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COMPRESSIBLE NAVIER-STOKES EQUATIONS FOR LOW MACH NUMBER APPLICATIONS

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Abstract. *In this paper, we present an overview concerning the simulation of low speed fluid flows using the compressible Navier-Stokes equations. This approach is appropriate in the context of complicated geometries and/or extreme flow conditions where the flow can vary from low subsonic to supersonic. Unfortunately, this approach has a major drawback: the observed convergence rate gets substantially slower and the solution produced can be of poor quality. We review how to overcome this type of stiffness by using preconditioning techniques and by changing the artificial dissipation in the spatial discretisation to improve the solution, and also the implication of this modification on the matrix timestep originally used to dramatically improve the convergence rate of our Navier-Stokes solver. Finally, before showing some numerical results for 2D and 3D applications, we reveal the influence of a low Mach number preconditioner on the boundary conditions: when the solution has almost converged to the steady state, and hence only low frequency waves remain, the decay of the disturbances does not always behave in the same manner as for the unpreconditioned equations, showing that care must be taken concerning the physical boundary conditions which must be applied for a specific problem.*

1 Introduction

Simulation of low speed fluid flow is very often achieved through the use of incompressible solvers. If one is interested in the steady state, a standard approach is to solve the time dependent equations using a Runge-Kutta scheme or a multistep method. However one of the main issues is to devise an efficient numerical method which takes into account the incompressibility constraint $\nabla.V = 0$ (where V denotes the velocity field), which does not contain any time derivatives. To overcome this difficulty, different techniques exist: addition of an artificial pressure time derivative within the continuity equation (and the momentum equation) with some multiplicative variable(s) [2, 21], or solution of a Poisson equation at each time step to obtain the pressure field [9, 24]. Thus, solving the incompressible equations is not as simple a task as it may seem.

In addition, there are applications where the use of the incompressible flow equations is inappropriate because the flow varies from low subsonic to transonic in different regions. This is particularly true in turbomachinery with very low speed cavity and bleed flows in certain regions, but it also occurs in low speed aerodynamics at high angle of attack where most of the flow has a low Mach number with localised regions containing shocks. Consequently, it seems appropriate to use the compressible equations even where the Mach number of the flow is small. However, in this approach there are some drawbacks too: the observed convergence rate gets substantially slower and the solution produced can be of poor quality, with pressure oscillations visible in contour plots. The slowdown is due to analytic stiffness arising from the inherent propagative disparities in the limit of vanishing Mach number, where the ratio of the convective speed to the acoustic speed approaches zero. This type of stiffness is often treated using preconditioning techniques [22, 23, 21]. By altering the acoustic speeds of the system such that all eigenvalues become of the same order this difficulty is completely alleviated. In addition, the solution can also be improved by changing the artificial dissipation in the spatial discretisation. Based on the preconditioned system, the relative scaling of different numerical smoothing terms can be improved, and the steady-state solution becomes more accurate. We have chosen to retain these techniques since one aspect of our work was to solve problems for a complete range of flow conditions from nearly incompressible to transonic and supersonic. Although the method is attractive and relatively easy to implement (it does not require any change of variables and major modifications in the current code), it is important to be aware of some inevitable problems that the programmer will have to come across. This paper intends to list and to review the reasons for these and to give the appropriate remedy.

The first of our concerns is the poor robustness at stagnation points due to local preconditioners designed for low Mach numbers. Darmofal and Schmid have shown that this lack of robustness is due to unlimited transient amplification of perturbations resulting from a degeneration of the structure of the eigenvectors of the preconditioned equations [4]. These becomes highly non-orthogonal as $M \to 0$. The most common technique to avoid this robustness problem is based on limiting the effect of preconditioning below a multiple of the freestream Mach number. This multiple is typically greater than one [11] and destroys the locality of the preconditioning, since the limit becomes more global. However there are problems where a reference Mach number is inappropriate (or non-existent) and where this types of limiting is difficult to realise. Examples of these type of flows would be a hypersonic flow around a blunt body (which would contain regions of subsonic flow) or flows in pumps and turbomachinery. A possible way to address this problem is then to base this limit on the local Mach number [13], or on strictly local information such as the pressure [3].

