Non-Reflecting Boundary Conditions for the Euler Equations

by

Michael Giles

CFDL-TR-88-1 February 1988

This research was supported by Air Force Office of Scientific Research contract F49620-78-C-0084, supervised by Dr. J. Wilson, and a research contract from Rolls-Royce, PLC, supervised by Dr. P. Stow.

Contents

Acknowledgements

Many people have helped in this research effort. In particular, I would like to thank Björn Engquist who suggested that I carefully examine the wellposedness of the non-reflecting boundary conditions, Lloyd N. Trefethen who helped me greatly in the well-posedness analysis of the degenerate case, and all of the members of the MIT CFDL for their constructive criticisms on the presentation of the material in this report.

1 Introduction

1.1 Statement of problem

When calculating a numerical solution to an unsteady, hyperbolic, partial differential equation on an infinite domain, it is normal to perform the calculation on a truncated finite domain. This raises the problem of choosing appropriate boundary conditions for this far-field boundary. Ideally these should prevent any non-physical reflection of outgoing waves, and should be straightforward to implement numerically. Also they must produce a well-posed analytic problem since this is a basic requirement for the corresponding numerical approximation to be consistent and stable.

The motivation for this report lies in the author's research in the calculation of turbomachinery flows. In some respects these flows are more complex than the flow past isolated airfoils. Whereas the far-field boundary for isolated airfoils can be taken to be many chords away from the airfoil, with turbomachinery the far-field boundary is typically less than one chord away from the blade. Consequently, whereas for isolated airfoils the steady-state far-field can be modelled as a vortex correction to the free-stream flow, in turbomachinery the far-field contains a significant component of several different spatial wavenumbers. This is particularly true for flows which are supersonic in the flow direction but subsonic in the axial direction, in which case shocks propagate indefinitely and can be reflected by improper boundary conditions. Thus one of the two aims of this report is the correct formulation of steady-state non-reflecting boundary conditions which will not produce artificial reflections of steady waves such as shock waves.

The other aim of the report is the formulation of accurate non-reflecting boundary conditions for unsteady waves. Here again isolated airfoils generally have fewer problems. The reason is that the primary concern for isolated airfoils is unsteady flow caused by either the airfoil's motion (airfoil flutter or aileron flutter) or a fluid-dynamic instability (transonic buzz or stall). In either case the unsteadiness originates in the vicinity of the airfoil and radiates outward. Typically the grids on which such calculations are performed becoming progressively coarser as the waves move out towards the far-field boundary, until a radius is reached at which the wavelength of the unsteady wave is of the order of a few mesh cells. At this point the numerical viscosity will dissipate the wave and so the unsteadiness will not reach the far-field boundary and accurate non-reflecting boundary conditions are unnecessary. One of the main concerns in turbomachinery is the unsteadiness caused by incoming shock waves and wakes from upstream blade rows. The need to retain an accurate representation of these incoming waves prevents the use of coarse grids in the far-field, and instead one must concentrate on accurate non-reflecting boundary conditions.

Unsteady flows can be split into two classes, nonlinear and linear, depending on the amplitude of the unsteadiness. If the amplitude is sufficiently small that the disturbances everywhere can be considered to be linear perturbations to a steady flow, then by the principle of superposition the solution can be decomposed into a sum of modes with different temporal frequencies and different inter-blade phase angles. Each of these modes can then be analyzed separately and so the problem is reduced to finding the complex amplitude of the harmonic disturbance. This can be achieved by either a direct method or a pseudo-time-marching method. In either case there is the same need for accurate boundary conditions, and it is found that because there is only a single frequency it is possible to construct the exact non-reflecting boundary conditions. In nonlinear unsteady flows there are regions where the amplitude of the unsteadiness is great enough for second order effects to become extremely important. This produces a coupling between the different frequencies, and so they cannot be separated. In the far-field however it is again assumed that the unsteady amplitudes are small so that linear theory can be applied. It is no longer possible to construct exact non-reflecting boundary conditions which can be implemented numerically, but approximate boundary conditions can be derived instead.

Fortunately in some respects the turbomachinery problem is simpler than the isolated airfoil problem. There are separate inflow and outflow boundaries, each with a trivial geometry. In this report x denotes the axial coordinate and y denotes the circumferential coordinate. The inflow boundary is at $x=0$ and the outflow is at $x=1$. Periodicity in y allows one to decompose the solution into its Fourier components, and the uniform orientation of the boundary relative to the flow field allows one to perform the analysis by considering linear perturbations to a uniform flow. With isolated airfoils it is harder to perform an eigenmode decomposition of the far-field because of the varying orientation of the boundary relative to the flow field; at some places it is an inflow boundary while at the others it is an outflow boundary and this varies as the solution develops.

1.2 History

The subject of non-reflecting boundary conditions for hyperbolic equations is less than twenty years old. The key paper which formed the firm, theoretical foundation for later research was a paper by Kreiss [1] in 1970 which examined the well-posedness of initial-boundary-value problems for hyperbolic systems in multiple dimensions. Well-posedness is the requirement that a solution exists, is unique, and is bounded in the sense that the magnitude of the solution (defined using some appropriate norm) divided by the magnitude of the initial and boundary data (defined using some other norm) is less than some function which depends on time but not on the initial and boundary data. Any hyperbolic system arising from a model of a physical problem ought to be well-posed and so it is critical that any far-field boundary conditions which are used to truncate the solution domain must give a well-posed problem. Higdon has written an excellent review [2] of the work of Kreiss and others and in particular gives a physical interpretation of the theory in terms of wave propagation which is used in this report.

In solving the Euler equations for unsteady flows it was quickly realized that for one-dimensional flow the correct non-reflecting boundary conditions could be established using hyperbolic characteristic theory. For example, in 1977 Hedstrom [3] derived both the linear and the nonlinear form of these boundary equations for the Euler equations, using an eigenvector approach which will be used in this report. The boundary conditions were also used in two and three-dimensional flow calculations by doing a local analysis normal to the far-field boundary and ignoring all tangential derivatives. This approach (referred to as the quasi-one-dimensional or normal one-dimensional boundary conditions or the method of characteristics) remains the most commonly used boundary condition in unsteady calculations.

Also in 1977, Engquist and Majda wrote an important paper [4] deriving a hierarchy of approximate non-reflecting boundary conditions for multi-dimensional problems, with the first order approximation being the one-dimensional approximation. Unfortunately, like the Kreiss paper, this paper was written for mathematicians specializing in the analysis of partial differential equations, assuming a familiarity with concepts, definitions and background literature which is not possessed by more applied mathematicians and engineers who are working in the area of computational fluid dynamics. Consequentially, over the last decade the important contributions of this paper have not been fully appreciated and implemented by the CFD community, and the higher order boundary conditions have only been implemented for acoustic and elastic wave propagation [4] and the scalar, unsteady potential equation [5]. A related approach to constructing approximate non-reflecting boundary conditions has been derived for the Euler equations by Gustafsson [6] in 1987.

When calculating linearized, harmonic unsteady flows, exact non-reflecting boundary conditions can be constructed. For the potential equation this was first done by Verdon *et. al.* [7] in 1975, and it is now the standard technique used by unsteady harmonic potential methods. Because the harmonic equations are solved directly (usually by a finite element method) the issue of well-posedness does not arise. In 1987 Hall [8] extended the technique for the Euler equations, again using a finite element method to solve the harmonic linearized, Euler equations.

Early methods for calculating the steady-state solution to the two-dimensional Euler equations for isolated airfoils used the free-stream conditions at the far-field boundary, and it was found that the boundary had to be placed 50-100 chord lengths away from airfoil to obtain the correct lift. Later methods found that the inclusion of a vortex correction due to the lift on the airfoil enabled the far-field boundary to be brought in to about 25 chords away. In 1986 Giles and Drela [9] showed that by including both the source effect due to the drag and the next order doublet terms the far-field radius could be further reduced to about 5 chords (suitably scaled by the Prandtl-Glauert factor for transonic flows). The same result was also achieved by Ferm [10] using an approach based upon the zero-frequency limit of the ideal non-reflecting boundary conditions. It is this viewpoint which links together the construction of boundary conditions for steady-state, single-frequency and general, unsteady flows, and it forms this basis for this report's unified approach to non-reflecting boundary conditions for the Euler equations.

1.3 Present paper

This report is intended for CFD researchers, who wish to implement accurate non-reflecting boundary conditions, and wish to understand the underlying theory. Most of the boundary conditions which will be derived have been implemented by the author in a program for the calculation of steady and unsteady flows in turbomachinery. The precise numerical details are presented in a separate report [11] but this report contains the full details of the analytic boundary conditions from which others can derive implementations which are appropriate to their numerical algorithms.

The first half presents a unified treatment of the construction of non-reflecting boundary conditions for hyperbolic systems. Four different types are derived; quasi-one-dimensional b.c.'s, exact single-frequency unsteady non-reflecting b.c.'s, exact steady-state non-reflecting b.c.'s, and approximate unsteady non-reflecting b.c.'s. It should be remembered that the term 'exact' refers to the solution of the linear problem. Since the linear problem is itself an approximation to the nonlinear problem there will be errors which are proportional to the square of the amplitude of the unsteadiness at the far-field boundaries. The theory section is essentially a condensed restatement of the previous work described in the last section. The one significant original contribution lies in the analysis of well-posedness. Kreiss [1] and subsequent investigators assumed for simplicity that there is no degeneracy in the eigenvalues of the hyperbolic system.

Unfortunately the Euler equations have a multiple root with distinct eigenvectors, and at a particular complex frequency there is an eigenvalue/eigenvector coalescence. The extensions to the well-posedness analysis which are necessary to treat these problems are presented, but lack the mathematical rigor of Kreiss' work.

The second half presents the application of the theory to the Euler equations. First the dispersion relation and the eigenvalues and eigenvectors are found. Then the quasi-one-dimensional, single-frequency, steady-state and second-order non-reflecting boundary conditions are formed. The well-posedness of the quasi-one-dimensional b.c.'s is assured due to a general energy analysis performed in the theory section. The well-posedness of the steady-state and single-frequency b.c.'s depends on the numerical method being used. If a direct solution method is used then the question does not arise. If a time-marching method is used then the analysis of the associated initial-boundary-value problem becomes too complicated to perform. Much of the second half is concerned with the well-posedness of the second-order, approximate b.c.'s derived from the general Engquist-Majda theory. An unexpected result is that the outflow b.c. is well-posed but the inlet b.c.'s are ill-posed. Modified inflow boundary conditions which are well-posed are then derived by an *ad hoc* method. A calculation of the reflection coefficients shows that the modified inflow boundary conditions are fourth order and the outflow boundary condition is second order. A corresponding modified outflow boundary condition is shown to give first order reflections for outgoing vorticity waves but fourth order reflections for outgoing pressure waves, and so might be a useful boundary condition in applications or regions where it is known that there are no outgoing vorticity waves. Finally a reference section lists all of the boundary conditions in a dimensional form suitable for implementation.

2 General analysis

2.1 Fourier analysis

Consider the following general unsteady, two-dimensional, hyperbolic partial differential equation.

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A}\frac{\partial \boldsymbol{U}}{\partial x} + \boldsymbol{B}\frac{\partial \boldsymbol{U}}{\partial y} = 0 \tag{1}$$

$\boldsymbol{U}$ is a N-component vector and $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant $N \times N$ matrices. Fourier analysis considers wave-like solutions of the form

$$\boldsymbol{U}(x, y, t) = \boldsymbol{u} e^{i(kx+ly-\omega t)}. \tag{2}$$

Substituting this into the differential equation gives

$$(-\omega \boldsymbol{I} + k\boldsymbol{A} + l\boldsymbol{B})\boldsymbol{u} = 0. \tag{3}$$

$$\Longrightarrow \quad \det(-\omega \boldsymbol{I} + k\boldsymbol{A} + l\boldsymbol{B}) = 0 \tag{4}$$

This equation is called the dispersion relation, and is a polynomial equation of degree N in each of ω, k, and l.

