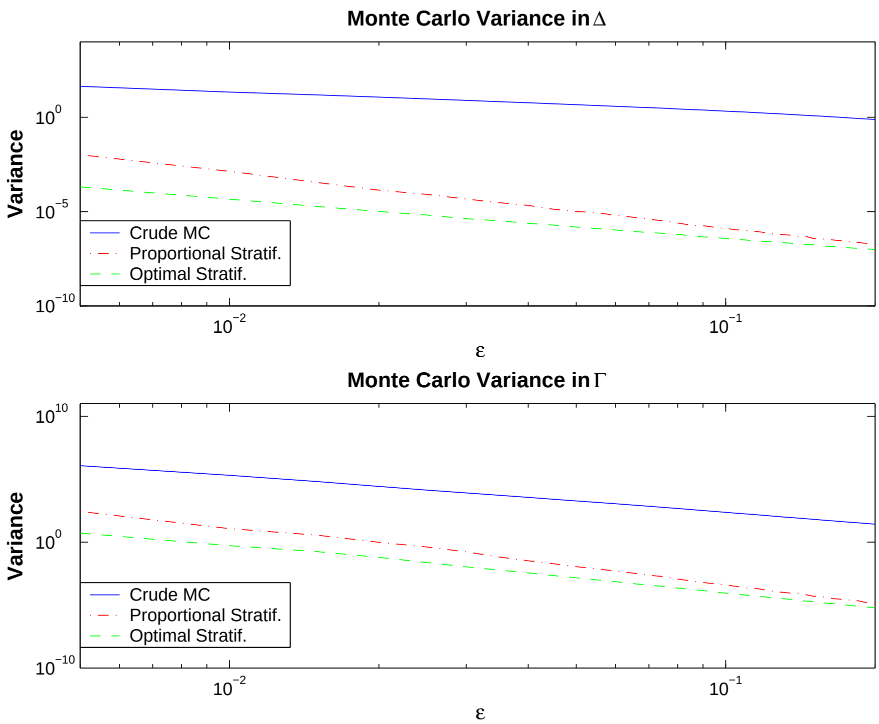

Figure 3.10: Digital Call - Monte Carlo variance, for total sample = 2000. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

[6]Note that the error in the case of crude Monte Carlo simulation varies smoothly as ϵ varies because we use the same sample for all simulation loops for all values of ϵ.

Monte Carlo with Adaptive Stratified Sampling - Algorithm

Inputs : I = #strata, n= total sample size , vector p with probabilities p_i

1.1 Use the Algorithm of table 3.2 to estimate the standard deviations
$\hat{\sigma}_{\Delta,i}$ and $\hat{\sigma}_{\Gamma,i}$ of each stratum A_i, for $i = 1, \ldots, I$.

1.2 Calculate the optimal number of samples per stratum $n^*_{\Delta,i}$ and $n^*_{\Gamma,i}$
according to the equation (3.74).

for $i = 1, \ldots, I$

for $j = 1, \ldots, n^*_{\Delta,i}$

1. Generate the variable $Z_{i,j}$ as in *Stratified Sampling Algorithm.*
2. Use $Z_{i,j}$ to calculate $S_{i,j}(T)$ by (3.70)
3. Calculate the value of Delta as
$\Delta_{i,j} = e^{-rT} \frac{1}{2\epsilon}\left[1 - \tanh^2\left(\frac{S_{i,j}(T)-I}{\epsilon}\right)\right]\frac{S_{i,j}(T)}{S(0)}$

end for

for $j = 1, \ldots, n^*_{\Gamma,i}$

1. Generate the variable $W_{i,j}$ as in *Stratified Sampling Algorithm.*
2. Use $W_{i,j}$ to calculate $S_{i,j}(T)$ by (3.70)
3. Calculate the value of Gamma as
$\Gamma_{i,j} = -e^{-rT} \tanh\left(\frac{S_{i,j}(T)-I}{\epsilon}\right)\left[1 - \tanh^2\left(\frac{S_{i,j}(T)-I}{\epsilon}\right)\right]\frac{1}{\epsilon^2}\left(\frac{S_{i,j}(T)}{S(0)}\right)^2$

end for

2.1 Calculate the means $\hat{\Delta}_i = \frac{1}{n^*_{\Delta,i}}\sum_{i=1}^{n_i}\Delta_{i,j}$ and $\hat{\Gamma}_i = \frac{1}{n^*_{\Gamma,i}}\sum_{i=1}^{n_i}\Gamma_{i,j}$

2.2 Calculate the variances $\sigma^2_{i,\Delta}$ and $\sigma^2_{i,\Gamma}$ of the $i - th$ stratum

end for

1.3 Calculate the stratified estimates $\hat{\Delta}_\epsilon = \sum_{i=1}^{I}\hat{\Delta}_i p_i$ and $\hat{\Gamma}_\epsilon = \sum_{i=1}^{I}\hat{\Gamma}_i p_i$

1.4 Calculate the total variances $\text{Var}(\hat{\Delta}_\epsilon) = \sum_{i=1}^{I} p_i^2 \frac{\sigma^2_{i,\Delta}}{n^*_{\Delta,i}}$ and $\text{Var}(\hat{\Gamma}_\epsilon) = \sum_{i=1}^{I} p_i^2 \frac{\sigma^2_{i,\Gamma}}{n^*_{\Gamma,i}}$

Table 3.3: Estimation of Greeks through stratified Monte Carlo with optimal allocation.

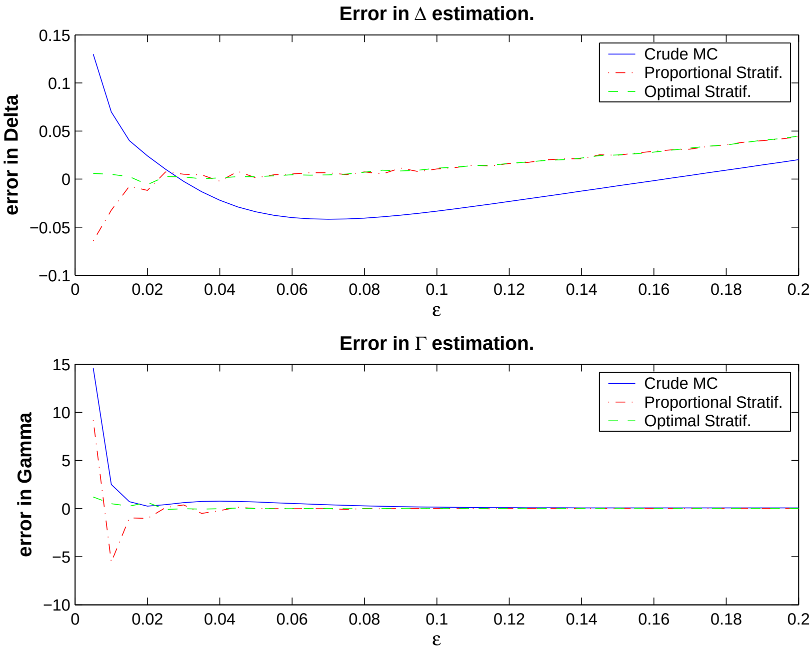

Figure 3.11: Digital Call - Monte Carlo error, for total sample = 2000. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

3.1.6 Comparison with Likelihood Ratio Estimators

In this section we compare our estimators of Gamma and Delta of digital option, with the Likelihood Ratio estimators. Before we give the numerical results, we review the derivation of likelihood ratio estimators. Suppose that the option payoff, is expressed as a function f of a random vector $X = (X_1, \ldots, X_m)$, where the components of X represent the underlying asset price at different times. Also we assume that θ is a parameter of the probability density of X. If we denote this density by $g_\theta(x)$, then the derivative with respect to θ is written as

$$a^{'}(\theta) = \frac{d}{d\theta} E[f(X)] = \int_{\mathbb{R}^m} f(x) \, \frac{d}{d\theta} \, g_\theta(x) \, dx \ . \tag{3.76}$$

Now if we multiply and divide the above integrand by $g_\theta(x)$, we obtain

$$a^{'}(\theta) = \int_{\mathbb{R}^m} f(x) \, \frac{g^{'}_\theta(x)}{g_\theta(x)} \, g_\theta(x) \, dx \ , \tag{3.77}$$

where we have written $g^{'}_\theta(x)$ for $dg_\theta/d\theta$. Thus the expression $f(X)\frac{g^{'}_\theta(x)}{g_\theta(x)}$ is an unbiased estimator of $a^{'}(\theta)$, while the quantity $g^{'}_\theta(x)/g_\theta(x)$ is known as the *score function*.

It can be shown (see [Gla04],[BK04]) that the likelihood ratio estimators for Delta and Gamma of digital call option are given by

$$\Delta_{LR} = e^{-rT}\mathbf{1}\{S_T \geq K\}\left(\frac{d}{S_0\sigma\sqrt{T}}\right)$$

$$\Gamma_{LR} = e^{-rT}\mathbf{1}\{S_T \geq K\}\left(\frac{d^2 - d\sigma\sqrt{T} - 1}{S_0^2\sigma^2 T}\right)$$

with

$$d = \frac{\ln(S_T/S_0) - (r - \sigma/2)^2 T}{\sigma\sqrt{T}} \quad . \tag{3.78}$$

Figure 3.12 shows the estimates of Delta and Gamma of the digital call option, which are obtained from our method (solid line with x) as well as the Likelihood Ratio method (dash-dot line with plus signs). We use the estimators of our methods after the first correction, such that the smoothing error to be $\mathcal{O}(\epsilon^4)$. Also, we note that every time, we use a value of parameter $\epsilon = 0.2$, to ensure that the variances and especially in Gamma estimate will not be so big. The estimates of both methods are compared with the exact values of Greeks (solid line). Figure 3.13 shows the error in estimates of the previous methods and finite-difference approximations as well. From the graphs of that figure, we can see that the estimates of Delta of our method, are competitive to that of the Likelihood Ratio method. This is consistent with the fact that the variances (see upper graph in figure 3.14) of the two estimators are almost identical, and much smaller than the variance of finite-difference estimator. In the case of Gamma, the estimates of our method seems to be worse than that of the Likelihood Ratio method, while the finite-difference approximations are very poor. The latter observations are explained from the variances of the three methods, see upper graph in figure 3.14.

