ERRATUM FOR "AN INVERSE THEOREM FOR THE GOWERS $U^{s+1}[N]$-NORM"

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Abstract. We correct some minor issues in our paper *An inverse theorem for the Gowers $U^{s+1}[N]$-norm*, Ann. Math **176** (2012), 1231–1372.

1. Introduction

In this erratum we correct some minor issues in [2]. These have come to light on account of the close study of [2] made by James Leng, Ashwin Sah and Mehtaab Sawhney in the course of preparing their work [3]. We are very grateful to James, Ashwin and Mehtaab for drawing these issues to our attention and for discussions relating to them.

To keep the erratum relatively short we assume familiarity with the concepts and notation of [2]. The primary issues in [2] are (with references here being to that paper):

- Insufficient care was taken with the top groups in various filtrations, for example we allow filtrations $G_0 \supset G_1 \supseteq \dots$ with $G_0 \neq G_1$, but this is best avoided for several reasons. One is that the equidistribution results of [1], cited in Appendix D, are stated with the condition $G_0 = G_1$. This turns out to be a minor issue, in that the main statements of [1] continue to hold without this condition. However, somewhat more annoying problems arise with top-order Taylor coefficients (for instance in the display after (12.7) on page 1310), with final groups in such filtrations not necessarily being central (for instance on page 1311, or Example 6.10) and with the fact that the degree of the filtration does not control the nilpotency class of G_0 (an issue in Appendix C).
- The proof of Proposition 8.3 is incorrect, and this proposition does not hold as stated. The problem is with the deduction of (8.4). When $\pi(g_0(n)^*\Gamma) \in B$, one can express $g_0(n) = \iota(x_n)t_n\gamma_n$ for some $x_n \in B$, $t_n \in G_{(1,0)}$, and $\gamma_n \in \Gamma$, and then
$$g(n,h) = \iota(x_n)t_n\gamma_n g_1(n)^h = \iota(x_n)t_n(\gamma_n g_1(n)\gamma_n^{-1})^h\gamma_n,$$
which when combined with (8.3) gives
$$\tilde{\chi}_{k,k'}(n) = e(h\xi(\gamma_n g_1(n)\gamma_n^{-1} \bmod \Gamma_{(1,0)}))\tilde{F}_{k,k'}(g_0(n)^*\Gamma)$$
rather than
$$\tilde{\chi}_{k,k'}(n) = e(h\xi(g_1(n) \bmod \Gamma_{(1,0)}))\tilde{F}_{k,k'}(g_0(n)^*\Gamma)$$
as claimed.
- Some minor issues in Sections 9–12 and Appendix E of the paper, at least one of which rises above the level of a mere typo.

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These three bullet points are addressed in Sections 2, 3 and 4 respectively, with the middle one of these being the most substantial. We note that all issues are also fixed in [3]; in particular, we remark that the authors of that paper independently found a fix for the second issue which is related to the one we shall present, though a little different in the details.

2. Filtrations

To fix the issues with filtrations, one should introduce some additional assumptions about the top order groups in the three main types of filtrations occurring in [2] (see page 1260). Specifically,

- When talking about a degree filtration on G, one should insist that $G_0 = G_1$;
- When talking about a degree-rank filtration on G, one should insist that $G_{(0,0)} = G_{(1,0)}$ (in addition to the conditions $G_{(d,0)} = G_{(d,1)}$ already imposed in [2, Definition 6.9]);
- When talking about a multidegree filtration on G, one should insist that $G_{\vec{0}}$ is generated by the groups $G_{\vec{e_i}}$.

Note that these notions are consistent with [3, Definition 2.4]. For the purposes of this section we call filtrations satisfying these additional conditions *taut*.

One now needs to check that all the filtrations introduced in [2] are taut, or else can be modified so that they are, and one also needs to confirm that the issues mentioned in the introduction cease to be issues when filtrations are assumed to be taut. We leave the latter to the reader.

Regarding the former, the problematic points are mostly various (closely related) places where equivalence of certain nilcharacters is proven: Lemma 12.1, and the proofs of Lemma E. 8 (iv) and (v), and the proof of Proposition E.9. In these places, the given constructions are not taut.