A critical step in the construction and analysis of boundary conditions is to separate the waves into incoming and outgoing modes. If ω is complex with $Im(\omega) > 0$ (giving an exponential growth in time) then the right-propagating waves are those for which $Im(k) > 0$. This is because the amplitude of the waves is proportional to $e^{Im(\omega)(t-x/c)}$ where $c = Im(\omega)/Im(k)$ is the apparent velocity of propagation of the amplitude.

If ω and k are real then a standard result in the analysis of dispersive wave propagation [12] is that the velocity of energy propagation is the group velocity defined by

$$\vec{c}_g = \begin{pmatrix} \frac{\partial \omega}{\partial k} \\ \\ \frac{\partial \omega}{\partial l} \end{pmatrix} \tag{5}$$

Hence for real ω the incoming waves are those which either have $Im(k) > 0$, or have real k and $\frac{\partial \omega}{\partial k} > 0$.

2.2 Eigenvectors

The right and left eigenvectors of an $N \times N$ matrix $\boldsymbol{C}$ are defined by

$$\boldsymbol{C}\boldsymbol{u}_n^R = \lambda_n \boldsymbol{u}_n^R, \qquad \boldsymbol{u}_n^L \boldsymbol{C} = \lambda_n \boldsymbol{u}_n^L \tag{6}$$

with λ_n being the corresponding eigenvalue. Note that the right eigenvector is a column vector, whereas the left eigenvector is a row vector. An equivalent definition is that the eigenvectors are the null-vectors of the singular matrix $\boldsymbol{C} - \lambda_n \boldsymbol{I}$.

$$(\boldsymbol{C} - \lambda_n \boldsymbol{I})\boldsymbol{u}_n^R = 0, \qquad \boldsymbol{u}_n^L(\boldsymbol{C} - \lambda_n \boldsymbol{I}) = 0 \tag{7}$$

An important property of the eigenvectors is that when the eigenvalues are distinct each left eigenvector is orthogonal to all of the right eigenvectors except the one corresponding to the same eigenvalue. The proof is as follows:

$$\begin{aligned} (\lambda_n - \lambda_m)\boldsymbol{v}_n^L \boldsymbol{u}_m^R &= (\boldsymbol{v}_n^L \boldsymbol{C})\boldsymbol{u}_m^R - \boldsymbol{v}_n^L(\boldsymbol{C}\boldsymbol{u}_m^R) \\ &= 0 \end{aligned} \tag{8}$$

If $m \neq n$ then $\lambda_m \neq \lambda_n$ and hence $\boldsymbol{v}_n^L \boldsymbol{u}_m^R = 0$. If there are identical eigenvalues then provided there is a complete set of N eigenvectors, $\boldsymbol{v}_n^L$ and $\boldsymbol{u}_m^R$ can still be constructed, using the Gram-Schmidt orthogonalization method [13], so that $\boldsymbol{v}_n^L \boldsymbol{u}_m^R = 0$ if $m \neq n$.

Turning now to the Fourier analysis begun in the last section, the right eigenvectors we require are the null-vectors of $(-\omega \boldsymbol{I} + k\boldsymbol{A} + l\boldsymbol{B})$.

$$(-\omega \boldsymbol{I} + k\boldsymbol{A} + l\boldsymbol{B})\boldsymbol{u}^R = 0 \tag{9}$$

$\boldsymbol{u}^R$ is a right eigenvector of $(k\boldsymbol{A} + l\boldsymbol{B})$ with eigenvalue ω. By premultiplying Eq. (9) by $\boldsymbol{A}^{-1}$ to obtain

$$(-\omega \boldsymbol{A}^{-1} + k\boldsymbol{I} + l\boldsymbol{A}^{-1}\boldsymbol{B})\boldsymbol{u}^R = 0, \tag{10}$$

it becomes apparent that $\boldsymbol{u}^R$ is also a right eigenvector of $(-\omega \boldsymbol{A}^{-1} + l\boldsymbol{A}^{-1}\boldsymbol{B})$ with eigenvalue $-k$. The significance of this latter property will become clear shortly.

There are two different sets of left eigenvectors which are important in this problem. The first set, labelled $\boldsymbol{u}^L$, are the left eigenvectors of $(k\boldsymbol{A} + l\boldsymbol{B})$ which are also the left null-vectors of $(-\omega \boldsymbol{I} + k\boldsymbol{A} + l\boldsymbol{B})$.

$$\boldsymbol{u}^L(-\omega \boldsymbol{I} + k\boldsymbol{A} + l\boldsymbol{B}) = 0 \tag{11}$$

The second set, labelled $\boldsymbol{v}^L$, are the left eigenvectors of $(-\omega \boldsymbol{A}^{-1} + l\boldsymbol{A}^{-1}\boldsymbol{B})$ which are also the left null-vectors of $(-\omega \boldsymbol{A}^{-1} + k\boldsymbol{I} + l\boldsymbol{A}^{-1}\boldsymbol{B}) = \boldsymbol{A}^{-1}(-\omega \boldsymbol{I} + k\boldsymbol{A} + l\boldsymbol{B})$.

$$\boldsymbol{v}^L \boldsymbol{A}^{-1}(-\omega \boldsymbol{I} + k\boldsymbol{A} + l\boldsymbol{B}) = 0 \tag{12}$$

By comparing Eqs. (11) and (12) it is clear that the two sets of left eigenvectors are related, with $\boldsymbol{v}^L \boldsymbol{A}^{-1}$ equal to (or a multiple of) $\boldsymbol{u}^L$, or alternatively $\boldsymbol{v}^L$ is equal to (or a multiple of) $\boldsymbol{u}^L \boldsymbol{A}$.

The difference between the two sets of left eigenvectors lies in their orthogonality relations with the right eigenvectors. Since u^L is a left eigenvector of $(k\boldsymbol{A}+l\boldsymbol{B})$, it is orthogonal to all of the right eigenvectors of the same matrix, except for the one with the same eigenvalue ω. The key point here is that the orthogonality is for the *same* k and l and *different* ω. Thus if ω_n and ω_m are two different roots of the dispersion equation for the same values of k and l, then the orthogonality relation is $\boldsymbol{v}^L(\omega_n, k, l)\boldsymbol{u}^R(\omega_m, k, l) = 0$. In contrast, since v^L is a left eigenvector of $(-\omega\boldsymbol{A}^{-1}+l\boldsymbol{A}^{-1}\boldsymbol{B})$, it is orthogonal to all of the right eigenvectors $\boldsymbol{u}^R$ with the *same* ω and l and *different* k. Thus if k_n and k_m are two different roots of the dispersion relation for the same values of ω and l, then the orthogonality relation is, $\boldsymbol{v}^L(\omega, k_n, l)\boldsymbol{u}^R(\omega, k_m, l) = 0$.

Normally in discussing wave motion one is concerned with propagation on an infinite domain, and so usually one considers a group of waves with the same k and l and different values of ω, in which case $\boldsymbol{u}^R$ and $\boldsymbol{u}^L$ would be the relevant right and left eigenvectors. In analyzing boundary conditions however, a general solution $\boldsymbol{U}$ at the boundary $x=0$ can be decomposed into a sum of Fourier modes with different values of ω and l. Each of these modes is then a collection of waves with the same values of ω and l and different values of k. Hence for the purpose of constructing non-reflecting boundary conditions it is $\boldsymbol{u}^R$ and $\boldsymbol{v}^L$ which are important.

$\boldsymbol{u}^L$ is still helpful in the current application as a convenient stepping stone in the construction of $\boldsymbol{v}^L$. The natural way to construct $\boldsymbol{v}^L$ is to calculate $(-\omega\boldsymbol{A}^{-1}+l\boldsymbol{A}^{-1}\boldsymbol{B})$ and find its left null-vector. This requires calculating $\boldsymbol{A}^{-1}$ however, which could be a laborious process. An easier way is to use the result obtained earlier that $\boldsymbol{v}^L$ is a multiple of $\boldsymbol{u}^L\boldsymbol{A}$. Let k_n be the n^{th} root of the dispersion relation for a given value of ω and l, and $\boldsymbol{u}^L_n$ be the corresponding left null-vector of $(k_n\boldsymbol{A}+l\boldsymbol{B})$. Then $\boldsymbol{v}^L_n$ can be defined by

$$\boldsymbol{v}^L_n = \left.\frac{k_n}{\omega}\right|_{l=0} \boldsymbol{u}^L_n \boldsymbol{A}. \tag{13}$$

The reason for choosing the constant of proportionality to be $\left.\frac{k_n}{\omega}\right|_{l=0}$ is that when $l=0$, $\boldsymbol{u}^L_n\boldsymbol{A} = \frac{\omega}{k_n}\boldsymbol{u}^L_n$ and hence $\boldsymbol{v}^L_n = \boldsymbol{u}^L_n$.

2.3 Non-reflecting b.c.'s for a single Fourier mode

Suppose that the differential equation is to be solved in the domain $x>0$, and one wants to construct non-reflecting boundary conditions at $x=0$ to minimize or ideally prevent the reflection of outgoing waves. At the boundary at $x = 0$, $\boldsymbol{U}$ can be decomposed into a sum of Fourier modes with different values of ω and l, so the analysis begins by considering just one particular choice of ω and l. In this case this most general form for $\boldsymbol{U}$ is

$$\boldsymbol{U}(x,y,t) = \left[\sum_{n=1}^{N} a_n \boldsymbol{u}_n^R e^{ik_n x}\right] e^{i(ly-\omega t)}. \tag{14}$$

k_n is the n^{th} root of the dispersion relation for the given values of ω and l, and $\boldsymbol{u}_n^R$ is the corresponding right eigenvector.

The ideal non-reflecting boundary conditions would be to specify that $a_n = 0$ for each n that corresponds to an incoming wave. Because of orthogonality,

$$\begin{aligned} \boldsymbol{v}_n^L \boldsymbol{U} &= \boldsymbol{v}_n^L \left[\sum_{m=1}^{N} a_m \boldsymbol{u}_m^R e^{ik_m x}\right] e^{i(ly-\omega t)} \\ &= a_n (\boldsymbol{v}_n^L \boldsymbol{u}_n^R) e^{ik_n x} e^{i(ly-\omega t)} \end{aligned} \tag{15}$$

and so an equivalent specification of non-reflecting boundary conditions is

$$\boldsymbol{v}_n^L \boldsymbol{U} = 0 \tag{16}$$

for each n corresponding to an incoming mode. $\boldsymbol{v}_n^L$ is the left eigenvector defined in the last section. This use of the left eigenvector illustrates its physical significance; because of the orthogonality relations, when applied to a general solution it "measures" the amplitude of a particular wave component. The significance of the right eigenvector is apparent in Eq. (14); it shows the variation in the primitive variables caused by a particular wave mode.

In principle these exact boundary conditions can be implemented in a numerical method. The problem is that $\boldsymbol{v}_n^L$ depends on ω and l and so the implementation would involve a Fourier transform in y and a Laplace transform in t. Computationally this is both difficult and expensive to implement and so instead we will consider four simpler variations which use different assumptions and approximations.

An observation is that by dividing the dispersion relation, Eq. (4), by ω we obtain

$$\det(-\boldsymbol{I} + \frac{k_n}{\omega}\boldsymbol{A} + \frac{l}{\omega}\boldsymbol{B}) = 0 \tag{17}$$

and so it is clear that $k_n/\omega, \boldsymbol{u}_n^R, \boldsymbol{u}_n^L$ and $\boldsymbol{v}_n^L$ are all functions of l/ω. Thus the variable $\lambda = l/\omega$ will play a key role in constructing all of the boundary conditions.

2.4 One-dimensional, unsteady b.c.'s

The one-dimensional, non-reflecting boundary conditions are obtained by ignoring all variations in the y-direction and setting $\lambda=0$. The corresponding right and left eigenvectors are important in defining and implementing the other boundary conditions, and so we label them $\boldsymbol{w}$.

$$\boldsymbol{w}_n^R = \boldsymbol{u}_n^R\Big|_{\lambda=0} \tag{18}$$

$$\boldsymbol{w}_n^L = \boldsymbol{u}_n^L\Big|_{\lambda=0} = \boldsymbol{v}_n^L\Big|_{\lambda=0} \tag{19}$$

The boundary condition, expressed in terms of the primitive variables, is

$$\boldsymbol{w}_n^L \boldsymbol{U} = 0 \tag{20}$$

for all n corresponding to incoming waves.