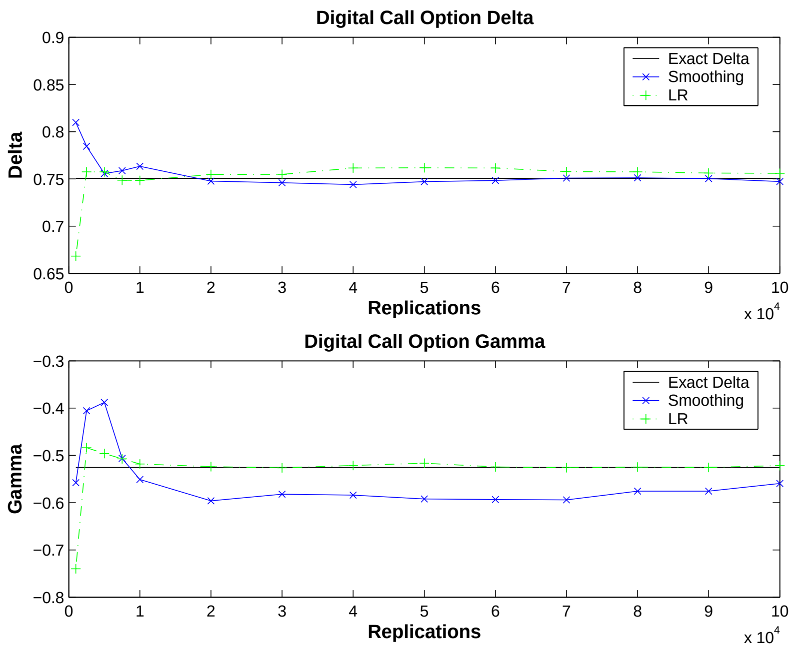

Figure 3.12: Digital Call - Pathwise (PW) vs Likelihood Ratio (LR) estimators, with $\Delta t = 1/1024$. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

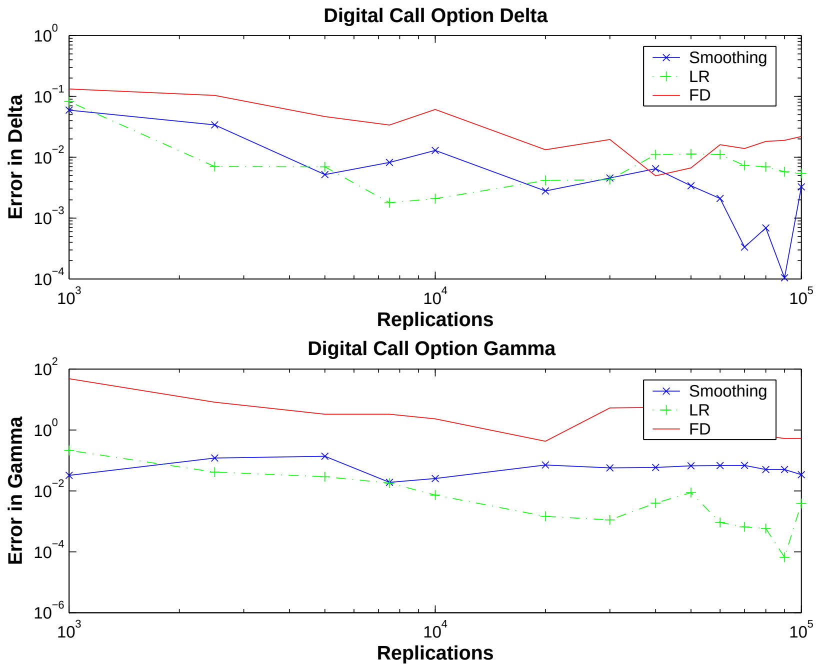

Figure 3.13: Digital Call - error in Pathwise and LR vs MC replications. $\Delta t = 1/1024$. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

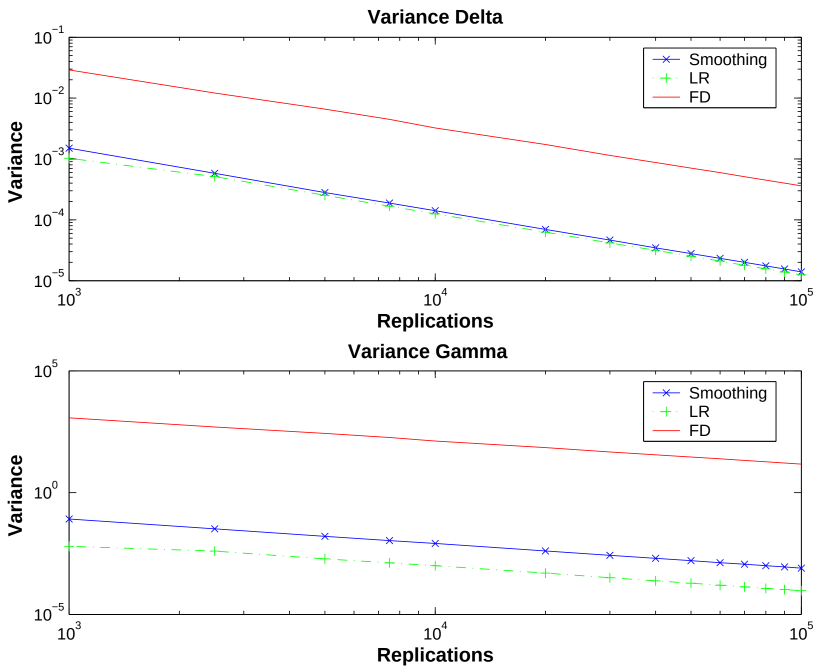

Figure 3.14: Digital Call - variance in Pathwise and LR vs number of MC replications. $\Delta t = 1/1024$.. Input parameters $K = 1$, $S_0 = 1$, $T = 1$, $\sigma = 0.5$, $r = 0.05$.

3.2 Hedging Barrier Options

In this section, we apply the method of payoff smoothing to path dependent options such as Barrier options. In particular, we consider the case of a *down-and-out call option*. This kind of barrier options pays off $S(T) - K$, unless the underlying asset price reaches a barrier $B < S(0)$ during the life of the option, otherwise pays off nothing. Thus we can write the discounted payoff of a down-and-out call option with continuously monitored barrier, as follows

$$Y = e^{-rT}(S(T) - K)^{+}\mathbf{1}\left\{\min_{0\leq t\leq T} S(t) > B\right\} \tag{3.79}$$

and therefore, its risk neutral price is given by

$$C_{do}(t) = e^{-r(T-t)}\mathbb{E}^{Q}[(S(T) - K)^{+}\mathbf{1}\left\{\min_{0\leq t\leq T} S(t) > B\right\} \ . \tag{3.80}$$

The payoff of this option depends on the whole path of the asset price and not only on the terminal value. In addition, the possibility of the option being knocked-out, makes its payoff discontinuous in the path of the asset price. Again, we can overcome these obstacles by choosing a continuous function that approximates the payoff function, and then applying the pathwise method in estimating the sensitivities of the option. Next, we will apply this smoothing technique in estimating the sensitivities of a down-and-out call option.

3.2.1 Smoothing the Down-and-Out Call Option Payoff

In this case, the payoff of the down-and-out call option is

$$P\big(S_{min}, S(T)\big) = H(S_{min} - B)R(S(T) - K) \ , \tag{3.81}$$

where $H(x)$ is the Heaviside function again, and

$$R(x) = \max(x, 0) = \int_{-\infty}^{x} H(s)ds \ . \tag{3.82}$$

This payoff is approximated by a smooth function

$$P_{\epsilon}\big(S_{min}, S(T)\big) = H_{\epsilon}(S_{min} - B)R_{\epsilon}(S(T) - K) \ , \tag{3.83}$$

with

$$S_{min} = \min_{0\leq t\leq T} S(t) \ . \tag{3.84}$$

Again the function

$$H_{\epsilon}(x) = h\left(\frac{x}{\epsilon}\right) \tag{3.85}$$

is a smooth approximation of Heaviside function $H(x)$ and

$$R_{\epsilon}(x) = \int_{-\infty}^{x} H_{\epsilon}(s)ds \ . \tag{3.86}$$

Thus, the price of the option is given by the following expectation

$$V_{\epsilon}\big(S_{min}, S(T)\big) = e^{-rT}E\big[H_{\epsilon}(S_{min} - B)R_{\epsilon}(S(T) - K)\big] \tag{3.87}$$

Parameter ϵ is a small positive number, which determines the smoothness of the function P_ϵ. Now, since the function P_ϵ is smooth, we can apply the pathwise method in order to estimate the Delta of the option, as

$$\Delta_\epsilon = \frac{\partial C_{do}}{\partial S(0)} = e^{-rT}\frac{\partial}{\partial S(0)}E\big[P_\epsilon\big(S_{min}, S(T)\big)\big] = e^{-rT}E\Big[\frac{\partial P_\epsilon(S_{min}, S(T))}{\partial S(0)}\Big] \ . \tag{3.88}$$

Using the chain rule we get

$$\frac{\partial P_\epsilon(S_{min}, S(T))}{\partial S(0)} = \frac{\partial H_\epsilon}{\partial S_{min}}\frac{dS_{min}}{dS(0)}R_\epsilon(S(T)-K) + \frac{\partial R_\epsilon}{\partial S(T)}\frac{dS(T)}{dS(0)}H_\epsilon(S_{min}-B) \tag{3.89}$$

and by substituting $\frac{dS_{min}}{dS(0)} = \frac{S_{min}}{S(0)}$ and $\frac{dS(T)}{dS(0)} = \frac{S(T)}{S(0)}$, we get

$$\frac{\partial P_\epsilon(S_{min}, S(T))}{\partial S(0)} = \frac{\partial H_\epsilon}{\partial S_{min}}\frac{S_{min}}{S(0)}R_\epsilon(S(T)-K) + \frac{S(T)}{S(0)}H_\epsilon(S(T)-K)H_\epsilon(S_{min}-B) \ . \tag{3.90}$$

Thus the last expectation of (3.88), can be written as

$$E\Big[\frac{\partial P_\epsilon(S_{min}, S(T))}{\partial S(0)}\Big] = \int_0^\infty\int_0^\infty \frac{\partial P_\epsilon(x,y)}{\partial S(0)}\, p(x,y)\, dx\, dy \ , \tag{3.91}$$

where $p(x, y)$ is the joint probability density function of $\big(S_{min}, S(T)\big)$. Hence, before calculating the expectation, we should first find this joint density function. First, note that the probability distribution $P(S_{min} \leq x|S(0), S(T))$ is known and it has the following form

$$P(S_{min} \leq x|S(0), S(T)) = \exp\Big[-\frac{2(\ln S(0) - \ln x)(\ln S(T) - \ln x)}{\sigma^2 \Delta t}\Big] \tag{3.92}$$

and thus by differentiating the above quantity, we obtain the conditional density $p(x|y)$. Also, we have seen in previous section that the density of $S(T)$ is given by

$$p(y) = \frac{1}{y\sqrt{2\pi\sigma^2 T}}\exp\Big[-\frac{\big(\log(y/S(0)) - (\mu - \frac{\sigma^2}{2})T\big)^2}{2\sigma^2 T}\Big] \ . \tag{3.93}$$

Now, we can calculate the unknown joint probability density function $p(x, y)$ as

$$\begin{aligned} p(x,y) &= p(y)p(x|y) \\ &= \tfrac{1}{y\sqrt{2\pi\sigma^2 T}}\exp\Big[-\tfrac{\big(\log(y/S(0))-(\mu-\frac{\sigma^2}{2})T\big)^2}{2\sigma^2 T}\Big] \\ &\times \tfrac{d}{dx}\bigg(\exp\Big[-\tfrac{2(\ln S(0)-\ln x)(\ln S(T)-\ln x)}{\sigma^2\Delta t}\Big]\bigg) \ . \end{aligned} \tag{3.94}$$

Similarly, the Gamma estimate of the option is given by

$$\Gamma_\epsilon = \frac{\partial^2 C_{do}}{\partial S^2(0)} = e^{-rT}\frac{\partial^2}{\partial S^2(0)}E\big[P_\epsilon\big(S_{min}, S(T)\big)\big] = e^{-rT}E\Big[\frac{\partial^2 P_\epsilon(S_{min}, S(T))}{\partial S^2(0)}\Big] \ , \tag{3.95}$$

with

$$\begin{aligned}\frac{\partial^2 P_\epsilon(S_{min}, S(T))}{\partial S^2(0)} &= \frac{\partial^2 H_\epsilon(S_{min}-B)}{\partial S_{min}^2}\,\frac{S_{min}^2}{S^2(0)}\,R_\epsilon(S(T)-K)\\ &+ \frac{2S_{min}S(T)}{S^2(0)}\,\frac{\partial H_\epsilon(S_{min}-B)}{\partial S_{min}}\,H_\epsilon(S(T)-K)\\ &+ \frac{S^2(T)}{S^2(0)}\,\frac{\partial H_\epsilon(S(T)-K)}{\partial S(T)}\,H_\epsilon(S_{min}-B)\ .\end{aligned}$$

The last expectation of equation (3.95) is given by the following double integral

$$E\Big[\frac{\partial^2 P_\epsilon(S_{min}, S(T))}{\partial S^2(0)}\Big] = \int_0^\infty \int_0^\infty \frac{\partial^2 P_\epsilon(x,y)}{\partial S^2(0)} p(x,y)\,dx\,dy\ , \tag{3.96}$$

where the joint probability density function $p(x, y)$ is given by (3.94). In the following section we describe how we can estimate the Delta and Gamma by calculating the expectations (3.92) and (3.96), respectively. As in the case of digital option, we use numerical integration as well as Monte Carlo simulation in order to calculate the expectations.