Finally, a last problem to be aware of concerns the physical boundary conditions which must be applied for the specific problem to be solved. The preconditioned equations can in some cases behave differently from the original equations, affecting the rate of convergence to steady state [5]: in conjunction with preconditioning, boundary conditions can even become reflective with a decay rate approaching zero.

Although these may appear to be significant problems, they are now well understood and good remedies have been developed. With these incorporated, the use of compressible flow methods for low Mach numbers works well, as will be demonstrated later.

2 Spatial discretisation

The low Mach number preconditioning is only concerned with the inviscid fluxes, so for clarity we will explain the ideas with reference to the one-dimensional Euler equations. We shall write these as

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = 0 \;, \tag{1}$$

where Q is the state vector, $Q = (\rho, \rho u, \rho E)^T$, and F is the flux vector. The propagation speeds are then the eigenvalues of the Jacobian matrix $A = \frac{\partial F}{\partial Q}$; $\lambda_1 = u, \lambda_{2,3} = u \pm c$, where c denotes the speed of sound. For a low speed flow ($|u| \ll c$), the huge variation in wave speeds becomes very important, and one effect is that the convergence to steady state becomes much slower. This is because the timestep is limited by the fastest wave speed, whereas the timescale for evolution to steady state is determined by the slowest wave speed.

The numerical smoothing is also affected when using characteristic smoothing. To see this, we consider the semi-discrete form of Equation (1) using a conservative discretisation with first-order upwinding,

$$\frac{\mathrm{d}Q_j}{\mathrm{d}t} + \frac{1}{\Delta x}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right) = 0, \tag{2}$$

where

$$F_{j+\frac{1}{2}} = \frac{1}{2}\left(F(Q_{j+1}) + F(Q_j)\right) - \frac{1}{2}\,|A|\,(Q_{j+1} - Q_j). \tag{3}$$

The matrix $|A|$ is defined as $T|\Lambda|T^{-1}$ where $|\Lambda|$ is the diagonal matrix of absolute eigenvalues of A, and T is the corresponding matrix of eigenvectors. Hence the largest components of $|A|$ have a magnitude proportional to $|u|{+}c$, making these smoothing terms rather large compared to characteristic fluxes which scale with u.

The objective of low Mach preconditioning is to modify the wave speeds so that they are all of the same order of magnitude. In that way, convergence is improved, and the re-scaling of the characteristic speeds also leads to a more accurate solution because of its effects on the numerical smoothing. The preconditioned 1D Euler equations are written as

$$P^{-1}\,\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = 0\ , \tag{4}$$

where P is the preconditioning matrix to be defined so that the new wave speeds, which are the eigenvalues of PA, are of a similar magnitude. The corresponding semi-discrete equations are then

$$P_j^{-1}\,\frac{\mathrm{d}Q_j}{\mathrm{d}t} + \frac{1}{\Delta x}\left(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}\right) = 0, \tag{5}$$

where

$$F_{j+\frac{1}{2}} = \frac{1}{2}\left(F(Q_{j+1}) + F(Q_j)\right) - \frac{1}{2}P^{-1}\ |PA|\ (Q_{j+1} - Q_j). \tag{6}$$

Note that the change in the numerical smoothing, which affects the steady state solution, is the replacement of $|A|$ by $P^{-1}|PA|$.

The preconditioning matrix is defined using the symmetrising variables

$$\mathrm{d}\tilde{Q} = \left(\mathrm{d}p/\rho c,\ \mathrm{d}u,\ \mathrm{d}p{-}c^2\mathrm{d}\rho\right)^T,$$

with respect to which the linearised 3D Euler equations have coefficients matrices which are symmetric [8, 1]. Using these variables, P is defined as

$$P = R\,\Gamma\ R^{-1}, \tag{7}$$

where R is the matrix for the transformation from symmetrised to conservative variables.