If the right and left eigenvectors are normalized so that

$$\boldsymbol{w}_m^L \boldsymbol{w}_n^R = \delta_{mn} \equiv \begin{cases} 1 & , \; m=n \\ 0 & , \; m\neq n \end{cases} \tag{21}$$

then they can be used to define a transformation between the primitive variables and the one-dimensional characteristic variables.

$$\boldsymbol{U} = \sum_{n=1}^{N} c_n \boldsymbol{w}_n^R, \tag{22}$$

where

$$c_n = \boldsymbol{w}_n^L \boldsymbol{U}. \tag{23}$$

Expressed in terms of the characteristic variables, the boundary condition is simply

$$c_n = 0 \tag{24}$$

for all n corresponding to incoming waves.

Numerical implementations of these boundary conditions usually extrapolate the outgoing characteristic variables, in addition to setting the incoming characteristic variables to zero. This gives a complete set of equations for the solution on the boundary using Eq. (22).

An observation, which will be needed later in the section on well-posedness, is that

$$\left(-\omega \boldsymbol{I} + k_n \boldsymbol{A}\right) \boldsymbol{w}_n^R = 0, \tag{25}$$

so

$$\boldsymbol{A}\boldsymbol{w}_n^R = \frac{\omega}{k_n}\boldsymbol{w}_n^R = \alpha_n \boldsymbol{w}_n^R. \tag{26}$$

Thus $\boldsymbol{w}_n^R$ is an eigenvector of $\boldsymbol{A}$ with eigenvalue $\alpha_n = \frac{\omega}{k_n}$. Furthermore,

$$\omega = \alpha_n k_n \quad \Rightarrow \quad \vec{c}_g = \begin{pmatrix} \alpha_n \\ 0 \end{pmatrix}, \tag{27}$$

so the incoming waves are those for which $\alpha_n > 0$.

2.5 Exact, two-dimensional, single-frequency b.c.'s

In the introduction it was pointed out that if the interior differential equation is linear then by the principle of linear superposition it is possible to split a general solution into a sum of different frequencies, and calculate them independently, each with its own forcing terms and boundary conditions. In this case it is possible to construct the exact non-reflecting boundary conditions for simple geometries in which the far-field boundary is at $x = 0$ and the solution is periodic in y, with period 2π. This is achieved by performing a Fourier transform in y along the boundary to decompose the solution into a sum of Fourier modes, and then using the exact non-reflecting boundary conditions for each Fourier mode.

The Fourier decomposition of $\boldsymbol{U}$ at the boundary can be written as

$$\boldsymbol{U}(0, y, t) = \sum_{-\infty}^{\infty} \hat{\boldsymbol{U}}_l(t) e^{ily}, \tag{28}$$

where

$$\hat{\boldsymbol{U}}_l(t) = \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{U}(0, y, t) e^{-ily}\, dy. \tag{29}$$

The boundary condition is then

$$\boldsymbol{v}_n^L(l/\omega) \hat{\boldsymbol{U}}_l = 0 \tag{30}$$

for each Fourier mode l and incoming wave n. Since ω and l are both known for each Fourier mode, this is simply an algebraic equation in the Fourier domain. Because the numerical implementation of these conditions is very dependent on the numerical algorithm being used for the interior equations, further discussion will be postponed to a later section which discusses the application to the Euler equations and turbomachinery problems in particular. One general comment however is that one of the principal difficulties is determining which waves are incoming and which are outgoing.

2.6 Exact, two-dimensional, steady b.c.'s

The exact, two-dimensional steady boundary conditions may be considered to be the limit of the single-frequency boundary conditions as $\omega \rightarrow 0$. Performing the same Fourier decomposition as in the last section, the boundary conditions for $l \neq 0$ are

$$\boldsymbol{s}_n^L \hat{\boldsymbol{U}}_l = 0 \tag{31}$$

for each incoming wave n, where

$$\boldsymbol{s}_n^L = \lim_{\lambda \rightarrow \infty} \boldsymbol{v}_n^L(\lambda). \tag{32}$$

The boundary condition for the $l = 0$ mode, which is the solution average at the boundary, is

$$\begin{aligned} & \boldsymbol{v}_n^L(0)\hat{\boldsymbol{U}}_0 = 0 \\ \Rightarrow \quad & \boldsymbol{w}_n^L \hat{\boldsymbol{U}}_0 = 0 \end{aligned} \tag{33}$$

for each incoming wave n. The right-hand-side of Eq. (33) can be modified by the user to specify the value of the incoming average characteristics. For example, in fluid flow calculations this is how the average inlet flow angle, stagnation enthalpy and entropy can be specified. The right-hand-side of Eq. (31) could also be modified, but in most applications the solution at $x = -\infty$ is assumed to be uniform and so the zero right-hand-side is correct.

Again, discussion of the numerical implementation of these boundary conditions will be postponed to a later section dealing with the application to the Euler equations and turbomachinery flows. Also, one of the principal difficulties is again determining which waves are incoming and which are outgoing.

2.7 Approximate, two-dimensional, unsteady b.c.'s

By dividing the dispersion relation by ω it is clear that $k_n/\omega, \boldsymbol{u}_n^R, \boldsymbol{u}_n^L, \boldsymbol{v}_n^L$ are all functions of l/ω. Thus a sequence of approximations can be obtained by expanding $\boldsymbol{v}_n^L$ in a Taylor series as a function of $\lambda = l/\omega$.

$$\boldsymbol{v}_n^L(\lambda) = \boldsymbol{v}_n^L\Big|_{\lambda=0} + \lambda \left.\frac{d\boldsymbol{v}_n^L}{d\lambda}\right|_{\lambda=0} + \tfrac{1}{2}\lambda^2 \left.\frac{d^2\boldsymbol{v}_n^L}{d\lambda^2}\right|_{\lambda=0} + \cdots \tag{34}$$

The first order approximation obtained by just keeping the leading term just gives the one-dimensional boundary conditions which have already been discussed. The second order approximation is

$$\begin{aligned} \overline{\boldsymbol{v}}_n^L(\lambda) &= \boldsymbol{v}_n^L\Big|_{\lambda=0} + \frac{l}{\omega}\left.\frac{d\boldsymbol{v}_n^L}{d\lambda}\right|_{\lambda=0} \\ &= \boldsymbol{u}_n^L\Big|_{\lambda=0} + \frac{l}{\omega}\left.\left[\frac{k_n}{\omega}\frac{d\boldsymbol{u}_n^L}{d\lambda}\boldsymbol{A}\right]\right|_{\lambda=0} \end{aligned} \tag{35}$$

The overbar denotes the fact that $\overline{\boldsymbol{v}}$ is an approximation to $\boldsymbol{v}$. This produces the boundary condition

$$\left(\boldsymbol{u}_n^L\Big|_{\lambda=0} + \frac{l}{\omega}\left.\left[\frac{k_n}{\omega}\frac{d\boldsymbol{u}_n^L}{d\lambda}\boldsymbol{A}\right]\right|_{\lambda=0} \right)\boldsymbol{U} = 0. \tag{36}$$

Multiplying by ω, and replacing ω and l by $i\dfrac{\partial}{\partial t}$ and $-i\dfrac{\partial}{\partial y}$ respectively gives,

$$\boldsymbol{u}_n^L\Big|_{\lambda=0}\frac{\partial \boldsymbol{U}}{\partial t} - \left.\left[\frac{k_n}{\omega}\frac{d\boldsymbol{u}_n^L}{d\lambda}\boldsymbol{A}\right]\right|_{\lambda=0}\frac{\partial \boldsymbol{U}}{\partial y} = 0 \tag{37}$$

This is a local boundary condition (meaning that it does not involve any global decomposition into Fourier modes) and so can be implemented without difficulty. As the equation is similar in nature to the original differential equation, having first order derivatives in both y and t, the numerical algorithm used for the interior equations can probably also be used for the boundary conditions.

2.8 Analysis of well-posedness

2.8.1 One-dimensional b.c.'s

It is relatively easy to prove that the one-dimensional b.c.'s are always well-posed, by using an energy analysis method. A key step in the proof is that because the system is hyperbolic there exists a transformation of variables under which the transformed $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices are symmetric, and so without loss of generality we can assume that $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric. The 'energy' is defined by

$$E(t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} |\boldsymbol{u}|^2 \, dx \, dy. \tag{38}$$

To ensure that this integral remains finite it will be assumed that $\boldsymbol{u}$ is zero outside some distance from the origin. The rate of change of the energy is given by

$$\begin{aligned}
\frac{dE}{dt} &= 2\int_{-\infty}^{\infty}\int_{0}^{\infty} \boldsymbol{u}^T \frac{\partial \boldsymbol{u}}{\partial t} \, dx \, dy \\
&= -2\int_{-\infty}^{\infty}\int_{0}^{\infty} \boldsymbol{u}^T \left(\boldsymbol{A}\frac{\partial \boldsymbol{u}}{\partial x} + \boldsymbol{B}\frac{\partial \boldsymbol{u}}{\partial y} \right) dx \, dy \\
&= -\int_{-\infty}^{\infty}\int_{0}^{\infty} \boldsymbol{u}^T \boldsymbol{A}\frac{\partial \boldsymbol{u}}{\partial x} + \frac{\partial \boldsymbol{u}^T}{\partial x}\boldsymbol{A}^T\boldsymbol{u} + \boldsymbol{u}^T\boldsymbol{B}\frac{\partial \boldsymbol{u}}{\partial y} + \frac{\partial \boldsymbol{u}^T}{\partial y}\boldsymbol{B}^T\boldsymbol{u} \, dx \, dy \\
&= -\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\partial}{\partial x}(\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u}) + \frac{\partial}{\partial y}(\boldsymbol{u}^T\boldsymbol{B}\boldsymbol{u}) \, dx \, dy \qquad (\text{since } \boldsymbol{A}=\boldsymbol{A}^T \text{ and } \boldsymbol{B}=\boldsymbol{B}^T) \\
&= -\int_{-\infty}^{\infty} \left(\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u}\Big|_{x=\infty} - \boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u}\Big|_{x=0} \right) dy \; - \int_{0}^{\infty} \left(\boldsymbol{u}^T\boldsymbol{B}\boldsymbol{u}\Big|_{y=\infty} - \boldsymbol{u}^T\boldsymbol{B}\boldsymbol{u}\Big|_{y=-\infty} \right) dx \\
&= \int_{-\infty}^{\infty} \boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u}\Big|_{x=0} \, dy
\end{aligned} \tag{39}$$

The evaluation of this integral requires some earlier results.

$$\begin{aligned}
\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u} &= \left(\sum_{n=1}^{N} c_n \boldsymbol{w}_n^R \right)^T \boldsymbol{A} \left(\sum_{n=1}^{N} c_n \boldsymbol{w}_n^R \right) \qquad (\text{using Eq. (22)}) \\
&= \sum_{n=1}^{N} \alpha_n c_n^2 \qquad (\text{using Eq. (26)})
\end{aligned} \tag{40}$$

where, as defined earlier, α_n is the n^{th} eigenvalue of $\boldsymbol{A}$, $\boldsymbol{w}_n^R$ is the corresponding right eigenvector which is also the transpose of the left eigenvector $\boldsymbol{w}_n^L$ since $\boldsymbol{A}$ is symmetric, and $c_n = \boldsymbol{w}_n^L \boldsymbol{u}$.

The final step is to note that the one-dimensional b.c. states that $c_n = 0$ for incoming waves, which are those for which $\alpha_n \geq 0$. Thus each term in the above sum is either zero or negative, and hence $\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u}$ is non-positive and the energy is non-increasing, proving that the initial-boundary-value problem is well-posed.

2.8.2 Approximate, two-dimensional b.c.'s

To analyze the well-posedness of the approximate, two-dimensional boundary conditions, one must use the theory developed by Kreiss [1]. As explained by Trefethen [14] and Higdon [2], the aim is to verify that there is no incoming mode which exactly satisfies the boundary condition.