3.2.2 Delta and Gamma Estimates

Again, we use the following smooth function

$$H_\epsilon(x) = \frac{1}{2}\left[\tanh\left(\frac{x}{\epsilon}\right) + 1\right] \tag{3.97}$$

to approximate the Heaviside function $H(x)$ and therefore

$$R_\epsilon(x) = \int_{-\infty}^{x} \frac{1}{2}\left[\tanh\left(\frac{s}{\epsilon}\right) + 1\right] ds\ . \tag{3.98}$$

Thus, we have the following derivatives

$$\begin{aligned}\frac{dH_\epsilon(x)}{dx} &= \frac{1}{2\epsilon}\left[1 - \tanh^2\left(\frac{x}{\epsilon}\right)\right],\\ \frac{d^2H_\epsilon(x)}{dx^2} &= -\frac{1}{\epsilon^2}\tanh\left(\frac{x}{\epsilon}\right)\left[1 - \tanh^2\left(\frac{x}{\epsilon}\right)\right],\\ \frac{dR_\epsilon(x)}{dx} &= H_\epsilon(x) = \frac{1}{2}\left[\tanh\left(\frac{x}{\epsilon}\right) + 1\right].\end{aligned}$$

Using the above results, we can find explicit forms of partial derivatives $\partial P_\epsilon(x, y)/\partial S(0)$ and $\partial^2 P_\epsilon(x, y)/\partial S^2(0)$ and therefore we can calculate the expectations (3.92) and (3.96) through numerical integration.

Alternatively, the expectations (3.92) and (3.96), which give the values of Delta and Gamma respectively, can be expressed as double integrals, as follows

$$\begin{aligned}\Delta_\epsilon &= e^{-rT}\int_0^1\int_0^1 \Big\{ \frac{S_{min}}{S(0)}\frac{dH_\epsilon(S_{min}-B)}{dS_{min}}\; R_\epsilon(S(T)-K) \\ &\qquad + \frac{S(T)}{S(0)}\; H_\epsilon\,(S_{min}-B)\;\; H_\epsilon(S(T)-K)\Big\}\, dU_1\, dU_2\ ,\end{aligned}$$

and

$$\begin{aligned}\Gamma_\epsilon &= e^{-rT}\int_0^1\int_0^1 \Big\{ \quad \frac{S_{min}^2}{S^2(0)}\;\frac{d^2H_\epsilon(S_{min}-B)}{dS_{min}^2}\; R_\epsilon(S(T)-K) \\ &\qquad + \frac{2S_{min}S(T)}{S^2(0)}\;\frac{dH_\epsilon(S_{min}-B)}{dS_{min}}\; H_\epsilon(S(T)-K) \\ &\qquad\quad + \frac{S(T)^2}{S^2(0)}\; H_\epsilon\,(S_{min}-B)\;\;\frac{dH_\epsilon(S(T)-K)}{dS(T)}\Big\}\, dU_1\, dU_2\ ,\end{aligned}$$

where

$$S(T) = S(0)\exp\Big((r-\tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,\Phi^{-1}(U_1)\Big) \tag{3.99}$$

and

$$S_{min} = \exp\left[\tfrac{1}{2}\Big(\log S(0) + \log S(T) - \sqrt{(\log S(0) - \log S(T))^2 - 2\sigma^2 T\log U_2}\Big)\right]. \tag{3.100}$$

This last equation comes from (2.40), which has been derived in section 2.2.2. This is a standard result for geometric Brownian interpolation, see also [ABR96].

We can calculate the above integrals either numerically or using Monte Carlo simulation. We did this in Matlab for a down-and-out call option, with input parameters $K = 3$, $S_0 = 9$, $T = 1$, $\sigma = 0.5$, $r = 0.05$ and barrier $B = 1$. Also, we estimated the above integrals through Monte Carlo simulation, using $M = 10^5$ replications and $N = 1024$ timesteps. Unlike in the case of digital option, where we simulated only the terminal value of the underlying asset, here we should simulate the whole path of the asset price. We consider the case of continuous monitoring of barrier, for which analytic formulas exist for both Delta and Gamma. Explicit formulas, for all the Greeks of barrier options, can be found in [Wys02]. Figure 3.15 shows the values of Delta and Gamma as the parameter ϵ varies.

From the graphs, we can see that the smaller the value of the parameter ϵ the better estimates of Greeks are obtained by numerical integration. On the other hand, when the ϵ is relatively big, the Monte Carlo estimates converge to those obtained by numerical integration. However, as ϵ tends to zero, the Monte Carlo estimates become very poor.

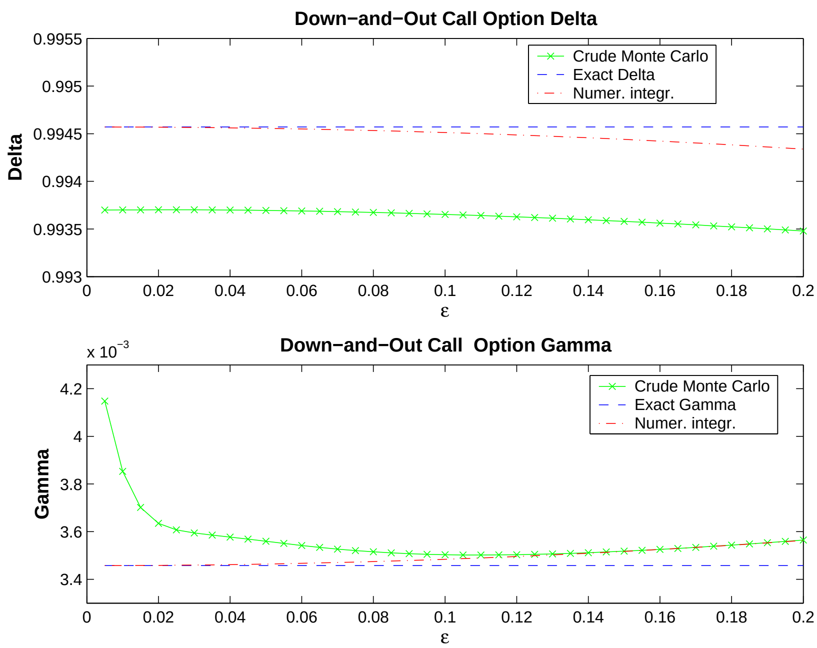

Figure 3.15: Delta and Gamma of a down and out call option, for $M = 10^5$, $N = 1024$. Input parameters : $K = 3$, $S_0 = 9$, $T = 1$, $\sigma = 0.5$, $r = 0.05$ and $B = 1$.

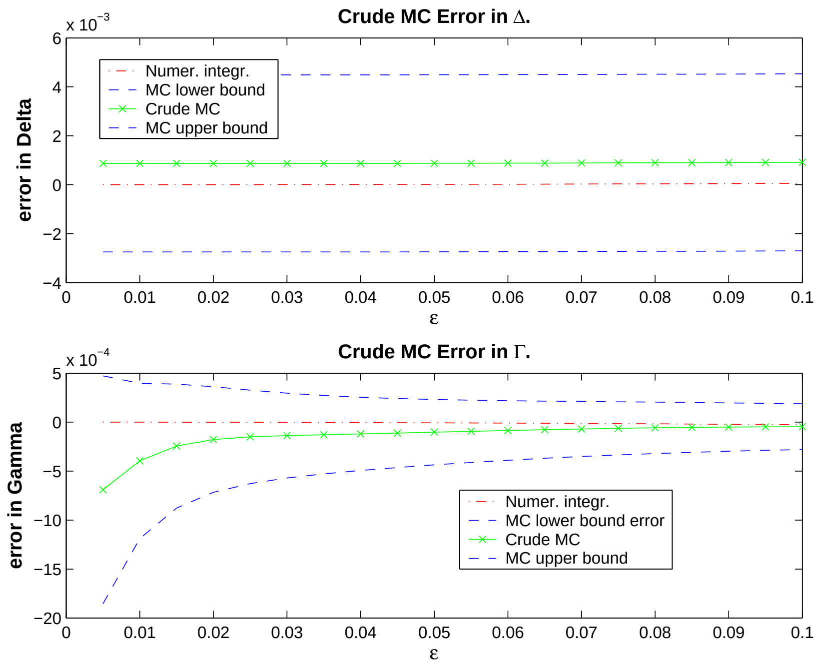

Figure 3.16: Error in Delta and Gamma estimations of a down and out call option, for $M = 10^5$, $N = 1024$. Input parameters : $K = 3$, $S_0 = 9$, $T = 1$, $\sigma = 0.5$, $r = 0.05$ and $B = 1$.

3.2.3 Asymptotic Analysis

Now, by doing asymptotic analysis as in digital options, we can show that the error for both Delta and Gamma is of order $\mathcal{O}(\epsilon^2)$.

The value of the option is given by the expected value of its discounted payoff, which is

$$V = e^{-rT} \int_0^\infty \int_0^\infty H(S_{min} - B) R(S(T) - K)\, p(S_{min}, S(T))\, dS_{min}\, dS(T) \ , \tag{3.101}$$

while the value of the option, which is estimated through the smooth approximation $P_\epsilon\,(S_{min}, S(T))$ of its payoff, is given by

$$V_\epsilon = e^{-rT} \int_0^\infty \int_0^\infty H_\epsilon(S_{min} - B) R_\epsilon(S(T) - K)\, p(S_{min}, S(T))\, dS_{min}\, dS(T) \ . \tag{3.102}$$

Thus we have that

$$\begin{aligned} V - V_\epsilon \;\; &= \;\; e^{-rT} \int_0^\infty \int_0^\infty [H(S_{min} - B) R(S(T) - K) \\ &\qquad - H_\epsilon(S_{min} - B) R_\epsilon(S(T) - K)]\; p(S_{min}, S(T))\, dS_{min}\, dS(T) \end{aligned}$$

which can be written in the following form

$$\begin{aligned} V - V_\epsilon \;\; &= \;\; e^{-rT} \int_0^\infty \int_0^\infty \Big([H(S_{min} - B) - H_\epsilon(S_{min} - B)]\, R(S(T) - K) \\ &\quad + [R(S(T) - K) - R_\epsilon(S(T) - K)]\, H_\epsilon(S_{min} - B) \Big)\, p(S_{min}, S(T))\, dS_{min}\, dS(T) \end{aligned}$$

or

$$\begin{aligned} V - V_\epsilon \;\; = \;\; e^{-rT} \Bigg\{ &\underbrace{\int_0^\infty \int_0^\infty \Big([H(S_{min} - B) - H_\epsilon(S_{min} - B)]\, R(S(T) - K) \Big)\, p(S_{min}, S(T))\, dS_{min}\, dS(T)}_{I_1} \\ &+ \underbrace{\int_0^\infty \int_0^\infty \Big([R(S(T) - K) - R_\epsilon(S(T) - K)]\, H_\epsilon(S_{min} - B) \Big)\, p(S_{min}, S(T))\, dS_{min}\, dS(T)}_{I_2} \Bigg\} \ . \end{aligned}$$