As mentioned in the Introduction and demonstrated in [23], because of their highly non-normal nature for low Mach numbers, many local preconditioners can transiently amplify perturbations by a factor of $1/M$ as $M \to 0$. Taking this fact into account, the preconditioner that we use is the Turkel preconditioner [21]

$$\Gamma = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for which the transient growth can be limited by a careful choice of the parameter ϵ. The eigenvalues of PA, which give the modified wave speeds, are

$$\begin{aligned} \lambda_1 &= u \\ \lambda_{2,3} &= \frac{1}{2}\left(1+\epsilon\right)u \pm \frac{1}{2}\tau \end{aligned}$$

with

$$\tau = \sqrt{(1-\epsilon)^2\, u^2 + 4\epsilon c^2}\ .$$

For good low Mach number preconditioning, choosing $\epsilon = O(M^2)$ ensures that the convective and acoustic wave speeds are of a similar magnitude, proportional to the flow speed [21]. Very often, it is required that ϵ be greater than some multiple of the square of the freestream Mach number [20, 19, 11]. Although this approach work well in many cases, it cannot be used for internal flows where the freestream Mach number is usually unknown, and it is also inappropriate when the freestream Mach number may be far from zero, but there are extensive flow regions in which the local Mach number is extremely small. Consequently, we prefer a different approach: looping over the grid edges, the biggest Mach number between two nodes connected by an edge is evaluated and kept for both nodes. Repeating the procedure four times defines small regions with a common maximum value. This evaluation remains local, and provides a smoother behaviour of the limiter than if this was only based on the nodal Mach number. In addition, this preconditioner is only for use at low Mach number, and is therefore switched off when not appropriate to reduce to the identity matrix. Thus, the final determination of ϵ is implemented as followed:

$$\epsilon = \min \ \left[1, \eta M_{max}^2\right] \ ,$$

where η is a free parameter set to 3.0 [11].

The extension to 2D and 3D follows the same approach, using a 1D characterstic treatment along each edge. In addition, for improved accuracy, the numerical smoothing flux which is used in practice is a blend of first and third differences, with a limiter ψ defined to give second order accuracy in smooth flow regions, and first order characteristic upwinding at shocks [13].

3 Convergence rate

All our research efforts to develop efficient numerical methods for a complete range of Mach number take place in the context of multigrid. However, for viscous flows, poor multigrid performance is not uncommon so that something needs to be added in order to improve the efficiency and the robustness of the code (a complete diagnosis of multigrid breakdown can be found in [17]). The method which has proven to be highly successful for structured and unstructured meshes [16, 13] combines a semi-coarsening multigrid strategy with a preconditioner, the so-called block-Jacobi preconditioner. It is based on a local linearisation of the 3D Navier-Stokes equations, with a first order upwind discretisation, and is built by extracting the terms corresponding to the central node. As the flux can be split into inviscid and viscous parts, it has contributions coming from both. As described in [16, 13], the inviscid part only depends on the absolute value of the jacobian, i.e. on $|A|$. As discussed earlier, in order to deal with low speed flows, we need to modify the artificial dissipation with a suitable low Mach number preconditioner Γ. This same modification is also incorporated into the block-Jacobi preconditioner.

In [14], we proved that the combined use of blok-Jacobi and low-Mach preconditioning is stable for the Euler equations subject to a CFL condition being satisfied. This is important in explaining the good multigrid behaviour observed in practical applications since the inviscid discretisation is the dominant part on coarse grids.

4 Boundary conditions

When the solution has almost converged to the steady state, the residual is due to low frequency waves which propagate up and down the domain and are not significantly affected by the numerical viscosity. These can only be dissipated through the interaction with the far-field boundary conditions. In general, when they arrive at one boundary, these waves are reflected and propagated in the other direction until they reach the other boundary, and so on. Ideally, one would like to have perfectly non-reflecting boundary conditions, absorbing these low frequency waves and resulting in a much faster convergence rate, but in two or three dimensions, these do not exist and consequently one only can expect an exponential decay of the amplitude of these waves. In [7], Giles has examined this process for the subsonic one-dimensional Euler equations by deriving the exact eigenmodes and eigenfrequencies of the initial boundary value problem and by determining the exponential decay rates for perturbations under different sets of boundary conditions. In [5], Darmofal, Moinier and Giles performed a similar analysis for the one-dimensional preconditioned Euler equations for low Mach number flows. We investigated the effect of two forms of low Mach number preconditioner (the Van Leer-Lee-Roe and Turkel [22, 21]) on the effectiveness of boundary condition in eliminating initial transients. The main results are the following:

