

### B3.4 Algebraic Number Theory, Hilary 2020

#### Exercises 4

**Question 1.** Find all quadratic fields for which the Minkowski bound is strictly less than 2. What is the class number of these fields?

**Question 2.** Show that  $\text{Cl}(K)$  is cyclic of order two, where  $K = \mathbb{Q}(\sqrt{-37})$ .

**Question 3.** Find  $\text{Cl}(K)$ , where  $K = \mathbb{Q}(\sqrt{-6})$ .

**Question 4.** Let  $K$  be a number field, other than  $\mathbb{Q}$ . Show that  $\Delta_K > 1$ .

**Question 5.** Find  $\text{Cl}(K)$ , where  $K = \mathbb{Q}(\sqrt{-34})$ .

**Question 6.** Find  $\text{Cl}(K)$ , where  $K = \mathbb{Q}(\sqrt{65})$ .

**Question 7.** Find all integer solutions to the equation  $y^2 + 74 = x^3$ . (You may assume that  $h_{\mathbb{Q}(\sqrt{-74})} = 10$ .)

**Question 8.** Show that the ring of integers in  $\mathbb{Q}(2^{1/3})$  is a principal ideal domain (any results about this field established on previous sheets may be used without proof).

**Question 9.** (i) Let  $\Lambda$  be a lattice in  $\mathbb{R}^2$  which has no nonzero vector  $x$  with  $\|x\| \leq 1$ . Show that  $\det(\Lambda) \geq \frac{1}{2}\sqrt{3}$ .

(ii) Deduce a (small) improvement to the Minkowski bound for imaginary quadratic fields.

(iii) Briefly comment on the implications for the Rabinowicz phenomenon.

ben.green@maths.ox.ac.uk