By setting $S_{min} = B + \epsilon x$ the first double integral I_1, becomes

$$I_1 \approx \int_0^\infty \int_{-\infty}^\infty \Big([H(\epsilon x) - H_\epsilon(\epsilon x)]\, R(S(T) - K) \Big)\, p(B + \epsilon x, S(T))\, \epsilon\, dx\, dS(T) \ . \tag{3.103}$$

Now, we have that

$$H_\epsilon(\epsilon x) - H(\epsilon x) = h(x) - H(x) \tag{3.104}$$

and thus integration with respect to x gives

$$\frac{1}{\epsilon}[R_\epsilon(\epsilon x) - R(\epsilon x)] = r(x) - R(x) \ , \tag{3.105}$$

where

$$r(x) = \int_{-\infty}^{x} h(s)\,ds \quad \text{and} \quad R(x) = \int_{-\infty}^{x} H(s)\,ds \ . \tag{3.106}$$

Hence,

$$R_\epsilon(\epsilon x) - R(\epsilon x) = \epsilon(r(x) - R(x)) \ . \tag{3.107}$$

The above result along with the fact that $r(x) - R(x) \to 0$ as $|x| \to \infty$, because $r(x) - R(x)$ is an even function (since the integral of an odd function is even) and it clearly goes to zero as $x \to -\infty$ because $h(x) - H(x)$ decays exponentially as $x \to -\infty$, see figures 3.18 and 3.17, means that I_1 can be written as follows

$$I_1 \approx \int_0^\infty \Big(\epsilon\,[R(x) - r(x)]\,R(S(T) - K)\Big)\,p(B + \epsilon x, S(T))\,\epsilon\,dS(T) \ . \tag{3.108}$$

Now expanding the Taylor series for the function $p(B + \epsilon x, S(T))$, we have

$$p(B + \epsilon x, S(T)) = p(B, S(T)) + \epsilon\,x\,\frac{dp}{dB} + \mathcal{O}(\epsilon^2) \tag{3.109}$$

and by substituting (3.109) in (3.108), we obtain

$$I_1 \approx \underbrace{\int_0^\infty \Big(\epsilon\,[R(x) - r(x)]\,R(S(T) - K)\Big)\,p(B, S(T))\,\epsilon\,dS(T)}_{\mathcal{O}(\epsilon^2)} + \mathcal{O}(\epsilon^4) \ .$$

since $r(x) - R(x)$ is an even function. Now by setting $S_{min} = B + \epsilon x$ and $S(T) = K + \epsilon y$, and using the above statements we obtain

$$I_2 \ \approx \ \underbrace{\int_{-\infty}^\infty \int_{-\infty}^\infty \Big(\epsilon\,[R(y) - r(y)]\,h(x)\Big)\ (p(B + \epsilon x, K + \epsilon y))\ \epsilon^2\,dx\,dy}_{\mathcal{O}(\epsilon^3)} \ + \mathcal{O}(\epsilon^5) \ ,$$

since by Taylor series expansion, we have

$$p(B + \epsilon x, K + \epsilon y) = p(B, K) + \epsilon y\frac{dp}{dB} + \epsilon x\frac{dp}{dK} + \mathcal{O}(\epsilon^2) \ . \tag{3.110}$$

From the above results, we have that $V - V_\epsilon \sim \mathcal{O}(\epsilon^2)$. Hence, the error of the Delta approximation, is given by

$$\Delta - \Delta_\epsilon = \frac{d}{dS_0}[V - V_\epsilon] \sim \mathcal{O}(\epsilon^2) \ , \tag{3.111}$$

while the error in Gamma estimation is given by

$$\Gamma - \Gamma_\epsilon = \frac{d^2}{dS_0^2}[V - V_\epsilon] \sim \mathcal{O}(\epsilon^2) \ . \tag{3.112}$$

Figure 3.19 shows the error in estimation of Delta and Gamma of the down-and-out option of the previous section through numerical integration, for different values of ϵ.

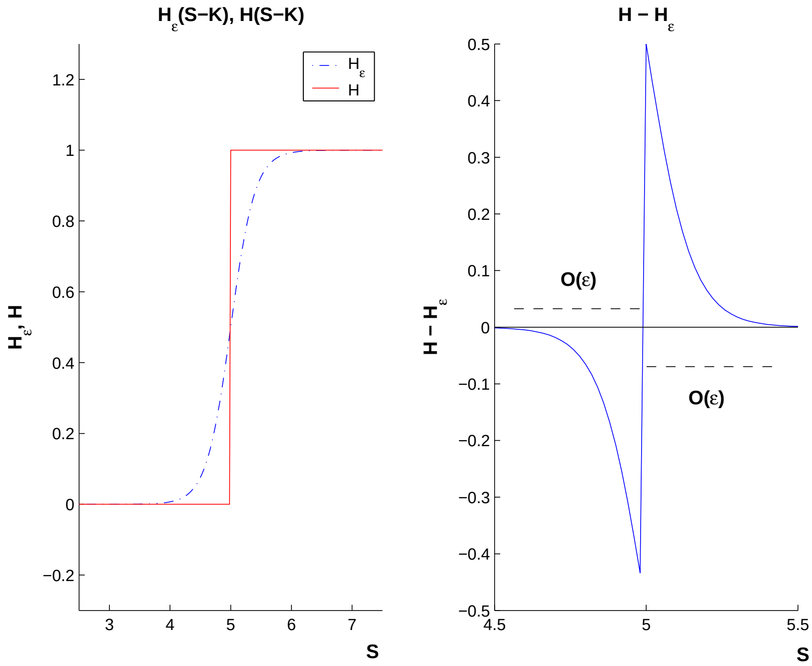

Figure 3.17: Approximation of Heaviside function.

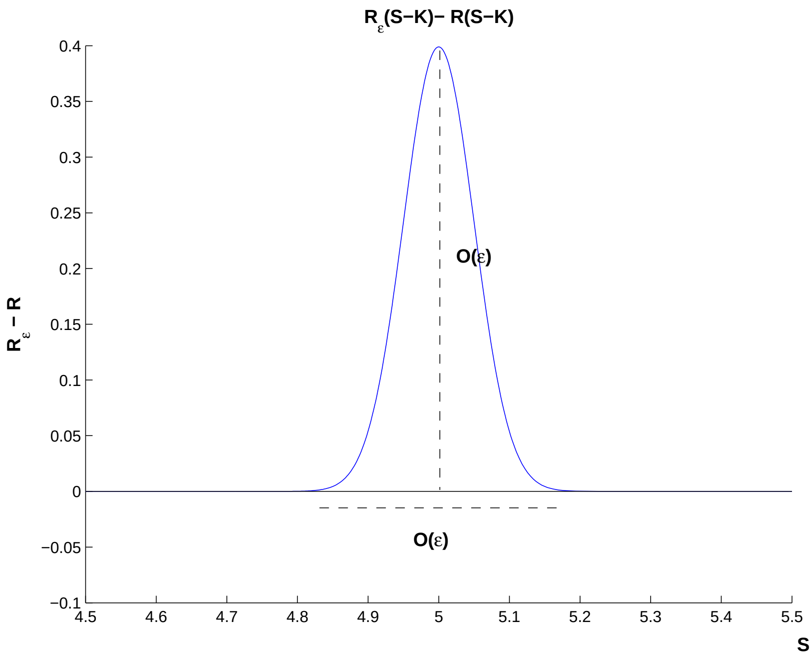

Figure 3.18: Graph of function $R_\epsilon(S - B) - R(S - B)$.

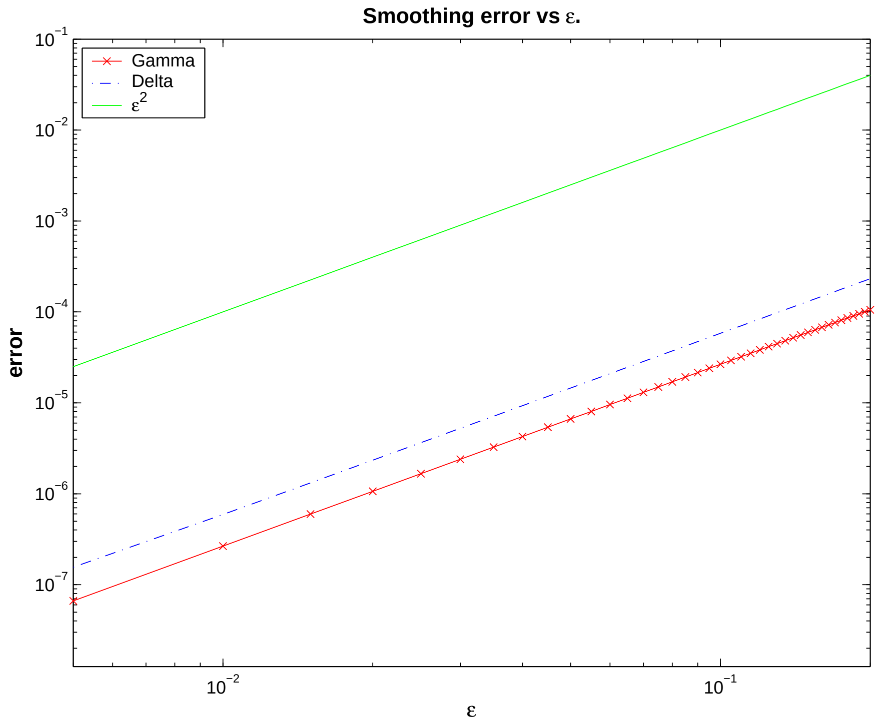

Figure 3.19: Error in Delta and Gamma estimations by numerical integration, of a down and out call option. Input parameters : $K = 3$, $S_0 = 9$, $T = 1$, $\sigma = 0.5$, $r = 0.05$ and $B = 1$.