- Boundary conditions based on the Riemann invariants of the Euler equations are found to be reflective in conjunction with preconditioning, whereas they are non-reflecting without it; the problem is most detrimental at low Mach numbers where the perturbation decay rate approaches zero.
- Boundary conditions which specify entropy and stagnation enthalpy at an inflow and pressure at an outflow are found to be non-reflective with the Van Leer/Lee/Roe preconditioning, and weakly non-reflective in the other case.
- The specification of velocity and density at the inflow and pressure at the outflow is found to be non-reflecting for the Van Leer/Lee/Roe preconditioning and weakly reflective for the Turkel preconditioning. However, for the unpreconditioned Euler equations they provide no damping of initial transients in the absence of numerical smoothing.

As this analysis showed, it is necessary, according to the preconditioner used, to be very careful when considering a particular type of boundary conditions since these could become completely inappropriate and jeopardise the convergence to steady state.

5 Results

In this section, we present a set of test cases ranging from 2D inviscid to 3D viscous to validate the accuracy and robustness of our compressible Navier-Stokes solver at low Mach number. All the calculations have been performed using a new multigrid method described in [15] which automatically generates coarse grids from the finest grid through an element collapsing algorithm. For the convergence comparisons we plot the L_2 norm of the residual vector (normalised by the initial residual in Fig. 3 (a)) during one application of the time-stepping scheme on the finest mesh in the multigrid cycle.

5.1 Inviscid 2D Naca0012 airfoil

The four cases that we first investigate are defined in Table 1. For Mach numbers smaller than 0.1, an unpreconditioned code is much slower (see Fig. 3 (b)) and produces, when it converges, solutions of poor quality with oscillations in the pressure and density contour plot (Fig. 1). These problems are fully addressed with a low Mach number preconditioner which preserves accuracy in the incompressible limit. This is shown in Fig. 2 where plots of the coefficient of pressure contours are depicted. Finally, convergence histories plotted together in Fig. 3 (a) show, as expected, that convergence is Mach independent.

Test	Geometry	M_∞	α
ELM1	NACA0012	0.1	0.0
ELM2	NACA0012	0.01	0.0
ELM3	NACA0012	0.001	0.0
ELM4	NACA0012	0.0001	0.0

Table 1: Euler test case definitions for low Mach number: airfoil, freestream Mach number, angle of attack in degrees.

5.2 Accuracy preservation by preconditioning

For an incompressible inviscid fluid, the velocity field and hence the pressure coefficient can be obtained by solving the potential flow via the Schwartz-Christoffel conformal mapping [12]. To demonstrate the accuracy discussed in the previous section, the computed pressure coefficient is here compared with that obtained with the Schwartz-Christoffel tool-box of Driscoll [6]. In Fig. 4 the comparison is presented for each case, showing the accuracy of the results obtained in the incompressible limit.

5.3 Inviscid flow over M6 wing

The first 3D test case is the inviscid flow over an ONERA M6 wing at a freestream Mach number of 0.001 and 0^o angle of attack. The fine grid has 173000 vertices and a