As explained earlier, an incoming mode is a solution of the interior differential equation which *either* is growing exponentially in time but decaying exponentially in space away from the boundary, *or* has a real frequency and a group velocity which is incoming. If this incoming mode also satisfies the boundary condition then in the first case there is an exponentially growing energy and in the second case there is a linear growth as the incoming mode moves into the interior.

If there are N' incoming waves then the generalized incoming mode may be written as

$$\boldsymbol{U}(x,y,t) = \left[\sum_{n=1}^{N'} a_n \boldsymbol{u}_n^R e^{ik_n x} \right] e^{i(ly-\omega t)}, \tag{41}$$

with $Im(\omega) \geq 0$. Substituting this into the N' non-reflecting boundary conditions produces a matrix equation of the form

$$\boldsymbol{C} \begin{pmatrix} a_1 \\ \vdots \\ a_{N'} \end{pmatrix} = 0 \tag{42}$$

where $\boldsymbol{C}$ is a $N' \times N'$ matrix whose elements are the products of the approximate left eigenvectors and the exact right eigenvectors.

$$\boldsymbol{C}_{mn} = \overline{\boldsymbol{v}}_m^L \boldsymbol{u}_n^R \tag{43}$$

Provided that the right eigenvectors are linearly independent, the requirement that there is no non-trivial incoming mode satisfying the boundary conditions is equivalent to the statement that there in no non-trivial solution to the above matrix equation. Thus the initial-boundary-value problem is well-posed if it can be proved that the determinant of $\boldsymbol{C}$ is non-zero for all real l and complex ω with $Im(\omega) \geq 0$.

If the right eigenvectors are linearly dependent then the theory needs to be modified. Suppose for simplicity that there are just two incoming waves and that $k_1 = k_2$ and $\boldsymbol{u}_1^R = \boldsymbol{u}_2^R$ at $\omega = \omega_{crit}$. A general incoming mode may be written as the sum of two incoming modes with amplitudes a_1', a_2'.

$$\boldsymbol{U}(x,y,t) = \left[a_1' \boldsymbol{u}_1^R e^{ik_1 x} + a_2' \frac{1}{\omega - \omega_{crit}} \left(\boldsymbol{u}_1^R e^{ik_1 x} - \boldsymbol{u}_2^R e^{ik_2 x} \right) \right] e^{i(ly-\omega t)} \tag{44}$$

By construction the second mode is finite in the limit $\omega \to \omega_{crit}$, and so if the limit is defined to be the value at $\omega = \omega_{crit}$ then this expression is the correct general solution to the eigenvalue problem

$$\omega \boldsymbol{U} = \boldsymbol{A}\frac{\partial \boldsymbol{U}}{\partial x} + l\boldsymbol{B}\boldsymbol{U} \tag{45}$$

subject to the condition $\boldsymbol{U} \to 0$ as $x \to \infty$.

The amplitudes $a_{1,2}$ and $a'_{1,2}$ are related by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \boldsymbol{T} \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix}, \tag{46}$$

where

$$\boldsymbol{T} = \begin{pmatrix} 1 & 1/(\omega - \omega_{crit}) \\ 0 & -1/(\omega - \omega_{crit}) \end{pmatrix}. \tag{47}$$

Substituting this into the boundary conditions gives

$$\boldsymbol{C}\boldsymbol{T} \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = 0. \tag{48}$$

By continuity, the requirement for well-posedness is that $\det(\boldsymbol{CT}) = \det(\boldsymbol{C})\det(\boldsymbol{T}) \not\to 0$ as $\omega \to \omega_{crit}$. Since $\det(\boldsymbol{T}) = O(\omega - \omega_{crit})^{-1}$, this requires that $\det(\boldsymbol{C}) = O(\omega - \omega_{crit})$, or equivalently that

$$\left.\frac{\partial}{\partial \omega}\det(\boldsymbol{C})\right|_{\omega_{crit}} \neq 0. \tag{49}$$

More generally, we conjecture that if the N' right eigenvectors collapse to N'' linearly independent eigenvectors at some ω_{crit} then the requirement for well-posedness is that $\det(\boldsymbol{C}) = O(\omega - \omega_{crit})^{N'-N''}$, or equivalently that

$$\left.\frac{\partial^{N'-N''}}{\partial \omega^{N'-N''}}\det(\boldsymbol{C})\right|_{\omega_{crit}} \neq 0. \tag{50}$$

Engquist and Majda conjectured that the second order approximation is always well-posed, but we will see in the next section that this is not true for the Euler equations. Trefethen and Halpern have proved that the boundary conditions for the scalar wave equation which come from the second and higher order Taylor series expansions in l^2/ω^2 are ill-posed [14]. Thus it seems likely that for the differential system of equations which we are considering the higher order Taylor series approximations may be ill-posed.

2.9 Reflection coefficients

The calculation of reflection coefficients is very similar to the well-posedness analysis. A general solution with a given frequency ω and wavenumber l can be written as a sum of incoming and outgoing modes.

$$\boldsymbol{U}(x,y,t) = \left[\sum_{n=1}^{N'} a_n \boldsymbol{u}_n^R e^{ik_n x} + \sum_{n=N'+1}^{N} a_n \boldsymbol{u}_n^R e^{ik_n x} \right] e^{i(ly-\omega t)} \tag{51}$$

Substituting this into the approximate non-reflecting boundary conditions gives

$$\boldsymbol{C} \begin{pmatrix} a_1 \\ \vdots \\ a_{N'} \end{pmatrix} + \boldsymbol{D} \begin{pmatrix} a'_N+1 \\ \vdots \\ a_N \end{pmatrix} = 0 \tag{52}$$

where $\boldsymbol{C}$ is the same matrix as in the well-posedness analysis and $\boldsymbol{D}$ is defined by

$$\boldsymbol{D}_{mn} = \overline{\boldsymbol{v}}_m^L \boldsymbol{u}_{N'+n}^R. \tag{53}$$

If the initial-boundary-value problem is well-posed $\boldsymbol{C}$ is non-singular and so Eq. (52) can be solved to obtain

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{N'} \end{pmatrix} = -\boldsymbol{C}^{-1}\boldsymbol{D} \begin{pmatrix} a_{N'+1} \\ \vdots \\ a_N \end{pmatrix}. \tag{54}$$

This equation relates the amplitudes of the incoming waves to the amplitude of the outgoing waves, so $-\boldsymbol{C}^{-1}\boldsymbol{D}$ is the matrix of reflection coefficients.

Since $\boldsymbol{v}_m^L \boldsymbol{u}_n^R = 0$ if $m \neq n$, the off-diagonal elements of $\boldsymbol{C}$ and the elements of $\boldsymbol{D}$ can be re-expressed as

$$\boldsymbol{C}_{mn} = (\overline{\boldsymbol{v}}_m^L - \boldsymbol{v}_m^L)\boldsymbol{u}_n^R, \quad m \neq n \tag{55}$$

$$\boldsymbol{D}_{mn} = (\overline{\boldsymbol{v}}_m^L - \boldsymbol{v}_m^L)\boldsymbol{u}_{N'+n}^R. \tag{56}$$

Because the elements on the diagonal of $\boldsymbol{C}$ are $O(1)$, $\boldsymbol{C}^{-1}$ is $O(1)$ and hence the order of magnitude of the reflection coefficents for $l/\omega \ll 1$ depends solely on the order of magnitude of $\boldsymbol{D}$. Using the one-dimensional approximation $\boldsymbol{w}_m^L - \boldsymbol{v}_m^L = O(l/\omega)$. Hence $\boldsymbol{D} = O(l/\omega)$ in general and the reflection coefficients will be $O(l/\omega)$. Similarly, using the approximate two-dimensional boundary condition gives reflection coefficients which are $O(l/\omega)^2$.

3 Application to Euler Equations

3.1 Non-dimensional linearized Euler equations

The two-dimensional Euler equations which describe an unsteady, inviscid, compressible flow are usually expressed in the following form based upon the conservation of mass, momentum and energy.

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho E u + pu \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho E v + pv \end{pmatrix} = 0 \tag{57}$$

ρ is the density, u and v are the two velocity components, and E is the total internal energy per unit mass. To complete the system of equations an equation of state is needed to define the pressure p. For an ideal gas this is

$$p = (\gamma - 1)(\rho E - \tfrac{1}{2}\rho(u^2 + v^2)), \tag{58}$$

with γ being the ratio of specific heats which is a constant.

Using Eq. (58) to eliminate E from Eq. (57), and rearranging substantially, yields the following 'primitive' form of the Euler equations.

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix} + \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & \frac{1}{\rho} \\ 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & u \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix} + \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & \frac{1}{\rho} \\ 0 & 0 & \gamma p & v \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix} = 0 \tag{59}$$

This equation is still nonlinear. The next step is to consider small perturbations from a uniform, steady flow, and neglect all but the first order linear terms. This produces a linear equation of the form analyzed in the theory section,

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{A}\frac{\partial \boldsymbol{U}}{\partial x} + \boldsymbol{B}\frac{\partial \boldsymbol{U}}{\partial y} = 0, \tag{60}$$

where $\boldsymbol{U}$ is the vector of perturbation variables

$$\boldsymbol{U} = \begin{pmatrix} \tilde{\rho} \\ \tilde{u} \\ \tilde{v} \\ \tilde{p} \end{pmatrix}, \tag{61}$$

and the coefficient matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant matrices based on the uniform, steady variables.

$$\boldsymbol{A} = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & \frac{1}{\rho} \\ 0 & 0 & u & 0 \\ 0 & \gamma p & 0 & u \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & \frac{1}{\rho} \\ 0 & 0 & \gamma p & v \end{pmatrix} \tag{62}$$

The analysis is greatly simplified if the unsteady perturbations and the steady variables in $\boldsymbol{A}$ and $\boldsymbol{B}$ are all non-dimensionalized using the steady density and speed of sound. With this choice of non-dimensionalization the final form of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\boldsymbol{A} = \begin{pmatrix} u & 1 & 0 & 0 \\ 0 & u & 0 & 1 \\ 0 & 0 & u & 0 \\ 0 & 1 & 0 & u \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} v & 0 & 1 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & 1 \\ 0 & 0 & 1 & v \end{pmatrix}, \tag{63}$$

and the variables u and v in the above matrices are now the Mach numbers in the x and y directions.

It should be mentioned that a number of approximation errors are being introduced in converting the nonlinear Euler equations into the linearized equations. For steady state calculations the error will be proportional to the square of the steady state perturbation at the inflow and outflow. These should be very small and may well be unnoticeable except for the case of an oblique shock at the outflow. For unsteady calculations there are two sources of error. The first which is similar to the steady state error is proportional to the square of the unsteady perturbation. The second is due to the possible nonuniformity of the underlying steady state solution to which the unsteady perturbation is being added. For the Fourier analysis to be valid requires that the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ be constant, and so they must be based on some average of the steady state solutiuon at the inflow and outflow. Thus there will be an error which is proportional to the product of the unsteady perturbation and the steady state nonuniformity. This will usually be negligible at the inflow where the steady state solution is almost uniform, but it may be the dominant error at the outflow where there are steady state nonuniformities due to wakes and shock losses. A final note is that these approximation errors are all second order effects and so are much smaller than the first order improvements we are obtaining with the nonreflecting boundary conditions. In fact they are probably similar in magnitude to the approximation errors involved in formulating the approximate non-reflecting boundary conditions for the general, unsteady, multi-frequency, two-dimensional problem.

3.2 Fourier analysis

Following the analytic theory described earlier, we first obtain the dispersion relation.

$$
\begin{aligned}
\det(-\omega \boldsymbol{I}+k\boldsymbol{A}+l\boldsymbol{B}) &= \det\begin{pmatrix} uk+vl-\omega & k & l & 0 \\ 0 & uk+vl-\omega & 0 & k \\ 0 & 0 & uk+vl-\omega & l \\ 0 & k & l & uk+vl-\omega \end{pmatrix} \\
&= (uk+vl-\omega)^2\left((uk+vl-\omega)^2-k^2-l^2\right) \\
&= 0 \qquad (64)
\end{aligned}
$$

The first two roots are clearly identical.

$$k_{1,2}=\frac{\omega-vl}{u} \qquad (65)$$

Let us assume that $u>0$ and we are concerned with defining boundary conditions for an inflow boundary at $x=0$ and an outflow boundary at $x=1$. If $Im(\omega)>0$ then $Im(k_{1,2})>0$, and if $Im(\omega)=0$ then $\frac{\partial\omega}{\partial k}=u>0$. In either case these satisfy the conditions for right-travelling waves, which are incoming waves at the inflow boundary and outgoing waves at the outflow boundary.