3.2.4 Monte Carlo Variance

In this section, we study the Monte Carlo variance. In general it holds

$$\mathrm{Var}[g(x)] = E[g^2(x)] - E[g(x)]^2 \tag{3.113}$$

First, we consider the variance of delta. In this case, we need to calculate the variance of the function $\partial P_\epsilon\left(S_{min}, S(T)\right)/\partial S(0)$, where

$$P_\epsilon\left(S_{min}, S(T)\right) = H_\epsilon\left(S_{min} - B\right) R_\epsilon\left(S(T) - K\right) \quad , \tag{3.114}$$

with

$$H_\epsilon(x) = \frac{1}{2}\left[\tanh\left(\frac{x}{\epsilon}\right) + 1\right]$$

and

$$R_\epsilon(x) = \int_{-\infty}^{x} H_\epsilon(s)ds \quad .$$

Thus by differentiating (3.114) with respect to $S(0)$, we get

$$\frac{\partial P_\epsilon(S_{min}, S(T))}{\partial S(0)} = \frac{dH_\epsilon\left(S_{min} - B\right)}{dS_{min}}\frac{S_{min}}{S(0)} + \frac{S(T)}{S(0)}H_\epsilon\left(S_{min} - B\right)H_\epsilon\left(S(T) - K\right) \tag{3.115}$$

As before, we have used the chain rule and the fact that $dS(T)/dS(0) = S(T)/S(0)$ and $dS_{min}/dS(0) = S_{min}/S(0)$. Thus the variance of delta is given by the following expression

$$\begin{aligned}\text{Var}\left[e^{-rT}\,\frac{dP_\epsilon}{dS(T)}\frac{dS(T)}{dS(0)}\right] &= e^{-2rT}\,E\left[\left(\frac{dH_\epsilon(S_{min}-B)}{dS_{min}}\frac{S_{min}}{S(0)}R_\epsilon\left(S(T)-K\right)\right.\right.\\ &\quad \left.\left.+\frac{S(T)}{S(0)}H_\epsilon\left(S_{min}-B\right)H_\epsilon\left(S(T)-K\right)\right)^2\right]\\ &\quad -e^{-2rT}\,E\left[\left(\frac{dH_\epsilon(S_{min}-B)}{dS_{min}}\frac{S_{min}}{S(0)}R_\epsilon\left(S(T)-K\right)\right.\right.\\ &\quad \left.\left.+\frac{S(T)}{S(0)}H_\epsilon\left(S_{min}-B\right)H_\epsilon\left(S(T)-K\right)\right)\right]^2 .\end{aligned} \tag{3.116}$$

Using similar arguments to those of the previous section, we can show that the variance of Monte Carlo estimate of Delta is of order $\mathcal{O}(1)$, i.e.

$$\text{Var}\left[e^{-rT}\,\Delta_\epsilon\right] = \text{Var}\left[e^{-rT}\frac{\partial P_\epsilon(S_{min},S(T))}{\partial S(0)}\right] \sim \mathcal{O}(1)\ . \tag{3.117}$$

This means that the variance of Δ_ϵ is independent of parameter ϵ, something which is verified from the experimental results that we provided in section 3.2.2.

In the case of Gamma estimation, we need to calculate the variance of the second derivative of P_ϵ with respect to $S(0)$. That is

$$\begin{aligned}\frac{\partial^2 P_\epsilon(S_{min},S(T))}{\partial S^2(0)} &= \frac{d^2H_\epsilon(S_{min}-B)}{dS^2_{min}}\;\frac{S^2_{min}}{S^2(0)}\;R_\epsilon(S(T)-K)\\ &+ \frac{2S_{min}S(T)}{S^2(0)}\;\frac{dH_\epsilon(S_{min}-B)}{dS_{min}}\;H_\epsilon(S(T)-K)\\ &+ \frac{S^2(T)}{S^2(0)}\;\frac{dH_\epsilon(S(T)-K)}{dS(T)}\;H_\epsilon(S_{min}-B)\ .\end{aligned}$$

Again, using similar procedure to that of the previous section, we can show that the variance of Gamma estimate is of order $\mathcal{O}(\epsilon^{-1})$, i.e.

$$\text{Var}\left[e^{-rT}\,\frac{\partial^2 P_\epsilon(S_{min},S(T))}{\partial S^2(0)}\right] \sim \mathcal{O}(\epsilon^{-1})\ . \tag{3.118}$$

Figure 3.20 shows the variance for both estimates of Delta and Gamma of the down-and-out call option. We can see that the variance in Delta estimate is constant as ϵ varies. On the other hand the variance in Gamma estimate behaves like ϵ^{-1}.

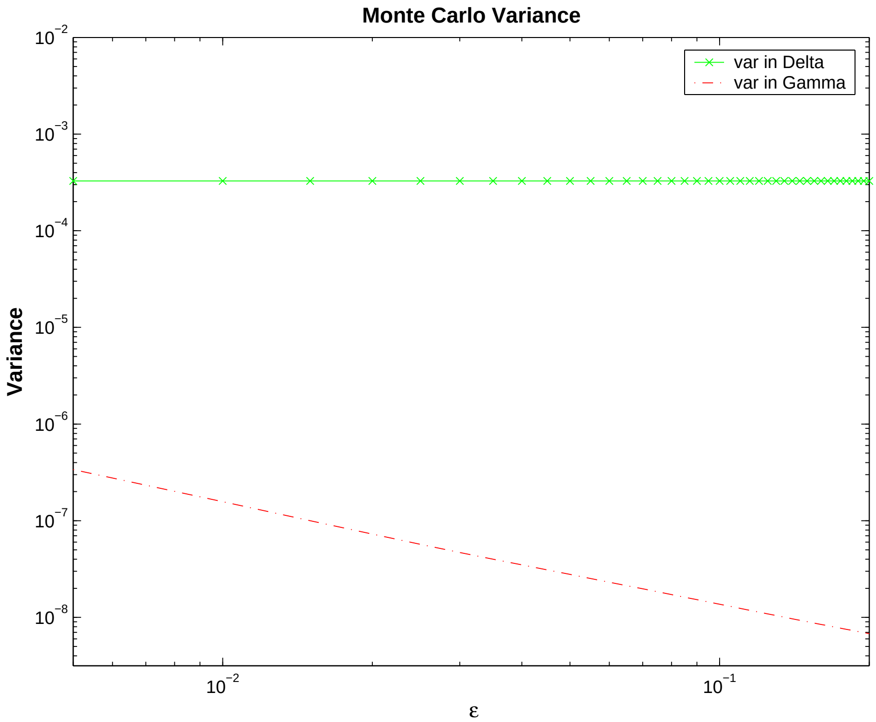

Figure 3.20: Variance of Delta and Gamma Monte Carlo estimators, for $M = 10^5$, $N = 1024$. Input parameters : $K = 3$, $S_0 = 9$, $T = 1$, $\sigma = 0.5$, $r = 0.05$ and $B = 1$.

3.2.5 Variance Reduction

Again, we can reduce the variance of the Monte Carlo estimators of Greeks, by using stratified sampling. In this case, in order to estimate the values of Greeks, we need to simulate both the terminal asset price $S(T)$ and its minimum during the life of the option. Thus we should simulate the whole asset price path. Using the Euler scheme, we have

$$S_{k+1} = S_k + r\, S_k\, \Delta t + \sigma\, S_k\, (W_{k+1} - W_k) \tag{3.119}$$

for $k = 0, \ldots, N-1$ and $(W_{k+1} - W_k) \sim N(0, \Delta t)$. Also we assume that the current stock price $S(0) = S_0$ is known and that $W_0 = 0$.

Now, much of the variability in the option's and Greeks values can be eliminated by stratifying the terminal asset price $S(T)$. We can do this by stratifying a number of Brownian paths. This is consequence of the fact that $S(T)$ is a monotone transformation[7] of $W(T)$. First, we consider the case of I equiprobable strata and a proportional allocation, namely $p_i = P(W(T) \in A_i) = 1/I$ and $n_i = p_i n$. Then, we can generate $I \times n_i$ Brownian paths, i.e. n_i paths from each stratum, stratified along $W(T)$. We can do this using the algorithm of table 3.4.

In the fourth step of that algorithm we use the notation Φ^{-1}, which denotes the inverse cumulative

[7]Remember that $S(T) = S(0)e^{(r-\sigma^2/2)T+\sigma W(T)}$.

Brownian Bridge Algorithm

Inputs : I =#strata, m s.t. 2^m = #timesteps
for $i = 1, \ldots, I$
set $n_i = n/I$
for $j = 1, \ldots, n_i$
1. generate $U \sim$ Unif$[0,1]$
2. calculate $V = \frac{i-1+U}{I}$, where $V \in [\frac{i-1}{I}, \frac{i}{I}]$
3. calculate $W_j(T) = \sqrt{T}\,\Phi^{-1}(V)$
4. given $W_{0,j} = 0$ and $W_{N,j} = W_j(T)$ calculate the intermediate values $W_{k,j}$ $k = 1, \ldots, N-1$, using the Brownian Bridge construction
end for
end for

Table 3.4: Generation of $I \times n_i$ Brownian paths stratified along $W(T)$.

standard normal distribution function[8].

Figure 3.21 shows K simulated Brownian motion paths using terminal stratification. The paths in figure were constructed using the above algorithm, with $I = 10$ and $n_i = 1$. It is worth studying the 4th step of the algorithm, which is given in table 3.4 and it refers to Brownian Bridge construction. It is possible by conditioning a Brownian motion on its endpoints to construct a Brownian bridge. Once we determine the value of W_N, we can sample the point $W_{\lfloor N/2 \rfloor}$, conditional on W_0 and W_N since it is known that (see page 84 in [Gla04])

$$\Big(W(t)|W(t_{k+1}), W(t_k)\Big) = N\left(\frac{(t_{k+1}-t)W(t_k) + (t-t_k)W(t_{k+1})}{(t_{k+1}-t_k)}, \sqrt{\frac{(t_{k+1}-t)(t-t_k)}{(t_{k+1}-t_k)}}\right) . \quad (3.120)$$

Thus, to sample the point $W_{\lfloor N/2 \rfloor}$, we may set

$$W_{\lfloor N/2 \rfloor} = \frac{(t_N - t_{\lfloor N/2 \rfloor})W_0 + (t_{\lfloor N/2 \rfloor} - t_0)W_N}{(t_N - t_0)} + \sqrt{\frac{(t_N - t_{\lfloor N/2 \rfloor})(t_{\lfloor N/2 \rfloor} - t_0)}{(t_N - t_0)}}\, Z , \quad (3.121)$$

where $Z \sim N(0,1)$. Similarly, in the next step we can sample $W_{\lfloor N/4 \rfloor}$ conditional on W_0 and $W_{\lfloor N/2 \rfloor}$ as well as $W_{\lfloor 3N/4 \rfloor}$ conditional on W_N and $W_{\lfloor N/2 \rfloor}$. By repeating this procedure, we compute all the components W_k, $k = 0, \ldots N$ of the Brownian path. This technique is known as *Brownian bridge construction*. For convenience, we assume that N is a power of 2, such that $N/2^m$ is always integer, where $m = \log_2 N$ is the total number of steps are needed to sample all the points of the Brownian path. An algorithm for the implementation of Brownian bridge construction, when the number of time indices is a power of 2, can be found in page 85 of Glasserman's book [Gla04].

[8]An implementation of C routine, which estimates the value of this function, is available on Peter J. Acklam's site http://www.math.uio.no/ jacklam .

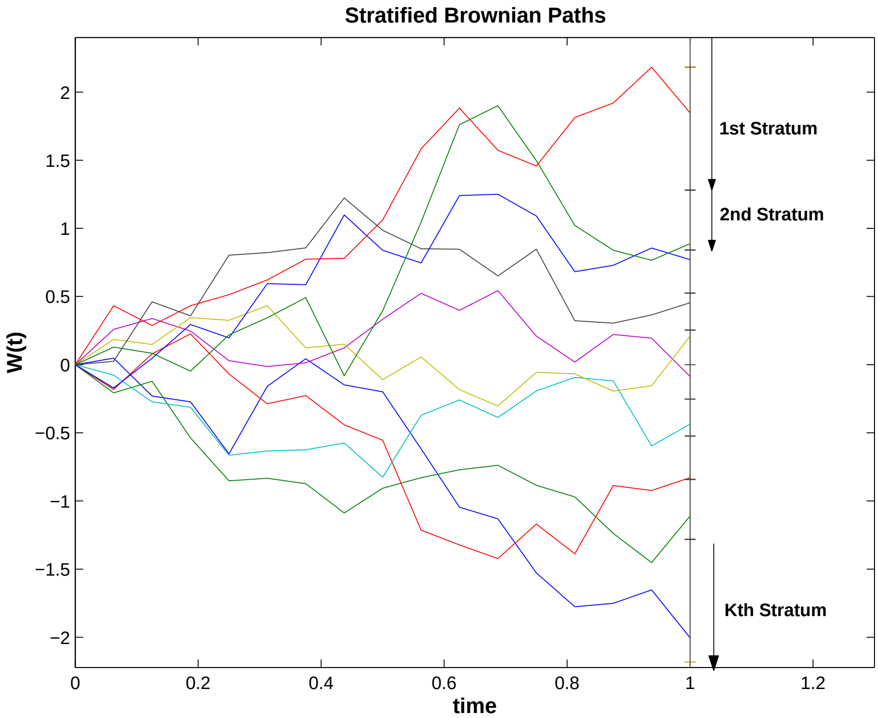

Figure 3.21: Simulation of K Brownian paths using terminal stratification.