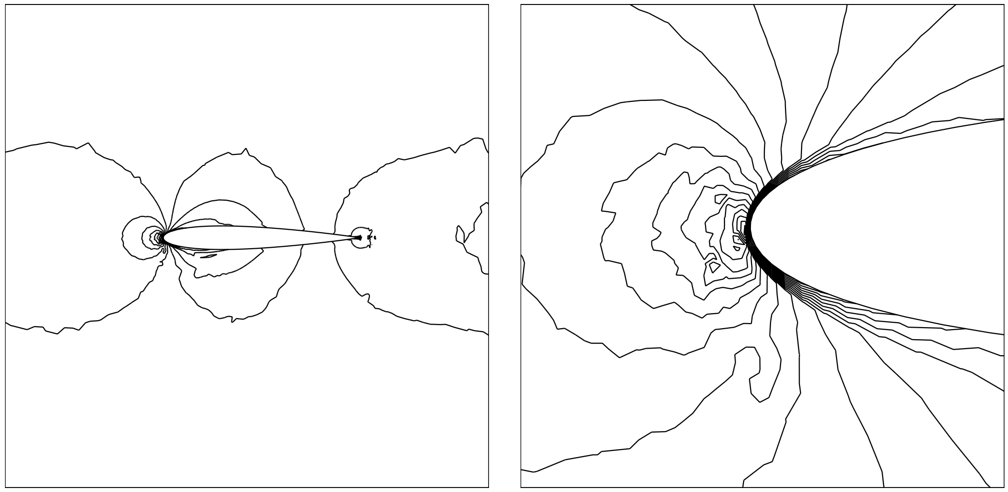


Figure 1: Computed pressure plot for $M_\infty = 0.01$ without low Mach number preconditioner. $\alpha = 0^o$.

sequence of 4 coarser grids is generated. We successively apply our solver with and without low Mach preconditioning and compare the results looking at the convergence, and the final solution. These are shown in Fig. 5 and 6. As for the 2D case, an unpreconditioned code is much slower and converge to a solution of very poor quality: in that case not only it shows oscillations in the pressure contour plot, but is also completely wrong. The low Mach number preconditioner fixed both problems.

5.4 Viscous flow over M6 wing

The last test case is a multigrid viscous calculation. It involves a finer grid of 306000 points over an ONERA M6 wing. To account for the effect of turbulence, the one equation turbulence model of Spalart and Allmaras [18] is utilised. This equation is discretised and solved in a same way as for the flow equations with the exception that the convective terms are discretised with a first order scheme. The freestream Mach number is 0.1, the incidence is 2.0 degrees, and the Reynolds number is 3 million. For this case only, the local maximum Mach number used in the definition of ϵ was taken as the highest nodal values for each edge. Convergence is depicted in Fig. 7. Fig. 8 illustrates the computed pressure obtained on this grid. As we can see, the solution shows smooth contour plot.

6 Conclusion

In this paper we rewieved which techniques are involved to modify a compressible Navier-Stokes solver in order to solve low-speed flows. This approach is very attractive because it does not involve many changes in the original code, gives the capability to use a single code over a very broad range of flow conditions and to simulate flows varying from low subsonic to transonic. We also presented recent results concerning the influence of low Mach preconditioning on the boundary conditions and highlighted the stability