The other two roots are given by

$$\left(1-u^2\right)k^2-2u(vl-\omega)k-(vl-\omega)^2+l^2=0 \qquad (66)$$

$$
\begin{aligned}
\Longrightarrow \quad k_{3,4} &= \frac{u(vl-\omega)\pm\sqrt{u^2(vl-\omega)^2+(1-u^2)(vl-\omega)^2-l^2(1-u^2)}}{1-u^2} \\
&= \frac{1}{1-u^2}\left(-u(\omega-vl)\pm\sqrt{(\omega-vl)^2-(1-u^2)l^2}\right) \\
&= \frac{(\omega-vl)(-u\pm S)}{1-u^2} \qquad (67)
\end{aligned}
$$

where

$$S=\sqrt{1-(1-u^2)l^2/(\omega-vl)^2} \qquad (68)$$

Hence the third and fourth roots are defined by,

$$k_3 = \frac{(\omega-vl)(-u+S)}{1-u^2} \qquad (69)$$

$$k_4 = \frac{(\omega-vl)(-u-S)}{1-u^2} \qquad (70)$$

Now we must be extremely careful with which branch of the complex square root function is used in defining S. If ω is real and S^2 is real and positive, then after some straightforward algebra we obtain

$$\frac{\partial k_3}{\partial \omega} = \frac{-u+1/S}{1-u^2} \quad \Longrightarrow \quad \frac{\partial \omega}{\partial k_3} = \frac{1-u^2}{-u+1/S} \tag{71}$$

$$\frac{\partial k_4}{\partial \omega} = \frac{-u-1/S}{1-u^2} \quad \Longrightarrow \quad \frac{\partial \omega}{\partial k_4} = \frac{1-u^2}{-u-1/S} \tag{72}$$

If $u > 1$ then $S^2 > 1$, and hence it follows that both $\frac{\partial \omega}{\partial k_3}$ and $\frac{\partial \omega}{\partial k_4}$ are positive and so both waves are right-running which is what one expects for supersonic flow. However there are very few turbomachines with axially supersonic flow and so henceforth we will assume that $0<u<1$.

If $u < 1$ and we take the positive real branch for S, then $0 < S < 1$ and hence $\frac{\partial \omega}{\partial k_3}$ is positive but $\frac{\partial \omega}{\partial k_4}$ is negative. Thus the third wave is a right-running wave, but the fourth is left-running. It can be proved that if ω and/or S are complex then one of the two roots for k has positive imaginary part, while the other has a negative imaginary part. To be consistent with the results when S^2 is real and positive, k_3 is defined to be the root with positive imaginary component so that it corresponds to a complex right-running wave, and k_4 remains a left-running wave.

3.3 Eigenvectors

3.3.1 Root 1: entropy wave

$$k_1 = \frac{\omega - vl}{u}, \qquad \omega = uk_1 + vl \tag{73}$$

Substituting for ω gives

$$-\omega \boldsymbol{I} + k_1 \boldsymbol{A} + l\boldsymbol{B} = \begin{pmatrix} 0 & k_1 & l & 0 \\ 0 & 0 & 0 & k_1 \\ 0 & 0 & 0 & l \\ 0 & k_1 & l & 0 \end{pmatrix} \tag{74}$$

and one can choose a right eigenvector

$$\boldsymbol{u}_1^R = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{75}$$

and a corresponding left eigenvector

$$\boldsymbol{u}_1^L = \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix}. \tag{76}$$

The vector $\boldsymbol{v}_1^L$, which is needed to construct the non-relecting boundary conditions, is calculated following the procedure given in the theory section.

$$\begin{aligned} \boldsymbol{v}_1^L &= \frac{1}{u}\boldsymbol{u}_1^L \boldsymbol{A} \\ &= \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{77}$$

This choice of eigenvectors corresponds to the entropy wave. This can be verified by noting that the only non-zero term in the right eigenvector is the density, so that the wave has varying entropy, no vorticity and constant pressure. Also, the left eigenvector 'measures' entropy in the sense that $\boldsymbol{u}_1^L \boldsymbol{U}$ is equal to the linearized entropy, $\delta p - \delta\rho$ (remembering that $c = 1$ because of the non-dimensionalization).

The eigenvectors are only determined to within an arbitrary factor; i.e., they may be multiplied by an arbitary constant or function of λ and they would still be eigenvectors. In the case of both this root and the other roots the arbitary factor was chosen to give the simplest possible form for the eigenvectors subject to the one restriction that at $\lambda = 0$, $\boldsymbol{u}^L \boldsymbol{u}^R = 1$. This restriction gives the orthonormal form for the vectors $\boldsymbol{w}$ which was assumed in the theory section, Eq. (21).

3.3.2 Root 2: vorticity wave

$$k_2 = \frac{\omega - vl}{u}, \qquad \omega = uk_2 + vl \tag{78}$$

By inspection, a second set of right and left eigenvectors for the multiple root is given by

$$\boldsymbol{u}_2^R = \begin{pmatrix} 0 \\ -ul/\omega \\ uk_2/\omega \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -u\lambda \\ 1-v\lambda \\ 0 \end{pmatrix}, \tag{79}$$

and

$$\begin{aligned} \boldsymbol{u}_2^L &= \begin{pmatrix} 0 & -ul/\omega & uk_2/\omega & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -u\lambda & 1-v\lambda & 0 \end{pmatrix} \end{aligned} \tag{80}$$

Hence,

$$\begin{aligned} \boldsymbol{v}_2^L &= \frac{1}{u}\boldsymbol{u}_2^L \boldsymbol{A} \\ &= \begin{pmatrix} 0 & -u\lambda & 1-v\lambda & -\lambda \end{pmatrix} \end{aligned} \tag{81}$$

This root corresponds to the vorticity wave, which can be verified by noting that the right eigenvector gives a wave with vorticity, but uniform entropy and pressure. Since the first two roots are a multiple root we must check that the chosen right and left eigenvectors satisfy the necessary orthogonality relations.

$$\boldsymbol{v}_1^L \boldsymbol{u}_2^R = 0 \tag{82}$$

$$\boldsymbol{v}_2^L \boldsymbol{u}_1^R = 0 \tag{83}$$

It is easily verified that these are correct.

The above choice of eigenvectors for the first and second roots is not unique. Any linear combinations of the eigenvectors is itself an eigenvector, and the only constraint is the required orthogonality conditions. The motivation for our particular choice is our knowledge of the distinct behavior of the entropy and vorticity variables in fluid dynamics, which leads us to suspect (correctly) that this will greatly simplify the algebra at later stages in our analysis.

3.3.3 Root 3: downstream running pressure wave

$$k_3 = \frac{(\omega - vl)(S-u)}{1-u^2} \tag{84}$$

The eigenvectors are derived by the usual method.

$$-\omega \boldsymbol{I} + k_3 \boldsymbol{A} + l \boldsymbol{B} = \begin{pmatrix} uk_3+vl-\omega & k_3 & l & 0 \\ 0 & uk_3+vl-\omega & 0 & k_3 \\ 0 & 0 & uk_3+vl-\omega & l \\ 0 & k_3 & l & uk_3+vl-\omega \end{pmatrix} \tag{85}$$

$$\boldsymbol{u}_3^R = \frac{1+u}{2\omega} \begin{pmatrix} \omega - uk_3 - vl \\ k_3 \\ l \\ \omega - uk_3 - vl \end{pmatrix} = \frac{1}{2(1-u)} \begin{pmatrix} (1-v\lambda)(1-uS) \\ (1-v\lambda)(S-u) \\ (1-u^2)\lambda \\ (1-v\lambda)(1-uS) \end{pmatrix} \tag{86}$$

$$\begin{aligned} \boldsymbol{u}_3^L &= \frac{1+u}{\omega} \begin{pmatrix} 0 & k_3 & l & \omega - uk_3 - vl \end{pmatrix} \\ &= \frac{1}{1-u} \begin{pmatrix} 0 & (1-v\lambda)(S-u) & (1-u^2)\lambda & (1-v\lambda)(1-uS) \end{pmatrix} \end{aligned} \tag{87}$$

$$\begin{aligned} \boldsymbol{v}_3^L &= \frac{1}{1+u} \boldsymbol{u}_3^L \boldsymbol{A} \\ &= \frac{1}{\omega} \begin{pmatrix} 0 & \omega - vl & ul & u(\omega - vl) + (1-u^2)k_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (1-v\lambda) & u\lambda & (1-v\lambda)S \end{pmatrix} \end{aligned} \tag{88}$$

This root corresponds to an isentropic, irrotational pressure wave, travelling downstream provided $u > -1$.

3.3.4 Root 4: upstream running pressure wave

$$k_4 = -\frac{(\omega - vl)(S+u)}{1-u^2} \tag{89}$$

The eigenvectors are derived by the usual method.

$$-\omega \boldsymbol{I} + k_4 \boldsymbol{A} + l \boldsymbol{B} = \begin{pmatrix} uk_4+vl-\omega & k_4 & l & 0 \\ 0 & uk_4+vl-\omega & 0 & k_4 \\ 0 & 0 & uk_4+vl-\omega & l \\ 0 & k_4 & l & uk_4+vl-\omega \end{pmatrix} \tag{90}$$

$$\boldsymbol{u}_4^R = \frac{1-u}{2\omega} \begin{pmatrix} \omega - uk_4 - vl \\ k_4 \\ l \\ \omega - uk_4 - vl \end{pmatrix} = \frac{1}{2(1+u)} \begin{pmatrix} (1-v\lambda)(1+uS) \\ -(1-v\lambda)(S+u) \\ (1-u^2)\lambda \\ (1-v\lambda)(1+uS) \end{pmatrix} \tag{91}$$

$$\begin{aligned} \boldsymbol{u}_4^L &= \frac{1-u}{\omega} \left(0 \quad k_4 \quad l \quad \omega - uk_4 - vl \right) \\ &= \frac{1}{1+u} \left(0 \quad -(1-v\lambda)(S+u) \quad (1-u^2)\lambda \quad (1-v\lambda)(1+uS) \right) \end{aligned} \tag{92}$$

$$\begin{aligned} \boldsymbol{v}_4^L &= -\frac{1}{1-u} \boldsymbol{u}_4^L \boldsymbol{A} \\ &= -\frac{1}{\omega} \left(0 \quad \omega - vl \quad ul \quad u(\omega - vl) + (1-u^2)k_4 \right) \\ &= \left(0 \quad -(1-v\lambda) \quad -u\lambda \quad (1-v\lambda)S \right) \end{aligned} \tag{93}$$

This root corresponds to an isentropic, irrotational pressure wave, travelling upstream provided $u < 1$.

3.4 One-dimensional, unsteady b.c.'s

If the computational domain is $0 < x < 1$, and $0 < u < 1$, then the boundary at $x=0$ is an inflow boundary with incoming waves corresponding to the first three roots, and the boundary at $x=1$ is an outflow boundary with just one incoming wave due to the fourth root.

When $\lambda=0$, $S=1$, and so the right eigenvectors $\boldsymbol{w}^R$ are

$$\boldsymbol{w}_1^R = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{w}_2^R = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{w}_3^R = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \quad \boldsymbol{w}_4^R = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \tag{94}$$

and the left eigenvectors $\boldsymbol{w}^L$ are

$$\begin{aligned} \boldsymbol{w}_1^L &= \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix} \\ \boldsymbol{w}_2^L &= \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \\ \boldsymbol{w}_3^L &= \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} \\ \boldsymbol{w}_4^L &= \begin{pmatrix} 0 & -1 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{95}$$

Hence the transformation to, and from, 1-D characteristic variables is given by the following two matrix equations.