After having calculated all the components of the Brownian path W_k, $k = 0, \ldots N$, we can continue, as in the standard Monte Carlo simulation, and compute the underlying prices S_k at each time step through the Euler scheme (3.119). Having constructed the whole asset price path, we can draw samples of both $S(T)$ and S_{min}. These values are used to calculate the Greeks for this single asset price path. We repeat the same procedure sufficiently many times and we calculate the mean of the discounted payoffs. The algorithm in table 3.5 summarizes the standard Monte Carlo algorithm, which can be used to estimate the greeks of a down-and-out call option. This algorithm is modified when we want to use Monte Carlo with stratified sampling with proportional or optimal allocation like the algorithms of tables 3.2 and 3.3, respectively.

Figure 3.22 shows the variance of the crude Monte Carlo simulation as well as the variance of the Monte Carlo with stratified sampling. We executed the experiments using a total sample size equal to 10^5. We can see that the use of stratified sampling results in a reduction of variance in both Delta and Gamma estimations. Also figure 3.23 shows the estimates of all Monte Carlo methods against the parameter ϵ. We can see that in Delta case, stratified sampling with both proportional and optimal allocation improves significantly the accuracy of the Monte Carlo method. In the case of Gamma estimation, we see that the stratified sampling with optimal allocation gives by far better approximations than that with proportional allocation. We can see that in this case the stratified sampling with proportional allocation does not help much when the ϵ is very small. However it gives better approximations than the standard Monte Carlo. Note that as the Monte Carlo sample increases we expect the Monte Carlo estimates to converge to the values which are obtained by the numerical integration (line with crosses) and not to the exact values (solid line) of Greeks.

Delta and Gamma Estimation Algorithm

Inputs : M =#paths, m s.t. 2^m = #timesteps

for $i = 1, \ldots, M$

1. Construct a Brownian path using the Brownian Bridge Algorithm
2. Use the values of the Brownian path components W_{k+1} and W_k to calculate the stock prices S_k at each time t_k, for $k = 0, 1, \ldots 2^m$.
3. Determine the final asset price S_T as well as its minimum value S_{min} over the whole path.
4. Calculate Δ_i and Γ_i as the dicounted first and second derivatives of P_ϵ with respect of $S(0)$, for this path.

end for

Calculate the means $\Delta_\epsilon = \frac{1}{M}\sum_{i=1}^{M}\Delta_i$ and $\Gamma_\epsilon = \frac{1}{M}\sum_{i=1}^{M}\Gamma_i$.

Table 3.5: Algorithm for call option pricing through Monte Carlo, with stratified sampling.

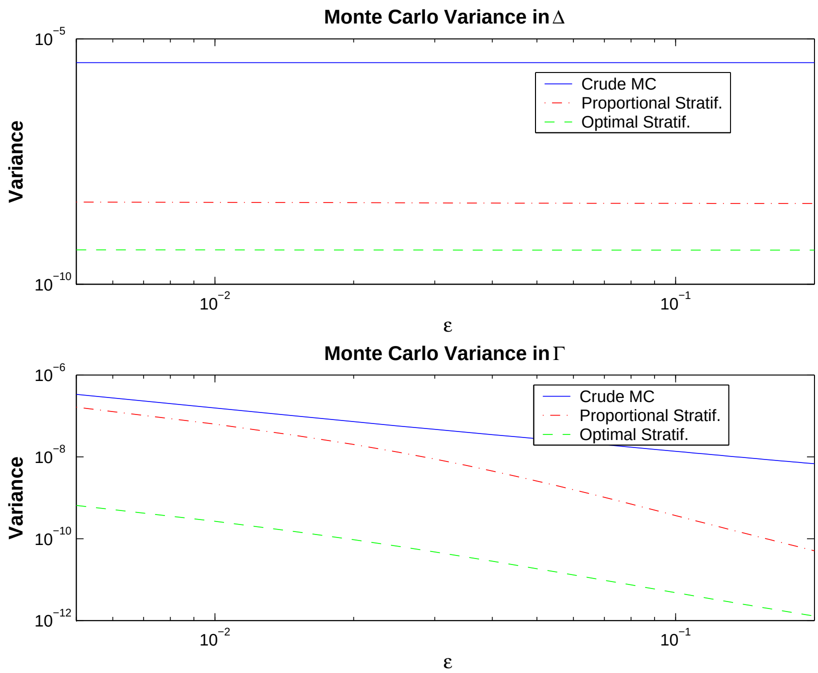

Figure 3.22: Monte Carlo variance, for $I = 1000$, $n_i = 100$ $N = 1024$. Input parameters : $K = 3$, $S_0 = 9$, $T = 1$, $\sigma = 0.5$, $r = 0.05$ and $B = 1$.

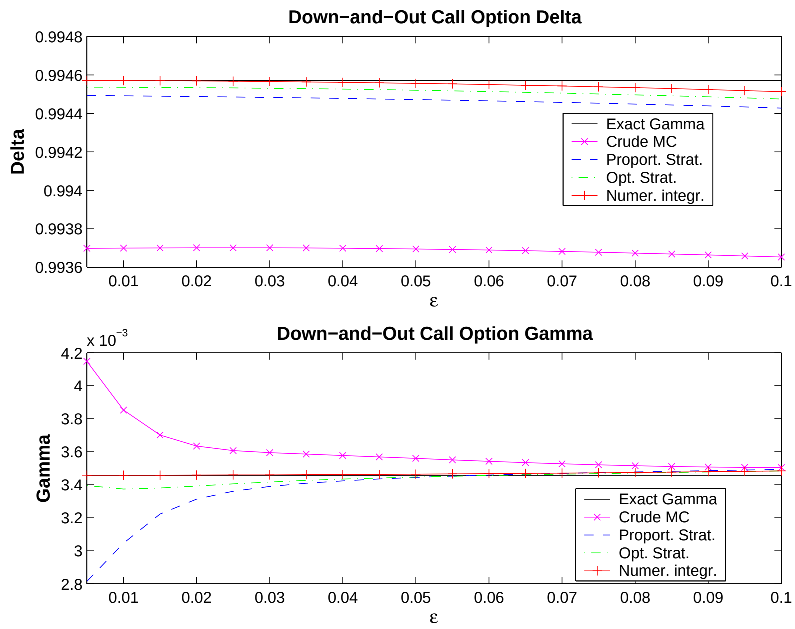

Figure 3.23: Monte Carlo error, for $I = 1000$, $n_i = 100$ $N = 1024$. Input parameters : $K = 3$, $S_0 = 9$, $T = 1$, $\sigma = 0.5$, $r = 0.05$ and $B = 1$.

3.2.6 Comparison with Likelihood Ratio Estimators

It can be shown (see [Gla04],[BG96],[BK04]) that the likelihood estimators of Delta and Gamma of a down-and-out Barrier[9] call option are given by

$$\Delta_{LR} = e^{-rT}(S_T - K)^+ \mathbf{1}\left\{\min_{0\le t\le T} S(t) > B\right\}\left(\frac{d_1}{S_0\sigma\sqrt{\Delta t}}\right)$$

$$\Gamma_{LR} = e^{-rT}(S_T - K)^+ \mathbf{1}\left\{\min_{0\le t\le T} S(t) > B\right\}\left(\frac{d_1^2 - d_1\sigma\sqrt{\Delta t} - 1}{S_0^2\sigma^2\Delta t}\right)$$

with

$$d_1 = \frac{\ln(S_1/S_0) - (r - \sigma/2)^2\Delta t}{\sigma\sqrt{\Delta t}} \quad . \tag{3.122}$$

Figure 3.24 shows the estimates of Delta and Gamma of the down-and-out barrier option of the previous sections, which are obtained from our method (solid line with x) as well as the Likelihood Ratio method (dash-dot line with plus signs). In this case, we choose ϵ to be fixed and equal to 0.1, i.e. $\epsilon = 0.1$. The Monte Carlo estimates are compared with the exact values of Greeks (solid line). Figures 3.25 and 3.26 show the error and the variance in estimates of the previous methods, respectively. From the graphs of that figure, we can see that the estimates of both Delta and Gamma of our method are much better than that of Likelihood Ratio method. In this case the finite-difference approximations are extremely bad and this is why we do not plot their results.

[9] Again, we consider the case of continuous monitoring of the barrier.

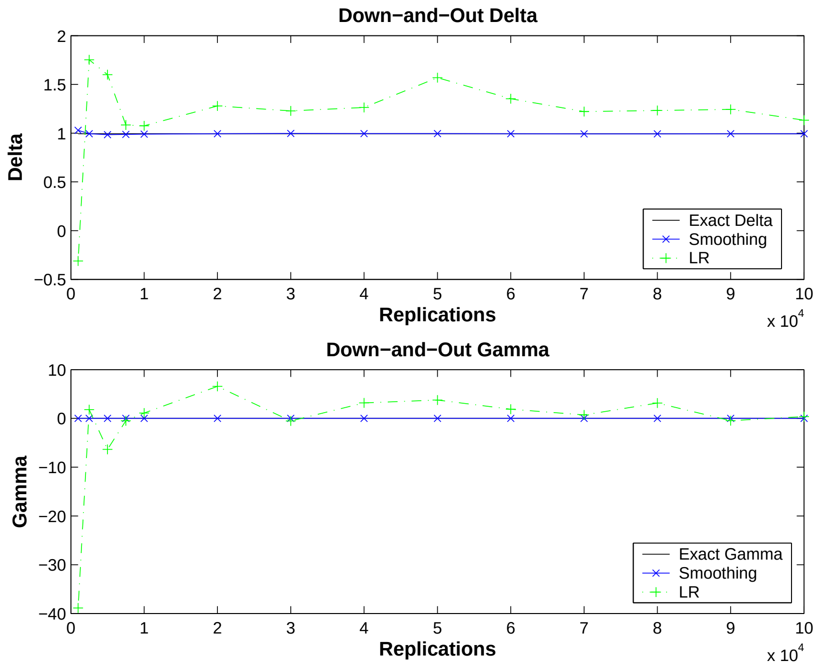

Figure 3.24: Comparison of Pathwise (smoothing) and Likelihood Ratio (LR) estimators. $\Delta t = 1/1024$. Input parameters : $K = 3$, $S_0 = 9$, $T = 1$, $\sigma = 0.5$, $r = 0.05$ and $B = 1$.