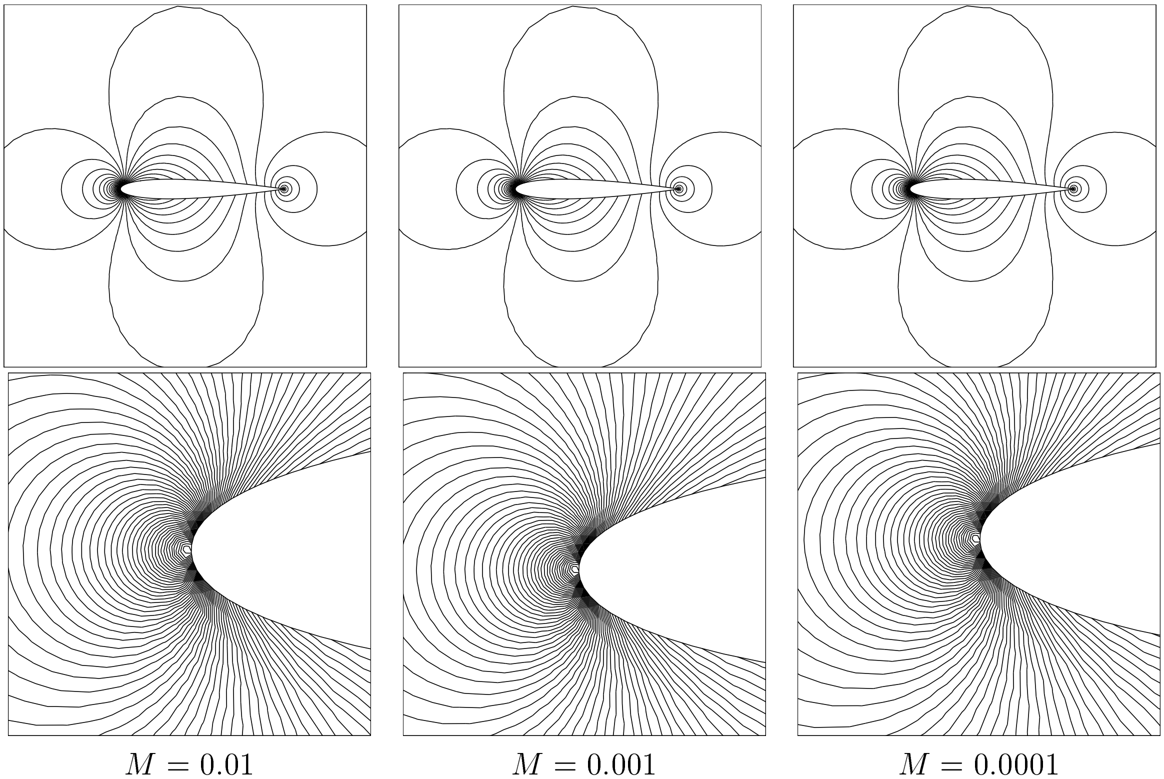
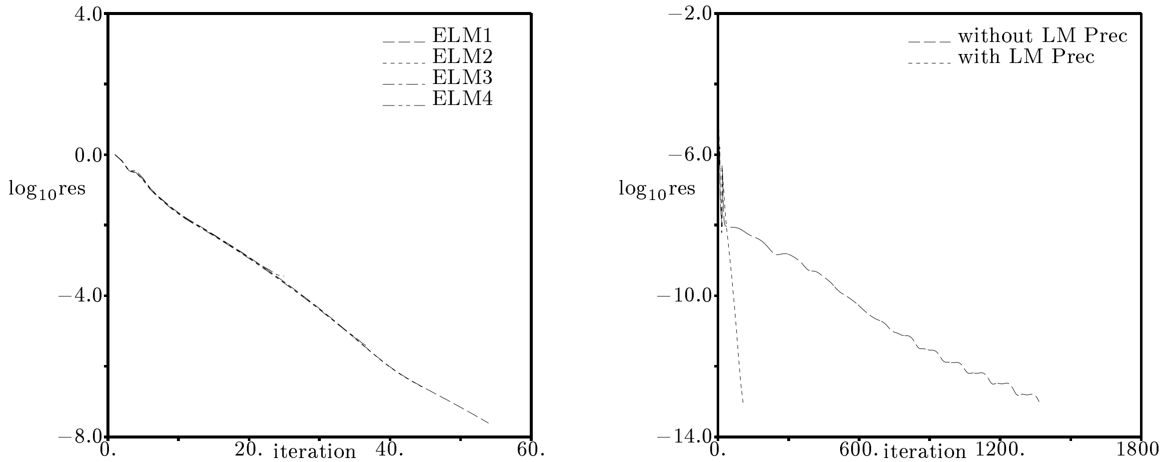


Figure 2: Computed pressure coefficient contours around a Naca0012 airfoil, overview and zoom around the leading edge. $\alpha = 0^o$.

guaranty for the Euler equations as long as a CFL condition is satisfied.

The numerical results presented demonstrated the efficiency and accuracy of our preconditioned scheme used in conjunction of multigrid. Convergence is improved as the final solution.



(a) Tests ELM1, ELM2, ELM3, ELM4. (b) Low Mach number preconditioner effect.