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \end{pmatrix} \tag{96}$$

$$\begin{pmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \end{pmatrix} = \begin{pmatrix} -1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{97}$$

$\delta\rho, \delta u, \delta v$ and δp are the perturbations from the uniform flow about which the Euler equations were linearized, and c_1, c_2, c_3 and c_4 are the amplitudes of the four characteristic waves. At the inflow boundary the correct unsteady, non-reflecting, boundary conditions are

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0, \tag{98}$$

while at the outflow boundary the correct non-reflecting boundary condition is

$$c_4 = 0 \tag{99}$$

The standard numerical method for implementing these is to calculate or extrapolate the outgoing characteristic values from the interior domain, and then use Eq. (97) to reconstruct the solution on the boundary.

3.5 Exact, two-dimensional, single-frequency b.c.'s

The construction of the exact, non-reflecting boundary conditions for a linear, two-dimensional, single-frequency solution begins by performing a Fourier decomposition of the flow field at the boundary. Whereas before we assumed that the field was periodic with period 2π, we will now be more specifically concerned with turbomachinery applications, and will consider a problem in which the pitch is P and the inter-blade phase angle is σ. The inter-blade phase angle is needed in the analysis of blades which are not moving in phase, but instead have a phase lag of σ between neighboring blades. Consequently the fluid motion also has a phase lag in the periodicity condition.

$$\boldsymbol{U}(x, y+P, t) = e^{i\sigma}\boldsymbol{U}(x,y,t) \tag{100}$$

In this case the Fourier decomposition of $\boldsymbol{U}$ at the boundary can be written as

$$\boldsymbol{U}(0,y,t) = \sum_{-\infty}^{\infty} \hat{\boldsymbol{U}}_m(t) e^{i l_m y}, \tag{101}$$

where

$$\hat{\boldsymbol{U}}_m(t) = \frac{1}{P}\int_0^P \boldsymbol{U}(0,y,t)e^{-i l_m y}\, dy, \tag{102}$$

and

$$l_m = \frac{2\pi m + \sigma}{P}. \tag{103}$$

Using the expressions for $\boldsymbol{v}^L$ derived earlier, the exact, two-dimensional, single-frequency, non-reflecting boundary conditions at the inflow are

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -u\lambda & 1-v\lambda & -\lambda \\ 0 & 1-v\lambda & u\lambda & (1-v\lambda)S \end{pmatrix} \hat{\boldsymbol{U}}_m = 0, \tag{104}$$

and at the outflow the boundary condition is

$$\begin{pmatrix} 0 & -(1-v\lambda) & -u\lambda & (1-v\lambda)S \end{pmatrix} \hat{\boldsymbol{U}}_m = 0. \tag{105}$$

λ and hence S are functions of l_m/ω and so depend on the mode number m.

How these conditions are implemented depends greatly on the numerical method being used to solve the linear, differential equations in the interior. The most efficient methods begin by expressing the flow field as the real part of a complex amplitude multiplying a complex oscillation in time.

$$\boldsymbol{U}(x,y,t) = \boldsymbol{U}e^{-i\omega t} \tag{106}$$

The direct class of methods now treats the amplitude $\boldsymbol{U}$ as a function of x and y only, and substitutes this definition into the equations of motion to obtain a spatial, partial differential equation for $\boldsymbol{U}$. This is then spatially discretized in some manner and solved by an iterative or direct matrix solution method. Following this approach the exact, non-reflecting boundary conditions can be introduced and solved directly. This was done by Hall, and the full details and some results are presented in his thesis [15] and a recent paper [8].

An alternative approach is to still treat the complex amplitude $\boldsymbol{U}$ as time-varying. In this way one obtains an unsteady, partial differential equation for $\boldsymbol{U}$ which can be solved by a time-marching method and integrated in time until $\boldsymbol{U}$ converges to a steady-state giving the correct complex amplitudes. This approach was developed by Ni for the isentropic, linearized Euler equations [16]. The exact, non-reflecting boundary conditions can still be used, but now one must be concerned with implementing them in a manner that gives the correct result in the converged steady-state but is also stable during the pseudo-time-evolution process. This requires that the associated initial-boundary-value problem be well-posed. To achieve this let us first express the non-reflecting boundary conditions in terms of the spatial Fourier transforms of the one-dimensional characteristic variables.

$$\hat{\boldsymbol{U}}_m = \begin{pmatrix} -1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{pmatrix} \tag{107}$$

Hence the inflow boundary condition is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-v\lambda & -\frac{1}{2}\lambda(u+1) & \frac{1}{2}\lambda(u-1) \\ 0 & u\lambda & \frac{1}{2}(1-v\lambda)(1+S) & -\frac{1}{2}(1-v\lambda)(1-S) \end{pmatrix} \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{pmatrix} = 0, \tag{108}$$

and the outflow equation is

$$\begin{pmatrix} 0 & -u\lambda & -\frac{1}{2}(1-v\lambda)(1-S) & \frac{1}{2}(1-v\lambda)(1+S) \end{pmatrix} \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{pmatrix} = 0, \tag{109}$$

These equations can now be solved to obtain the incoming characteristics as a function of the outgoing ones.

$$\begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{(1-u)\lambda}{(1+S)(1-v\lambda)}\hat{c}_4 \\ \dfrac{(1-u)^2\lambda^2}{(1+S)^2(1-v\lambda)^2}\hat{c}_4 \end{pmatrix} \tag{110}$$

$$\hat{c}_4 = \frac{2u\lambda}{(1-v\lambda)(1+S)}\hat{c}_2 + \frac{1-S}{1+S}\hat{c}_3 \tag{111}$$

In inverting the inflow matrix the following identity was used to eliminate $\lambda^2(1\pm u)$.

$$(1-u^2)\lambda^2 = (1-v\lambda)^2(1+S)(1-S) \tag{112}$$

Note that in the resulting inflow equation $(1+S)(1-v\lambda)$ remains finite as $(1-v\lambda)\to 0$, and so the reflection coefficient is never infinite.

It has already been proved that if the incoming one-dimensional characteristics are set to zero then the initial-boundary-value problem is well-posed. This suggests that the evolutionary process for this problem will be well-posed if we lag the updating of the incoming characteristics.

$$\frac{\partial}{\partial t}\begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{pmatrix} = \alpha \begin{pmatrix} -\hat{c}_1 \\ \dfrac{(1-u)\lambda}{(1+S)(1-v\lambda)}\hat{c}_4 - \hat{c}_2 \\ \dfrac{(1-u)^2\lambda^2}{(1+S)^2(1-v\lambda)^2}\hat{c}_4 - \hat{c}_3 \end{pmatrix} \tag{113}$$

$$\frac{\partial \hat{c}_4}{\partial t} = \alpha\left(\frac{2u\lambda}{(1-v\lambda)(1+S)}\hat{c}_2 + \left(\frac{1-S}{1+S}\right)\hat{c}_3 - \hat{c}_4\right) \tag{114}$$

This is the one numerical boundary condition in this report which the author has not yet implemented, and so the correct value for α is not clear. Choosing too large a value for α may lead to ill-posedness and numerical instability. Choosing too small a value will lead to a poor convergence rate. Some numerical experimentation may be needed to obtain the best value for α. As before, the outgoing characteristics can be obtained by extrapolation or calculation from the interior solution, and then the solution on the boundary can be reconstructed by first converting back to the Fourier-transformed primitive variables, and finally back to the primitive variables in the physical domain.

3.6 Exact, two-dimensional, steady b.c.'s

The exact, two-dimensional steady boundary conditions are essentially the exact, two-dimensional single-frequency boundary conditions in the limit $\omega \to 0$. Again one begins by Fourier transforming the solution along the boundary, and then constructing boundary conditions for each Fourier mode. If the mode number m is non-zero, then

$$\begin{aligned}\lim_{\lambda\to\infty} S(\lambda) &= \sqrt{1-\frac{1-u^2}{v^2}} \\ &= -\frac{\beta}{v}\end{aligned} \tag{115}$$

where

$$\beta = \begin{cases} i\,\mathrm{sign}(l)\sqrt{1-u^2-v^2}\,, & u^2+v^2<1 \\ -\mathrm{sign}(v)\sqrt{u^2+v^2-1}\,, & u^2+v^2>1 \end{cases} \tag{116}$$

The reason for the choice of sign functions in the definition of β, is that for supersonic flow S must be positive, as discussed when S was first defined, and for subsonic flow S must be consistent with $Im(k_3)>0$. To check that we have satisfied this latter condition, the steady-state values of the four wavenumbers are

$$\begin{aligned}k_1 &= -\frac{vl}{u} \\ k_2 &= -\frac{vl}{u} \\ k_3 &= -\frac{vl(-u+S)}{1-u^2} = \frac{uvl+\beta l}{1-u^2} \\ k_4 &= -\frac{vl(-u-S)}{1-u^2} = \frac{uvl-\beta l}{1-u^2}.\end{aligned} \tag{117}$$

If $u^2+v^2<1$ then

$$Im(k_3) = \frac{|l|\sqrt{1-u^2-v^2}}{1-u^2} > 0 \tag{118}$$

and so the condition is indeed satisfied.

It should be remembered that in discussing the supersonic flow condition we are still assuming that the flow is axially subsonic, $u<1$, and so there are three incoming characteristics at the inflow boundary and one incoming characteristic at the outflow boundary.

The next step is to construct the steady-state left eigenvectors $\boldsymbol{s}^L$. Since it is permissible to multiply the eigenvectors by any function of λ, we will slightly modify the definition given in the theory section in order to keep the limits finite as $\lambda \to \infty$.

$$
\begin{aligned}
\boldsymbol{s}_1^L &= \lim_{\lambda\to\infty} \boldsymbol{v}_1^L &&= \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix} \\
\boldsymbol{s}_2^L &= \lim_{\lambda\to\infty} \tfrac{1}{\lambda}\boldsymbol{v}_2^L &&= \begin{pmatrix} 0 & -u & -v & -1 \end{pmatrix} \\
\boldsymbol{s}_3^L &= \lim_{\lambda\to\infty} \tfrac{1}{\lambda}\boldsymbol{v}_3^L &&= \begin{pmatrix} 0 & -v & u & \beta \end{pmatrix} \\
\boldsymbol{s}_4^L &= \lim_{\lambda\to\infty} \tfrac{1}{\lambda}\boldsymbol{v}_4^L &&= \begin{pmatrix} 0 & v & -u & \beta \end{pmatrix}
\end{aligned}
\tag{119}
$$

Using these vectors, the exact, two-dimensional, steady-state, non-reflecting boundary conditions at the inflow are

$$
\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -u & -v & -1 \\ 0 & -v & u & \beta \end{pmatrix} \hat{\boldsymbol{U}}_m = 0, \tag{120}
$$

and at the outflow the boundary condition is

$$
\begin{pmatrix} 0 & v & -u & \beta \end{pmatrix} \hat{\boldsymbol{U}}_m = 0. \tag{121}
$$

For subsonic flow, β depends on l and hence the mode number m. For supersonic flow, β does not depend on l and so the boundary conditions are the same for each Fourier mode other than $m\!=\!0$.