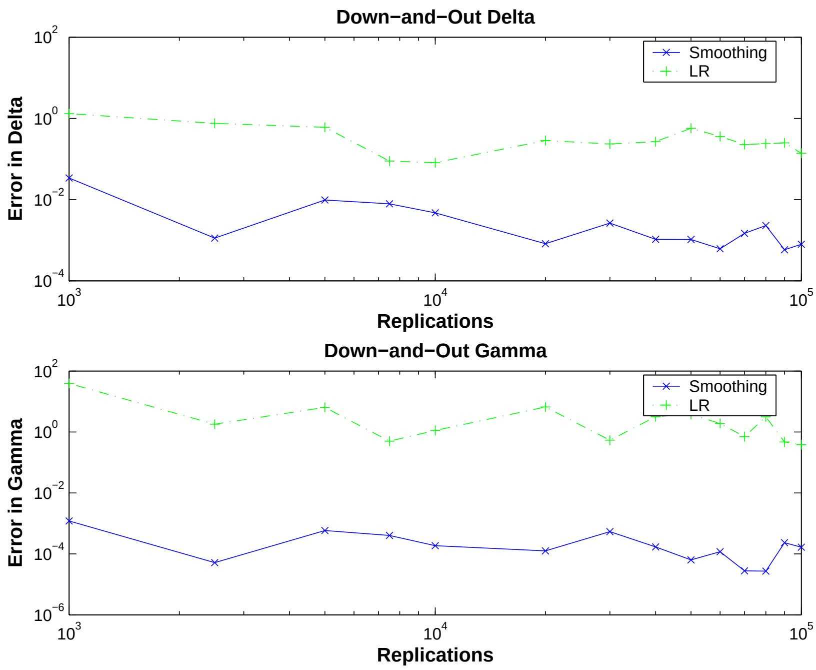

Figure 3.25: Error in estimates with $\Delta t = 1/1024$. Input parameters : $K = 3$, $S_0 = 9$, $T = 1$, $\sigma = 0.5$, $r = 0.05$ and $B = 1$.

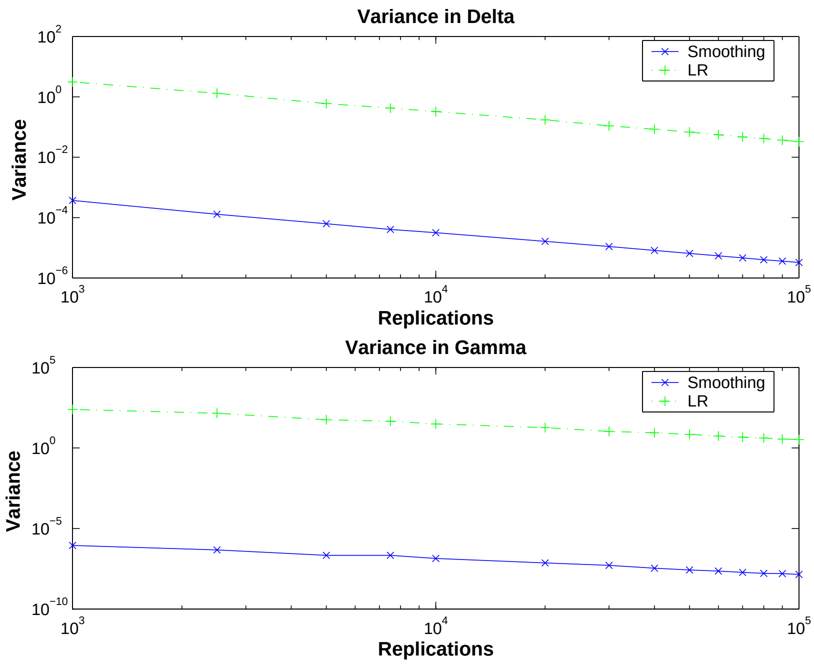

Figure 3.26: Variance in estimates with $\Delta t = 1/1024$. Input parameters : $K = 3$, $S_0 = 9$, $T = 1$, $\sigma = 0.5$, $r = 0.05$ and $B = 1$.

Chapter 4

Conclusion

In this thesis, we study how Monte Carlo simulation can be applied in pricing and hedging financial options. We focus on valuation and on Greeks estimation of exotic options such as Barrier, Lookback and Digital options.

In particular, in the first part of this work we consider enhanced Monte Carlo methods for pricing path-dependent options, which reduce the discretization error - the bias in Monte Carlo estimates that results from time-discretization of stochastic differential equations. We apply a similar correction for pricing continuously monitored barrier options, to that Broadie, Glasserman and Kou applied for pricing discretely monitored barrier options in [BGK97]. We shift the barrier B by a quantity $-B\beta\sigma\sqrt{\Delta t}$ if $B > S(0)$ and $+B\beta\sigma\sqrt{\Delta t}$ if $B < S(0)$ and then we apply the regular Monte Carlo method to price the options. Moreover, we use a probabilistic method, which allows us to simulate the extremals of the geometric Brownian motion, given the endpoints of that process for a single time interval. A similar method is applied in pricing a Lookback option, the value of which depends on the maximum or minimum of an underlying asset price over the life of the option. These methods are suggested in [ABR96], [BDZ97] and [BCI99]. We implemented the above methods in C, and we compare the latter methods with the crude Monte Carlo in terms of weak convergence rate. The numerical results from the experiments confirm the superiority of those methods against the standard Monte Carlo.

In the second part, we propose a Monte Carlo method for estimating sensitivities of derivative securities, which is based on smoothing the discontinuities of the discounted payoff of an option. This smoothing technique allows us to apply the pathwise method to estimate the Greeks. We apply this method in order to estimate the Delta and Gamma of Digital and Barrier options, for which the pathwise method is otherwise inapplicable. Extensive numerical results (see section 7.4 [Gla04]) indicate that the pathwise method, when applicable, provides the best estimates of sensitivities. Compared with the finite-difference methods, pathwise estimates require less computational effort and they directly estimate derivatives rather than differences. Furthermore, compared with the likelihood ratio method, pathwise estimators usually have smaller variance - often much smaller. The main limitation of the use of the likelihood ratio method is the need for explicit knowledge of a density. In cases in which the transition density of the underlying price process is not explicitly known, ideas from Malliavin Calculus can be used. For example, for Asian options, stochastic volatility models or interest rate models the densities are not explicitly known. Kohatsu-Higa and Montero give a basic introduction to Malliavin Calculus and its applications within the area of Monte Carlo simulation in Finance, in [KHM04]. Several authors, including Benhamou [Ben00] and Fournié et.al. [FLLT99], used ideas from Malliavin Calculus to estimate greeks.

As we have mentioned in this thesis, we use a smooth function to approximate the discontinuous payoff of an option and then we apply the pathwise method to estimate the delta and gamma through Monte Carlo simulation. Particularly, we estimate the delta and gamma of a digital call option and down-and-out barrier option. Moreover, we carry out asymptotic analysis in order to determine the error that the smoothing introduces in our estimations and we show how we can reduce this error. Also, we study how the smoothing parameter ϵ affects the variance of Monte Carlo estimates. In particular, we show that when ϵ becomes too small the Monte Carlo error becomes huge. On the other hand, when ϵ becomes too big the smoothing error, which is introduced, becomes big as well. Thus, there is a tradeoff between the Monte Carlo and smoothing error. This makes a matter of investigation the value of ϵ that we should use, so that the total error is minimized. Finally, we use the stratified sampling method to reduce the variance and thus to improve further the efficiency of Monte Carlo method. We show that using stratified with proportional allocation of sampling we can achieve a significant variance reduction, while further variance reduction is achieved using optimal allocation. Numerical results support the theoretical analysis.

The major advantage of the proposed technique is that it extends the pathwise method in problems in which is otherwise inapplicable. This allows us to exploit all the advantages of the pathwise method in estimating sensitivities. Furthermore, the simplicity of the method makes it easily applicable in several problems. Also, using a variance reduction method such as the stratified sampling, we can significantly improve the convergence rate of the Monte Carlo estimates.

This research is in its early stages and much work remains to be done. The method could be applied in more complex or multidimensional problems, in which its advantages can be more apparent. In general, sensitivity estimation presents both theoretical and practical challenges to Monte Carlo simulation.

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Appendix A

Appendix

A.1 Transition Density of a Brownian motion with Upstream Absorbing Barrier

We first consider a Brownian motion W_t that starts at zero, with constant volatility σ and zero drift. Initially, we will apply the reflection principle to calculate the probability $P\left\{m^T \leq b, W_T \geq x\right\}$, where $x \geq 0$, $b \leq 0$ and m^T is the minimum value of the Brownian motion over $[0,T]$. Now if the event $\left\{m^T \leq b, W_T \geq x\right\}$ occurs, then for some time t_b we have that $W(t_b) = b$ and there is certainly some value of t for which W_t is less than or equal to b. The Brownian motion therefore descends at least as far as b and then comes up back to the level x, see figure A.1. Suppose that instead of continuing the Brownian motion after time t_b, we restart it and replace it by its value reflected in the level b. We define this random process as

$$W'_t = \begin{cases} W_t & \text{, for } t < t_b \\ 2b - W_t & \text{, for } t \geq t_b \end{cases} \tag{A.1}$$

Then the event $\{W_T \geq x\}$ becomes $\{W'_T \leq 2b - x\}$. However, the event $\{W'_T \leq 2b - x\}$ occurs only if $m^T \leq b$ occurs also and therefore, we have that the event $\left\{m^T \leq b, W_T \geq x\right\}$ is equivalent to $\{W'_T \leq 2b - x\}$. Also by the reflection principle we have

$$W'_{t_b+s} - W'_{t_b} = -\left(W_{t_b+s} - W_{t_b}\right) \quad \text{, for } s \geq 0, \tag{A.2}$$

where t_b is stopping time and therefore it only depends on the path history $\left\{W_0^t : 0 \leq t \leq t_b\right\}$ and it does not affect the Brownian motion at later times. By the Markov strong property we argue that the two Brownian increments have the same distribution. Thus we have that

$$\begin{aligned} P\left\{m^T \leq b, W_T \geq x\right\} &= P\left\{W'_T \leq 2b - x\right\} \\ &= P\left\{W_T \leq 2b - x\right\} \\ &= N\left(\tfrac{2b-x}{\sigma\sqrt{T}}\right) \end{aligned} \tag{A.3}$$

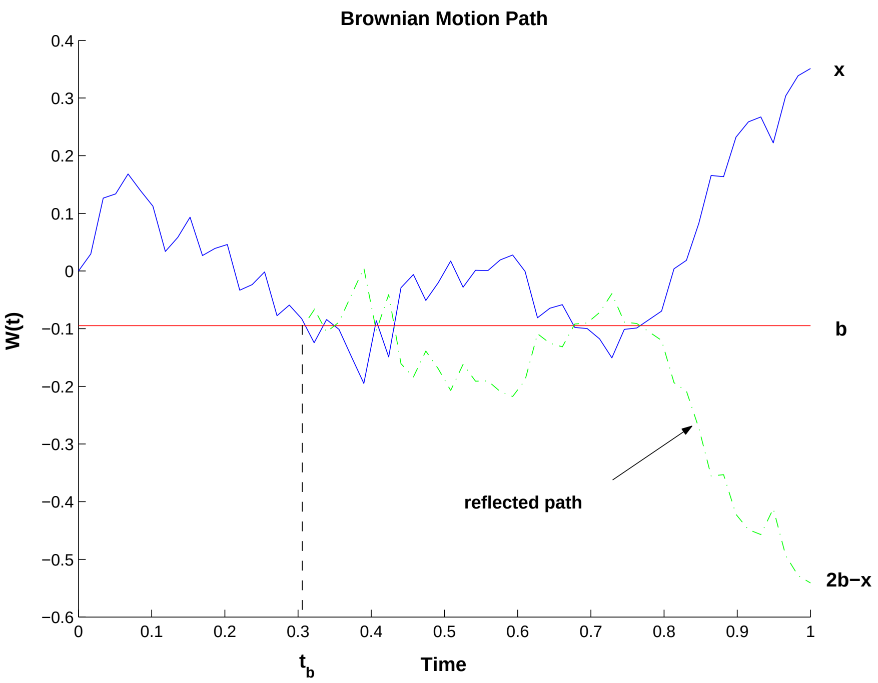

Figure A.1: Brownian motion path.