The way to fix this in each case is to use a trick introduced in [1] (in the degree filtration case) to show that any nilsequence in the sense of [2] can be rewritten as a nilsequence in which the underlying filtration is taut. In particular the notion of equivalence of nilcharacters χ, χ' in [2, Definition 6.22] is the same as the *a priori* stronger one in which one insists on tautness in the filtration underlying the nilsequence $\chi \otimes \overline{\chi}'$.

For simplicity we illustrate this trick with standard nilsequences but the extension to the limit formulation used in [2] is routine. In the degree filtration case, given $F(g(n)\Gamma)$ one writes it as $F'(g'(n)\Gamma)$ where $F'(x) = F(\{g(0)\}x)$ and $g'(n) = \{g(0)\}^{-1}(g(n)g(0)^{-1})\{g(0)\}$, where here $\{g(0)\}\Gamma = g(0)\Gamma$ but $\{g(0)\}$ is bounded, noting that F' is still Lipschitz, but g' now takes values in G_1. Thus one may replace the filtration $(G_i)_{i\geqslant 0}$ by $(G'_i)_{i\geqslant 0}$, where $G'_0 := G_1$ and $G'_i = G_i$ for $i \geqslant 1$, noting that this filtration is now taut. For more details (and a discussion of the other filtrations, which may be handled in the same way) see [3, Appendix D].

The issue in Appendix C (which, in any case, is somewhat orthogonal to the main paper) is very slightly different in that it does not involve the notion of equivalence of nilcharacters, but it could be resolved in exactly the same way. However, it is best to simply remove the problem by insisting on the tautness condition $G_0 = G_1$ as part of the definition of filtered nilmanifold ([2, Definition 4.1]).

3. A revised version of [2, Proposition 8.3]

Let $\chi \in \Xi^{(1,s-1)}_{\mathrm{Multi}}({}^*\mathbb{Z}^2)$. The claim of [2, Proposition 8.3] is that we can approximate χ in the uniform norm by χs which are 'linearised', the definition here being [2, Definition 8.1]. In [2], Proposition 8.3 is only used in the proof of Proposition 7.3. An inspection of the argument shows that the only consequence of the definition of [2, Definition 8.1] that we use is that, if χ is linearised, then one has a factorization

$$\begin{aligned}\chi(h_1, n) \otimes \chi(h_2, n + h_1 - h_4) \otimes \bar{\chi}(h_3, n) \otimes \bar{\chi}(h_4, n + h_1 - h_4)\\ = \psi(n) \otimes \psi(n + h_1 + h_4) \otimes \bar{\psi}(n) \otimes \bar{\psi}(n + h_1 - h_4) c_{h_1,h_2,h_3,h_4}(n) \qquad (3.1)\end{aligned}$$

for all $n \in \mathbb{Z}$, where $\psi \in L^\infty(\mathbb{Z} \to \bar{\mathbb{C}}^\omega)$ is a limit function and $c_{h_1,h_2,h_3,h_4} \in L^\infty(\mathbb{Z} \to \mathbb{C})$ is a degree $\leqslant s - 2$ nilsequence.

In this erratum, we do not establish (3.1), but rather the following related statement, which (by a minor modification of the arguments in [2, Section 8]) still suffices for the deduction of [2, Proposition 7.3].

Proposition 3.1. *Fix h, k. Then each component of $\chi(h + k, n) \otimes \overline{\chi}(h, n)$ can be approximated, up to arbitrarily small error in L^∞, by a finite sum of functions of the form $\psi_k(n)\psi'_{k,h}(n)$, where $\psi_k \in \Xi^{s-1}({}^*\mathbb{Z})$ and $\psi'_{k,h}(n)$ is a degree at most $(s-2)$ nilsequence in n, and the functions ψ_k and $\psi'_{k,h}$ as well as the number of terms K in the sum behave in a limit fashion on h, k.*