As with the single-frequency boundary conditions we now transform from primitive variables into characteristic variables. The inflow boundary condition becomes

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -v & -\frac{1}{2}(1+u) & -\frac{1}{2}(1-u) \\ 0 & u & \frac{1}{2}(\beta-v) & \frac{1}{2}(\beta+v) \end{pmatrix} \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{pmatrix} = 0, \tag{122}
$$

and the outflow equation becomes

$$
\begin{pmatrix} 0 & -u & \frac{1}{2}(\beta+v) & \frac{1}{2}(\beta-v) \end{pmatrix} \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{pmatrix} = 0, \tag{123}
$$

Solving to obtain the incoming characteristics as a function of the outgoing ones gives

$$
\begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\left(\dfrac{\beta+v}{1+u}\right)\hat{c}_4 \\ \left(\dfrac{\beta+v}{1+u}\right)^2 \hat{c}_4 \end{pmatrix}, \tag{124}
$$

and

$$\hat{c}_4 = \left(\frac{2u}{\beta-v}\right)\hat{c}_2 - \left(\frac{\beta+v}{\beta-v}\right)\hat{c}_3. \tag{125}$$

In inverting the inflow matrix we twice used the following identity.

$$(1+u)(1-u) = -(\beta+v)(\beta-v) \tag{126}$$

To ensure the well-posedness of the evolutionary process, these equations are again lagged.

$$\frac{\partial}{\partial t}\begin{pmatrix}\hat{c}_1\\ \hat{c}_2\\ \hat{c}_3\end{pmatrix} = \alpha \begin{pmatrix} -\hat{c}_1\\ -\left(\frac{\beta+v}{1+u}\right)\hat{c}_4 - \hat{c}_2\\ \left(\frac{\beta+v}{1+u}\right)^2\hat{c}_4 - \hat{c}_3 \end{pmatrix} \tag{127}$$

$$\frac{\partial \hat{c}_4}{\partial t} = \alpha\left(\left(\frac{2u}{\beta-v}\right)\hat{c}_2 - \left(\frac{\beta+v}{\beta-v}\right)\hat{c}_3 - \hat{c}_4\right) \tag{128}$$

Numerical experience indicates that a suitable choice for α is $1/P$. This completes the formulation of the boundary conditions for all of the Fourier modes except $m=0$, which corresponds to $l=0$ which is the average mode. For this mode the user specifies the changes in the incoming one-dimensional characteristics in order to achieve certain average flow conditions. For example at the inflow the three incoming characteristics can be determined by specifiying the average entropy, flow angle and stagnation enthalpy, and at the outflow boundary the one incoming characteristic can be determined by specifying the average exit pressure. Full details of this numerical procedure are given in a separate report [11], which also illustrates the effectiveness of these steady-state nonreflecting boundary conditions. It also tackles the problems caused by the fact that because we have used a linear theory we can get second-order non-uniformities in entropy and stagnation enthalpy across the inflow boundary. These are undesirable, and can be avoided by modifying one of the inflow boundary conditions, and replacing another by the constraint of uniform stagnation enthalpy. The report also shows how the same boundary condition approach can be used to match together two stator and rotor calculations, so that the interface is treated in an average, conservative manner.

3.7 Approximate, two-dimensional, unsteady b.c.'s

3.7.1 Second-order b.c.'s

Following the theory presented earlier, the second order non-reflecting boundary conditions are obtained by taking the second-order approximation to the left eigenvectors $\boldsymbol{v}^L$ in the limit $\lambda \approx 0$. In this limit $S \approx 1$ and so one obtains the following approximate eigenvectors.

$$
\begin{aligned}
\overline{\boldsymbol{v}}_1^L &= \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix} \\
\overline{\boldsymbol{v}}_2^L &= \begin{pmatrix} 0 & -u\lambda & 1-v\lambda & -\lambda \end{pmatrix} \\
\overline{\boldsymbol{v}}_3^L &= \begin{pmatrix} 0 & 1-v\lambda & u\lambda & 1-v\lambda \end{pmatrix} \\
\overline{\boldsymbol{v}}_4^L &= \begin{pmatrix} 0 & -(1-v\lambda) & -u\lambda & 1-v\lambda \end{pmatrix}
\end{aligned}
\tag{129}
$$

Actually, the first two eigenvectors are exact since the only approximation which has been made is $S \approx 1$ in the third and fourth eigenvectors. Consequently, the inflow boundary conditions will be perfectly non-reflecting for both of the incoming entropy and vorticity characteristics.

The second step is to multiply by ω and replace ω by $-\frac{\partial}{\partial t}$ and l by $\frac{\partial}{\partial y}$. This gives the inflow boundary condition

$$
\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \frac{\partial \boldsymbol{U}}{\partial t} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & u & v & 1 \\ 0 & v & -u & v \end{pmatrix} \frac{\partial \boldsymbol{U}}{\partial y} = 0, \tag{130}
$$

and the outflow boundary condition

$$
\begin{pmatrix} 0 & -1 & 0 & 1 \end{pmatrix} \frac{\partial \boldsymbol{U}}{\partial t} + \begin{pmatrix} 0 & -v & u & v \end{pmatrix} \frac{\partial \boldsymbol{U}}{\partial y} = 0. \tag{131}
$$

For implementation purposes it is preferable to rewrite these equations using one-dimensional characteristics.

$$
\frac{\partial}{\partial t} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & v & \frac{1}{2}(1+u) & \frac{1}{2}(1-u) \\ 0 & -u & v & 0 \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \tag{132}
$$

$$
\frac{\partial c_4}{\partial t} + \begin{pmatrix} 0 & u & 0 & v \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0. \tag{133}
$$

The outgoing characteristics can be extrapolated or calculated from the interior, and the incoming characteristics can be calculated by integrating these equations in time using an appropriate method, which in many cases could probably be the same as is used for the interior, partial differential equations. Before using these conditions however, we must check whether or not they form a well-posed initial-boundary-value problem. If they do not then no matter how they are implemented they will produce a divergent solution on a sufficiently fine grid.

3.7.2 Analysis of well-posedness

The well-posedness of the second approximation non-reflecting boundary conditions can be analyzed using the theory discussed earlier. To simplify the analysis we shift to a frame of reference which is moving with speed v in the y-direction. The transformed equations of motion and boundary conditions then correspond to $v=0$ which simplifies the algebra, and well-posedness in this frame of reference is clearly both necessary and sufficient for well-posedness in the original frame of reference.

At the inflow boundary there are three incoming waves and the generalized incoming mode is

$$\boldsymbol{U}(x,y,t) = \left[\sum_{n=1}^{3} a_n \boldsymbol{u}_n^R e^{ik_n x}\right] e^{i(ly-\omega t)} \tag{134}$$

with $Im(\omega) \geq 0$. Using the assumption that $v=0$ the wave numbers are given by

$$k_1 = k_2 = \frac{\omega}{u} \tag{135}$$

$$k_3 = \frac{\omega(S-u)}{1-u^2} \tag{136}$$

where

$$S = \sqrt{1-(1-u^2)\lambda^2} \tag{137}$$

with the correct square root being taken in the definition of S to ensure that if ω and S are both real then S is positive, and if ω or S is complex then $Im(k_3)>0$. Following the procedure presented in the theory section, we obtain the critical matrix $\boldsymbol{C}$.

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+u^2\lambda^2 & 0 \\ 0 & 0 & \frac{1}{2}(S+1+u(1+u)\lambda^2) \end{pmatrix} \tag{138}$$

If $\omega=+iu|l|$ (satisfying the condition that $Im(\omega)\geq 0$), then $\lambda^2=-1/u^2$ and $S=1/u$ (with the correct branch of the square root being taken to ensure that $Im(k_3)\geq 0$).

Hence, for this value of ω,

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{139}$$

so there is clearly a non-trivial incoming mode, $\boldsymbol{U}(x,y,t) = a_2 \boldsymbol{u}_2^R e^{i(k_2 x + ly - \omega t)}$, and the inflow boundary conditions are ill-posed.

It may appear that there is a second incoming mode, $\boldsymbol{U}(x,y,t) = a_3 \boldsymbol{u}_3^R e^{i(k_3 x + ly - \omega t)}$, but this is actually a multiple of the first, because when $\omega = iu|l|$, $k_3 = k_2$ and $\boldsymbol{u}_3^R$ is a multiple of $\boldsymbol{u}_2^R$. This degenerate situation was discussed in the theory section, and in this case it is easily verified that in the neighborhood of $\omega_{crit} = iu|l|$, $\det(\boldsymbol{C}) = O(\omega - \omega_{crit})^2$ and the initial-boundary-value problem is ill-posed with just one ill-posed mode.

At the outflow the generalized incoming mode is

$$\boldsymbol{U}(x,y,t) = \boldsymbol{u}_4^R e^{ik_4 x} e^{i(ly - \omega t)} \tag{140}$$

with $Im(\omega) \geq 0$. Since $v = 0$ the wave number is given by

$$k_4 = -\frac{\omega(S+u)}{1-u^2} \tag{141}$$

Again the correct square root must be taken in the definition of S to ensure that if ω and S are both real then S is positive, and if ω or S is complex then $Im(k_4) < 0$. Since there is now only one incoming mode, the matrix $\boldsymbol{C}$ is simply a scalar.

$$C = \frac{1}{2}(S + 1 - u(1-u)\lambda^2) \tag{142}$$

The outflow boundary conditions are ill-posed *if* there is a solution to

$$S = -1 + u(1-u)\lambda^2 \tag{143}$$

Squaring this equation gives

$$1 - (1-u^2)\lambda^2 = 1 - 2u(1-u)\lambda^2 + u^2(1-u)^2\lambda^4 \tag{144}$$

Solving for λ gives $\lambda^2 = -1/u^2$. This implies that $\omega = +iu|l|$ and $S = 1/u$, as with the inflow analysis. However, when these values are substituted back into Eq. (143) we obtain

$$\frac{1}{u} = -1 - \frac{1-u}{u} = -\frac{1}{u} \tag{145}$$

This inconsistent equation contradicts the supposition that there is a incoming mode which satisfies the boundary conditions, and so we conclude that the outflow boundary condition is well-posed.

3.7.3 Modified boundary conditions

To overcome the ill-posedness of the inflow boundary conditions we modify the third inflow boundary condition. To do this we note that we have been overly restrictive in requiring $\boldsymbol{v}_3^L$ to be orthogonal to $\boldsymbol{u}_1^R$ and $\boldsymbol{u}_2^R$. Since the first two inflow boundary conditions already require that $a_1=a_2=0$, we only really require that $\boldsymbol{v}_3^L$ is orthogonal to $\boldsymbol{u}_4^R$. Thus we propose a new definition of $\overline{\boldsymbol{v}}_3^L$ which is equal to $(\overline{\boldsymbol{v}}_3^L)_{old}$ plus λ times some multiple of the leading order term in $\overline{\boldsymbol{v}}_2^L$.

$$\overline{\boldsymbol{v}}_3^L = \begin{pmatrix} 0 & 1 & u\lambda & 1 \end{pmatrix} + \lambda m \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \tag{146}$$

The variable m will be chosen to minimize $\overline{\boldsymbol{v}}_3^L \boldsymbol{u}_4^R$, which controls the magnitude of the reflection coefficient, and at the same time will produce a well-posed boundary condition. The motivation for this approach is that the second approximation to the scalar wave equation is well-posed and produces fourth order reflections [4].

Substituting definitions gives

$$\overline{\boldsymbol{v}}_3^L \boldsymbol{u}_4^R = \frac{1}{2}\left(\frac{1-u}{1+u}\right)\left(-S + 1 + (1+u)(m+u)\lambda^2\right). \tag{147}$$

Now $S(\lambda) = 1 - \frac{1}{2}(1-u^2)\lambda^2 + O(\lambda^4)$, so the reflection coefficient is fourth order if m is chosen such that $m+u = -\frac{1}{2}(1-u)$. Thus the new form for $\overline{\boldsymbol{v}}_3^L$ is

$$\overline{\boldsymbol{v}}_3^L = \begin{pmatrix} 0 & 1 & -\frac{1}{2}\lambda & 1 \end{pmatrix} \tag{148}$$

To prove the well-posedness of this new boundary condition we examine the new $\boldsymbol{C}$ matrix which is obtained.

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+u^2\lambda^2 & 0 \\ 0 & -\frac{1}{2}(1+u)\lambda & \frac{1}{2}(S+1-\frac{1}{2}(1-u^2)\lambda^2) \end{pmatrix} \tag{149}$$

$\det(\boldsymbol{C})=0$ implies that either

$$-S = 1+\frac{1}{2}(1-u^2)\lambda^2, \tag{150}$$

or

$$\lambda^2 = -1/u^2. \tag{151}$$

Examining the first possibility, squaring both sides gives

$$1-(1-u^2)\lambda^2 = 1-(1-u^2)\lambda^2 + \tfrac{1}{4}(1-u^2)^2\lambda^4, \tag{152}$$

which implies that $\lambda = 0$. In this case $S=1$ and so Eq. (150) becomes $-1=1$ which is inconsistent. Thus the first possibility does not lead to an ill-posed mode.