Next, we can apply the Girsanov Theorem to find the above joint distribution in case where the Brownian motion has non-zero drift. We suppose that under the measure Q, W_t is a Brownian motion with drift rate a. Now we change the measure from Q to Q' such that W_t becomes Brownian motion with zero drift under Q'. So we can write

$$\begin{aligned} P\left\{m^T \leq b, W_T \geq x\right\} &= E_Q\left[\mathbf{1}_{\{m^T \leq b\}}\mathbf{1}_{\{W_T \geq x\}}\right] \\ &= E_{Q'}\left[\mathbf{1}_{\{m^T \leq b\}}\mathbf{1}_{\{W_T \geq x\}}\exp\left(\frac{aW_T}{\sigma^2} - \frac{a^2 T}{2\sigma^2}\right)\right] \quad , \end{aligned} \tag{A.4}$$

where the term $\exp\left(\frac{aW_T}{\sigma^2} - \frac{a^2 T}{2\sigma^2}\right)$ is the Radon-Nykodym derivative. For the derivation of this, see chapter 4 in [Kwo98] as well as in chapter 8 in [Jos03]. Thus by applying the reflection principle for the driftless Brownian motion under Q', we obtain

$$\begin{aligned} P\left\{m^T \le b, W_T \ge x\right\} &= E_{Q'}\left[\mathbf{1}_{\{m^T \le b\}}\mathbf{1}_{\{W_T \ge x\}} \exp\left(\tfrac{aW_T}{\sigma^2} - \tfrac{a^2T}{2\sigma^2}\right)\right] \\ &= E_{Q'}\left[\mathbf{1}_{\{2b-W_T > x\}} \exp\left(\tfrac{a(2b-W_T)}{\sigma^2} - \tfrac{a^2T}{2\sigma^2}\right)\right] \\ &= e^{\frac{2ab}{\sigma^2}} E_{Q'}\left[\mathbf{1}_{\{W_T < 2b-x\}} \exp\left(-\tfrac{aW_T}{\sigma^2} - \tfrac{a^2T}{2\sigma^2}\right)\right] \\ &= e^{\frac{2ab}{\sigma^2}} \int_{-\infty}^{2b-x} \tfrac{1}{\sqrt{2\pi\sigma^2 T}} e^{\frac{z^2}{2\sigma^2 T}} \exp\left(-\tfrac{aW_T}{\sigma^2} - \tfrac{a^2T}{2\sigma^2}\right) dz \\ &= e^{\frac{2ab}{\sigma^2}} \int_{-\infty}^{2b-x} \tfrac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\tfrac{(z+aT)^2}{2\sigma^2 T}\right) dz \\ &= e^{\frac{2ab}{\sigma^2}} N\left(\tfrac{2b-x+aT}{\sigma\sqrt{T}}\right) \end{aligned} \tag{A.5}$$

and by applying the law of total probabilities, we get

$$\begin{aligned} P\left\{m^T > b, W_T \ge x\right\} &= P\left\{W_T \ge x\right\} - P\left\{m^T \le b, W_T \ge x\right\} \\ &= N\left(\tfrac{-x+aT}{\sigma\sqrt{T}}\right) - e^{\frac{2ab}{\sigma^2}} N\left(\tfrac{2b-x+aT}{\sigma\sqrt{T}}\right) \ . \end{aligned} \tag{A.6}$$

The above result can be extended to the case where the barrier B is above the initial value of Brownian motion W_t. We denote by M^T the maximum of Brownian motion over the time interval $[0, T]$. Then we can write

$$M^T = \max_{0 \le t \le T} \{\sigma Z_t + at\} = - \min_{0 \le t \le T} \{-\sigma Z_t - at\} \ , \tag{A.7}$$

where Z_t is the standard Brownian motion. Since $-Z_t$ has the same distribution as Z_t, the distribution of the maximum of W_t with drift a is the same as that of the negative of the minimum of W_t with negative drift. Thus by replacing $-a$ for a, $-B$ for b and $-y$ for x in (A.5), we obtain

$$P\left\{M^T \ge B, W_T \le y\right\} = e^{\frac{2aB}{\sigma^2}} N\left(\frac{y - 2B - aT}{\sigma\sqrt{T}}\right) \ , \tag{A.8}$$

with $B > \max(y, 0)$. Thus by differentiating with respect to y we obtain the following density function

$$P\left\{M^T \ge B, W_T \in dy\right\} = e^{\frac{2aB}{\sigma^2}} \frac{1}{\sigma\sqrt{T}} \ n\left(\frac{y - 2B - aT}{\sigma\sqrt{T}}\right) \tag{A.9}$$

and finally the transition function $p_B(x, t; x_0, t_0)$ for the restricted Brownian motion which hits the barrier B at some time $t_b \in [t_0, t]$ is found to be

$$p_B(x, t; x_0, t_0) = e^{\frac{2a(B-x_0)}{\sigma^2}} \frac{1}{\sigma\sqrt{t-t_0}} \ n\left(\frac{(x - x_0) - 2(B - x_0) - a(t - t_0)}{\sigma\sqrt{t - t_0}}\right) \tag{A.10}$$

and therefore we obtain the following probability

$$p\left\{t_b \in [t_0, t], W_{t_0} = x_0, W_t = x\right\} = e^{\frac{2a(B-x_0)}{\sigma^2}} \frac{1}{\sigma\sqrt{t-t_0}} \ n\left(\frac{x + x_0 - 2B - a(t - t_0)}{\sigma\sqrt{t - t_0}}\right) \ . \tag{A.11}$$

A.2 Delta and Gamma of Down-and-Out Barrier

The explicit formula for the Delta of Down-and-Out barrier option is given by

$$\Delta = N(X) - \frac{2(r-\sigma^2/2)}{\sigma^2 S_0}\left(\frac{B}{S_0}\right)^{2\lambda-2}\left[S_0\left(\frac{B}{S_0}\right)^2 N(y) - e^{-rT}N(y-\sigma\sqrt{T})\right] + \left(\frac{B}{S_0}\right)^{2\lambda} N(y) \quad \text{(A.12)}$$

with

$$\begin{aligned} X &= \frac{\log\left(\frac{S_0}{K}\right) + (r+\sigma^2/2)T}{\sigma\sqrt{T}} \\ y &= \frac{\log\left(\frac{B^2}{S_0 K}\right) + (r+\sigma^2/2)T}{\sigma\sqrt{T}} \\ \lambda &= \frac{1}{2} + \frac{r}{\sigma^2}\ . \end{aligned}$$

Also, the explicit formula for the Gamma is the following

$$\Gamma = \frac{n(X)}{S_0\sigma\sqrt{T}} - \frac{2(r-\sigma^2/2)}{\sigma^2 S_0}\left(\frac{C}{S_0} - D\right) + \frac{B^{2\lambda}}{S_0^{2\lambda+1}}\left[2\lambda N(y) - \frac{n(y)}{\sigma\sqrt{T}}\right] \quad \text{(A.13)}$$

with

$$\begin{aligned} C &= \left(\frac{B}{S_0}\right)^{2\lambda-2}\left[S_0\left(\frac{B}{S_0}\right)^2 N(y) - e^{-rT}N(y-\sigma\sqrt{T})\right] \\ D &= -\frac{2(r-\sigma^2/2)}{\sigma^2 S_0}C - \left(\frac{B}{S_0}\right)^{2\lambda} N(y) \end{aligned}$$

and X, y, λ are defined as before.

A complete list, with explicit formulas of Greeks for all kind of barrier options, is given in [Wys02].

A.3 The Brownian Bridge

A Brownian bridge[1] is a continuous-time stochastic process whose probability distribution is the conditional probability distribution of a Wiener process W(t) (a mathematical model of Brownian motion) given the condition that initially $W(0) = a$ and finally $W(T) = b$.

Suppose that $Z(t)$ is a standard Brownian process. Let the points $Z(t_i)$, $Z(t_k)$ are known and we want to draw the point $Z(t_j)$ with $t_i < t_j < t_k$ conditional on $Z(t_i)$, $Z(t_k)$. If now $x, y \sim N(0,1)$

[1]See http://en.wikipedia.org/wiki/Brownian_bridge for a definition.

and we know the point $Z(t_i) = Z_i$, then we can generate the points $Z(t_k) = Z_k$, $Z(t_j) = Z_j$ as follows

$$
\begin{aligned}
Z_j &= Z_i + x\sqrt{t_j - t_i} && \text{(A.14)} \\
Z_k &= Z_j + y\sqrt{t_k - t_j} = Z_i + x\sqrt{t_j - t_i} + y\sqrt{t_k - t_j} \quad . && \text{(A.15)}
\end{aligned}
$$

However, we could generate the Z_k directly as

$$Z_k = Z_i + z\sqrt{t_k - t_i} \quad , \tag{A.16}$$

where $z \sim N(0,1)$. Since, we have shown that if we know Z_i we know Z_k as well, this means that the above method for generating Z_j is constrained. By (A.15) and (A.16), we have that

$$y = \frac{z\sqrt{t_k - t_i} - x\sqrt{t_j - t_i}}{\sqrt{t_k - t_j}} \quad . \tag{A.17}$$

The probability density of drawing the pair of standard random variables (x, y) given z, is

$$n(x, y|z) = \frac{n(x)n(y)}{n(z)} \quad , \tag{A.18}$$

since x, y are independent[2]. From equation (A.17), we can write $y = y(x, z)$. Thus (A.18) becomes

$$n(x, y(x, z)|z) = \frac{n(x)n(y(x, z))}{n(z)} \quad , \tag{A.19}$$

and therefore

$$n(x, y(x, z)|z) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x^2 + y^2 - z^2)}{2}\right] \quad . \tag{A.20}$$

Now substituting (A.17) in (A.20) and doing some algebra, we obtain

$$n(x|z) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x - \mu z)^2}{2\sigma^2}\right] \quad , \tag{A.21}$$

with

$$\mu = \sqrt{\frac{t_j - t_i}{t_k - t_i}} \quad , \quad \sigma = \sqrt{\frac{t_k - t_j}{t_k - t_i}} \quad , \tag{A.22}$$

i.e. x is normally distributed with mean μz and variance σ^2. Now, since

$$z = \frac{Z_k - Z_i}{\sqrt{t_k - t_i}} \quad , \tag{A.23}$$

we can write

$$x = \frac{\sqrt{t_j - t_i}}{t_k - t_i}(Z_k - Z_i) + \phi\sqrt{\frac{t_k - t_j}{t_k - t_i}} \quad , \tag{A.24}$$

where $\phi \sim N(0,1)$. Finally by substituting (A.24) into (A.14), we obtain

$$Z_j = \frac{t_k - t_j}{t_k - t_i} Z_i + \frac{t_j - t_i}{t_k - t_i} Z_k + \phi\sqrt{\frac{(t_j - t_i)(t_k - t_j)}{t_k - t_i}} \quad . \tag{A.25}$$

This last equation is known as the Brownian Bridge. This elegant derivation of the Brownian Bridge is based on P.A. Forsyth's notes [For05].

[2] A Brownian motion has independent successive increments.