Figure 3: Convergence history comparison for low Mach number flow around a Naca0012 airfoil.

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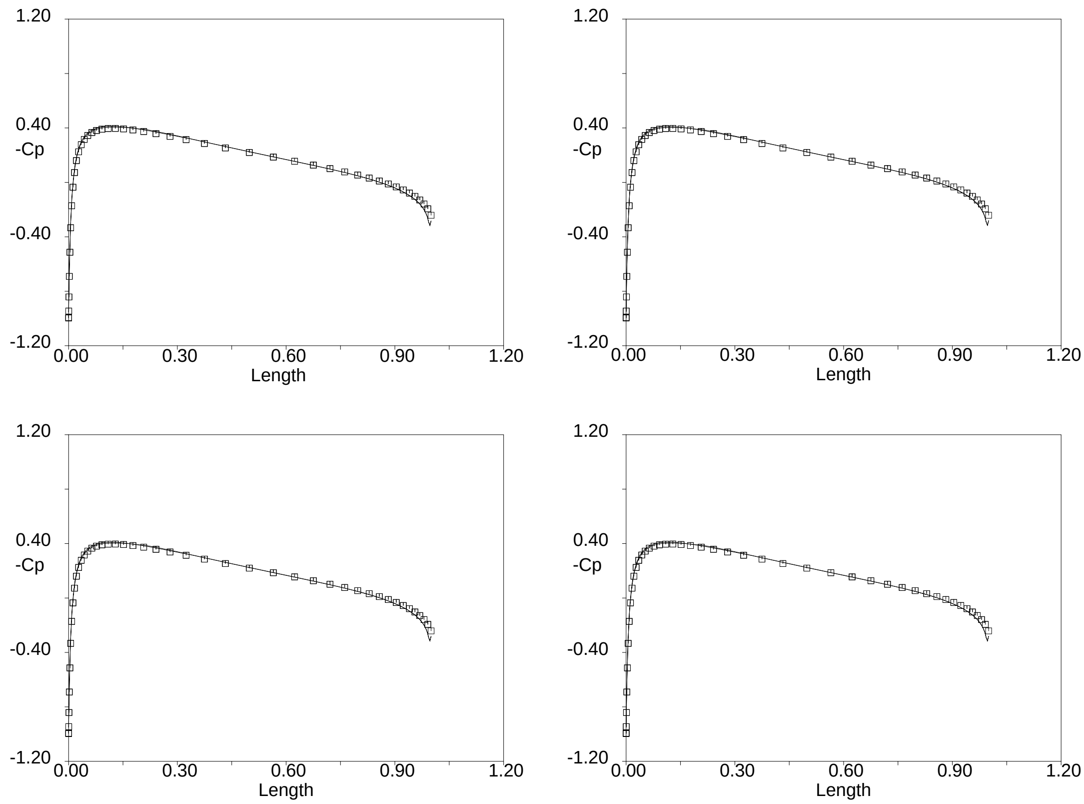

Figure 4: Computed pressure coefficient contours around a Naca0012 airfoil. Comparison with the potential flow solution. $M_\infty = 0.1, 0.01, 0.001, 0.0001$. $\alpha = 0^o$.

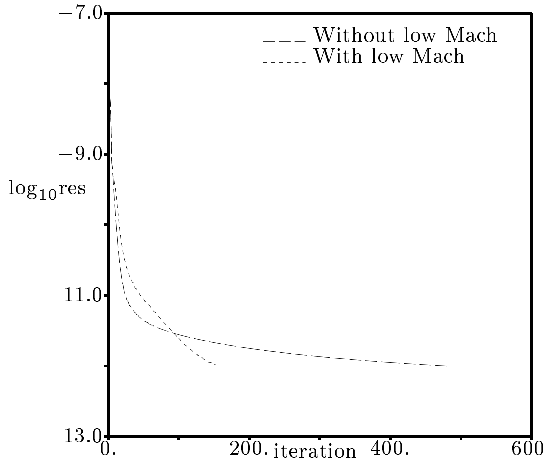

Figure 5: Convergence history comparison for a low Mach number flow over the M6 wing. Inviscid calculation. $M = 0.0001$, $\alpha = 0^o$.