The second possibility corresponds to $\omega = iu|l|$. At this frequency the second and third eigenvectors become degenerate and so we must apply the extended theory again.

$$\det(\boldsymbol{C}) = (1+iu\lambda)(1-iu\lambda)(S+1-\frac{1}{2}(1-u^2)\lambda^2) \tag{153}$$

$$\implies \quad \frac{\partial}{\partial\omega}\det(\boldsymbol{C})\Big|_{\omega_{crit}} = \frac{(1+u)^2}{u^2\omega_{crit}} \neq 0 \tag{154}$$

Thus the problem is well-posed at the critical degenerate frequency, and this concludes the proof of well-posedness.

Transforming back into the original frame of reference in which $v \neq 0$, the modified inflow boundary conditions are

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \frac{\partial \boldsymbol{U}}{\partial t} + \begin{pmatrix} -v & 0 & 0 & v \\ 0 & u & v & 1 \\ 0 & v & \frac{1}{2}(1-u) & v \end{pmatrix} \frac{\partial \boldsymbol{U}}{\partial y} = 0. \tag{155}$$

There is also a corresponding modified outflow boundary condition which is

$$\begin{pmatrix} 0 & -1 & 0 & 1 \end{pmatrix} \frac{\partial \boldsymbol{U}}{\partial t} + \begin{pmatrix} 0 & -v & \frac{1}{2}(1+u) & v \end{pmatrix} \frac{\partial \boldsymbol{U}}{\partial y} = 0. \tag{156}$$

It is easily proved that this boundary condition gives a well-posed problem. The properties of this condition, and why one might wish to use it instead of the second order approximation, are discussed in the next section.

Finally, it is helpful to express these boundary conditions in their one-dimensional characteristic form.

$$\frac{\partial}{\partial t}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & v & \frac{1}{2}(1+u) & \frac{1}{2}(1-u) \\ 0 & \frac{1}{2}(1-u) & v & 0 \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \tag{157}$$

$$\frac{\partial c_4}{\partial t} + \begin{pmatrix} 0 & \frac{1}{2}(1+u) & 0 & v \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0. \tag{158}$$

3.7.4 Reflection coefficients

The reflection matrix for the modified inflow boundary conditions is

$$
\begin{aligned}
&-\boldsymbol{C}^{-1}\boldsymbol{D} \\
&= -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+u^2\lambda^2 & 0 \\ 0 & -\frac{1}{2}(1+u)\lambda & \frac{1}{2}(S+1-\frac{1}{2}(1-u^2)\lambda^2) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}\left(\frac{1-u}{1+u}\right)\left(-S+1-\frac{1}{2}(1-u^2)\lambda^2\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \left(\dfrac{1-u}{1+u}\right)\dfrac{S-1+\frac{1}{2}(1-u^2)\lambda^2}{S+1-\frac{1}{2}(1-u^2)\lambda^2} \end{pmatrix} \qquad (159)
\end{aligned}
$$

There are three things to note in the above result. Firstly, the outgoing pressure wave produces no reflected entropy or vorticity waves. Secondly, the reflected pressure wave has an amplitude which is $O(l/\omega)^4$. Lastly, at the cutoff frequency, at which $\lambda^2 = 1/(1-u^2)$ and the x-component of the group velocity is zero, the pressure wave reflection coefficient is $-(1-u)/(1+u)$.

The reflection coefficient matrix for the second order outflow boundary condition is

$$
\begin{aligned}
-\boldsymbol{C}^{-1}\boldsymbol{D} &= -\frac{2}{(-S-1+u(1-u)\lambda^2)}\begin{pmatrix} 0 & 0 & \frac{1}{2}\left(\frac{1+u}{1-u}\right)(S-1+u(1-u)\lambda^2) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \left(\dfrac{1+u}{1-u}\right)\dfrac{S-1+u(1-u)\lambda^2}{S+1-u(1-u)\lambda^2} \end{pmatrix} \qquad (160)
\end{aligned}
$$

Again there are three things to note. The outgoing entropy and vorticity waves produce no reflections, the outgoing pressure wave produces a second order reflection, and at the cutoff frequency the reflection coefficient is $-(1+u)/(1-u)$. The product of the pressure wave reflection coefficients at the cutoff frequency is 1, which is to be expected because at the cutoff frequency both pressure waves have zero group velocity in the x-direction and so it is impossible to discriminate between them.

The reflection coefficient matrix for the modified outflow boundary condition is

$$
\begin{aligned}
-\boldsymbol{C}^{-1}\boldsymbol{D} &= -\frac{2}{(-S-1+\frac{1}{2}(1-u^2)\lambda^2)}\begin{pmatrix} 0 & \frac{1}{2}(1-u)\lambda & \frac{1}{2}\left(\frac{1+u}{1-u}\right)(S-1+\frac{1}{2}(1-u^2)\lambda^2) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \dfrac{\frac{1}{2}(1-u)\lambda}{S+1-\frac{1}{2}(1-u^2)\lambda^2} & \left(\dfrac{1+u}{1-u}\right)\dfrac{S-1+\frac{1}{2}(1-u^2)\lambda^2}{S+1-\frac{1}{2}(1-u^2)\lambda^2} \end{pmatrix} \qquad (161)
\end{aligned}
$$

This differs from the second order outflow condition in that now the outgoing pressure wave produces a fourth order reflection, but the outgoing vorticity wave produces a first order reflection. Thus this boundary condition is preferable only in situations where it is known that there is no outgoing vorticity wave. As an example, in the far-field of an oscillating transonic airfoil there will be an outgoing vorticity wave only at the outflow boundary directly behind the airfoil because the only vorticity generation mechanisms are a shock and the unsteady Kutta condition. Thus one might use the second order boundary condition directly behind the airfoil, and the modified boundary condition on the remainder of the outflow far-field boundary.

3.8 Dimensional boundary conditions

For convenience, this section lists all of the boundary conditions in the original dimensional variables.

a) Transformation to, and from, one-dimensional characteristic variables.

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -c^2 & 0 & 0 & 1 \\ 0 & 0 & \rho c & 0 \\ 0 & \rho c & 0 & 1 \\ 0 & -\rho c & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \end{pmatrix} \tag{162}$$

$$\begin{pmatrix} \delta\rho \\ \delta u \\ \delta v \\ \delta p \end{pmatrix} = \begin{pmatrix} -\frac{1}{c^2} & 0 & \frac{1}{2c^2} & \frac{1}{2c^2} \\ 0 & 0 & \frac{1}{2\rho c} & -\frac{1}{2\rho c} \\ 0 & \frac{1}{\rho c} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \tag{163}$$

b) One-dimensional, unsteady b.c.'s.

Inflow:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \tag{164}$$

Outflow:

$$c_4 = 0 \tag{165}$$

c) Exact, two-dimensional, single-frequency b.c.'s.

Inflow:

$$\frac{\partial}{\partial t}\begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{pmatrix} = \alpha \begin{pmatrix} -\hat{c}_1 \\ \dfrac{(c-u)\lambda}{(1+S)(c-v\lambda)}\hat{c}_4 - \hat{c}_2 \\ \dfrac{(c-u)^2\lambda^2}{(1+S)^2(c-v\lambda)^2}\hat{c}_4 - \hat{c}_3 \end{pmatrix} \tag{166}$$

Outflow:

$$\frac{\partial \hat{c}_4}{\partial t} = \alpha \left(\frac{2u\lambda}{(c-v\lambda)(1+S)}\hat{c}_2 + \left(\frac{1-S}{1+S}\right)\hat{c}_3 - \hat{c}_4 \right) \tag{167}$$

where

$$\lambda = \frac{cl}{\omega}, \tag{168}$$

and

$$S = \sqrt{1 - \frac{(c^2-u^2)\lambda^2}{(c-v\lambda)^2}}. \tag{169}$$

d) Exact, two-dimensional, steady b.c.'s.

Inflow:

$$\frac{\partial}{\partial t}\begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{pmatrix} = \alpha \begin{pmatrix} -\hat{c}_1 \\ -\left(\frac{c\beta+v}{c+u}\right)\hat{c}_4 - \hat{c}_2 \\ \left(\frac{c\beta+v}{c+u}\right)^2 \hat{c}_4 - \hat{c}_3 \end{pmatrix} \tag{170}$$

Outflow:

$$\frac{\partial \hat{c}_4}{\partial t} = \alpha\left(\left(\frac{2u}{c\beta-v}\right)\hat{c}_2 - \left(\frac{c\beta+v}{c\beta-v}\right)\hat{c}_3 - \hat{c}_4\right) \tag{171}$$

where

$$\beta = \begin{cases} i\,\mathrm{sign}(l)\sqrt{1-M^2}\,, & M<1 \\ -\mathrm{sign}(v)\sqrt{M^2-1}\,, & M>1 \end{cases} \tag{172}$$

e) Fourth order, two-dimensional, unsteady, inflow b.c.

$$\frac{\partial}{\partial t}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & v & \frac{1}{2}(c+u) & \frac{1}{2}(c-u) \\ 0 & \frac{1}{2}(c-u) & v & 0 \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \tag{173}$$

f) Second order, two-dimensional, unsteady, outflow b.c.

$$\frac{\partial c_4}{\partial t} + \begin{pmatrix} 0 & u & 0 & v \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0. \tag{174}$$

g) First/fourth order, two-dimensional, unsteady, outflow b.c.

$$\frac{\partial c_4}{\partial t} + \begin{pmatrix} 0 & \frac{1}{2}(c+u) & 0 & v \end{pmatrix} \frac{\partial}{\partial y}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0. \tag{175}$$

References

[1] H.-O. Kreiss. Initial Boundary Value Problems for Hyperbolic Systems. *Communications on Pure and Applied Mathematics*, 23:277–298, 1970.

[2] R. L. Higdon. Initial-Boundary Value Problems for Linear Hyperbolic Systems. *SIAM Review*, 28:177–217, 1986.

[3] G. W. Hedstrom. Nonreflecting Boundary Conditions for Nonlinear Hyperbolic Systems. *Journal of Computational Physics*, 30:222–237, 1979.

[4] B. Engquist and A. Majda. Absorbing Boundary Conditions for the Numerical Simulation of Waves. *Mathematics of Computation*, 31:629–651, July 1977.

[5] D. Kwak. Nonreflecting Far-Field Boundary Conditions for Unsteady Transonic Flow Computation. *AIAA Journal*, 19:1401–1407, 1981.

[6] B. Gustafsson. *Far Field Boundary Conditions for Time-Dependent Hyperbolic Systems.* Center for Large Scale Scientific Computation CLaSSiC-87-16, Stanford University, Feb 1987.

[7] J. M. Verdon. *Linearized Unsteady Aerodynamic Theory.* Technical Report R85-151774-1, UTRC, Nov 1985.

[8] K. C. Hall and E. F. Crawley. *Calculation of Unsteady Flows in Turbomachinery Using the Linearized Euler Equations.* Proceedings of Fourth Symposium on Unsteady Aerodynamics and Aeroelasticity of Turbomachines and Propellers, Sept 1987.

[9] M. B. Giles, M. Drela, and W. T. Thompkins. *Newton Solution of Direct and Inverse Transonic Euler Equations.* AIAA Paper 85-1530, July 1985.

[10] L. Ferm. *Open Boundary Conditions for External Flow Problems.* Dept. of Computing Science Report 108, Uppsala University, Feb 1987.

[11] M. B. Giles. *UNSFLO: A Numerical Method for Calculating Unsteady Stator/Rotor Interaction.* Technical Report TR-86-6, MIT Computational Fluid Dynamics Laboratory, 1986.

[12] G. B. Whitham. *Linear and Nonlinear Waves.* John Wiley & Sons, 1974.

[13] J. Mathews and R. L. Walker. *Mathematical Methods of Physics.* W. A. Benjamin, Inc., 1970.

[14] L. N. Trefethen and L. Halpern. Well-Posedness of One-Way Wave Equations and Absorbing Boundary Conditions. *Mathematics of Computation*, 47:421–435, 1986.

[15] K. C. Hall. *A Linearized Euler Analysis of Unsteady Flows in Turbomachinery.* PhD thesis, M.I.T., Sept 1987.

[16] R.-H. Ni and F. Sisto. *Numerical Computation of Nonstationary Aerodynamics of Flat Plate Cascades in Compressible Flow.* ASME Paper 75-GT-5, 1975.