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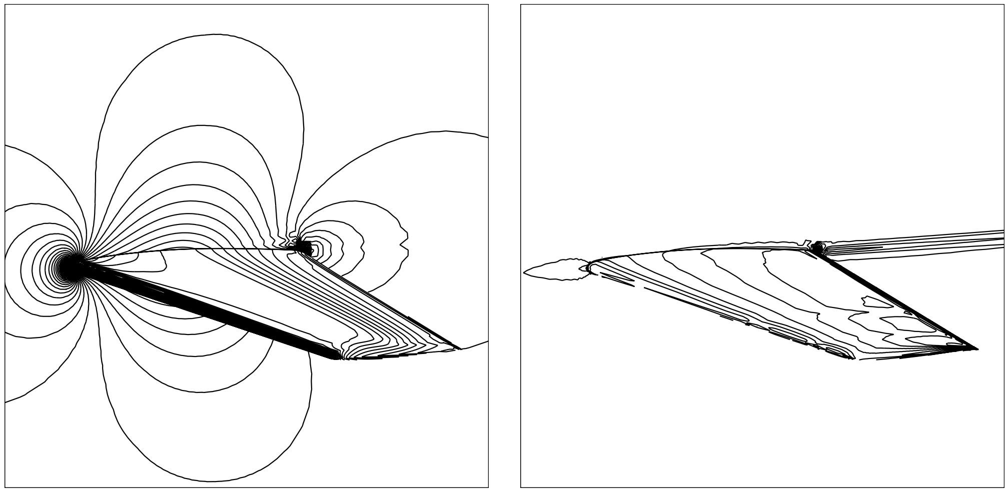

Figure 6: Inviscid flow over the M6 wing: computed pressure plot for $M_\infty = 0.0001$ with and without low Mach number preconditioner. $\alpha = 0^o$.

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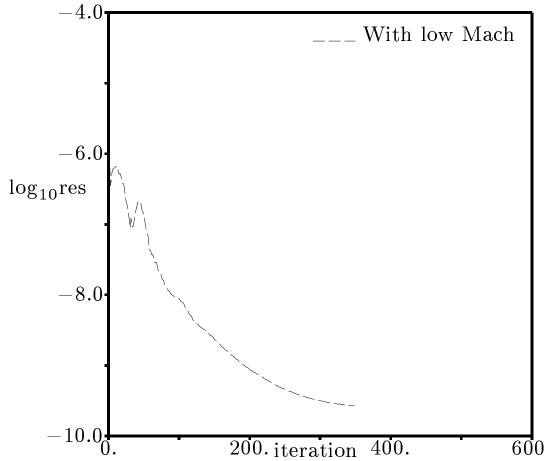

Figure 7: Convergence history comparison for a low Mach number flow over the M6 wing. Viscous calculation. $M = 0.1$, $\alpha = 0^o$, $Re = 3 \times 10^6$.

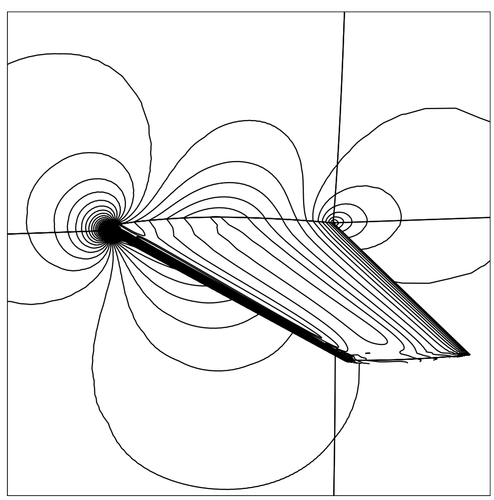

Figure 8: Viscous flow over the M6 wing: computed pressure plot for $M_\infty = 0.1$ with low Mach number preconditioner. $\alpha = 2^o$, $Re = 3 \times 10^6$.