Proof. We first apply [2, Lemma E.8 (iv)] with $H = \mathbb{Z}^2$ having the multidegree filtration, and taking the shift h in that statement to be $(k, 0) \in {}^*\mathbb{Z}^2$. Note that $(k, 0) \in ({}^*\mathbb{Z}^2)_{(1,0)}$ (see [2, Definition 6.17]) and that $(1, 0) \succ (0, 0)$, so [2, Lemma E.8 (iv)] applies, and we conclude that χ and $\chi(\cdot+(k, 0))$ are equivalent in the sense of [2, Definition 6.22], that is to say $\chi(h+k, n)\otimes\overline{\chi}(h, n)$ is a nilsequence of degree strictly less than $(1, s - 1)$ in the multidegree ordering. The set of elements of $\mathbb{N}_0^2$ which are strictly less than $(1, s - 1)$ in the multidegree ordering is $J \cup J'$, where J is the elements of degree $\preceq (0, s-1)$, and J' the elements of degree $\preceq (1, s-2)$. Therefore (with notation as in [2, Definition 6.19], and with $\overline{\mathbb{C}}$ denoting the bounded elements of the nonstandard complex numbers $\mathbb{C}^*$, as in [2, Appendix A]), each component of $\chi(h + k, n) \otimes \overline{\chi}(h, n)$ lies in $\mathrm{Nil}^{\subset J\cup J'}({}^*\mathbb{Z}^2 \to \overline{\mathbb{C}})$.

Now we apply the splitting lemma [2, Lemma E.4] to this situation. This tells us that, up to an arbitrarily small error in L^∞, each component of $\chi(h + k, n) \otimes \overline{\chi}(h, n)$ can be written as a finite linear combination of expressions of the form $\psi_k(h, n)\psi'_k(h, n)$, where $\psi_k(h, n) \in \mathrm{Nil}^{\subset J}({}^*\mathbb{Z}^2 \to \overline{\mathbb{C}})$ and $\psi'_k(h, n) \in \mathrm{Nil}^{\subset J'}({}^*\mathbb{Z}^2 \to \overline{\mathbb{C}})$. Note in particular that $\psi_k(h, n)$ does not depend on h and $\psi_k(n) := \psi_k(h, n)$ lies in $\mathrm{Nil}^{s-1}({}^*\mathbb{Z} \to \overline{\mathbb{C}})$. By a vertical Fourier expansion (Lemma [2, E.5]) we may further assume that each $\psi_k(n)$ is a nilcharacter in $\Xi^{s-1}({}^*\mathbb{Z})$. Finally, note that for fixed h, $\psi'_k(h, n)$ (considered as a function of n only) lies in $\mathrm{Nil}^{s-2}({}^*\mathbb{Z} \to \overline{\mathbb{C}})$.

That ψ_k and $\psi'_{k,h}$ depend in a limit fashion on k, h follows from [2, Lemma A.11] (or one can proceed by inspection of the proof of Stone-Weierstrass). □

The deduction of [2, Proposition 7.3] from (the incorrect) [2, Proposition 8.3] is given in [2, Section 8], from the bottom of page 1275 to the top of page 1277. By fairly minor modifications of the argument, we now deduce [2, Proposition 7.3] using Proposition 3.1.

We start in the same way, that is to say by noting that, by hypothesis, for each $h \in H$ we can find a scalar nilsequence $\mathbf{c}_h$ of degree $\leqslant s-2$ such that

$$|\mathbb{E}_{n\in[N]}\Delta_h f(n)\overline{\chi(h,n)} \otimes \overline{\chi_h(n)\mathbf{c}_h(n)}| \gg 1,$$

where here recall the convention (in force from page 1251 onwards in [2]) that N is a fixed unbounded limit integer, and H is a dense subset of $[[N]] := [-N,N]$ (see [2, Appendix A] for the definitions here). By [2, Corollary A.12], we may ensure that ψ_h varies in a limit fashion on h, Applying [2, Corollary A.6], this lower bound is uniform in h.

It is convenient to write $\mathbf{c}_{\mathrm{params}}(n)$ for a scalar degree $\leqslant (s-2)$ nilsequence depending in a limit fashion on the parameters, which will vary over limit intervals such as $[N]$; different instances of $\mathbf{c}$ may denote different nilsequences.

Therefore there is some $b \in L^\infty[[N]]$ such that

$$|\mathbb{E}_{h\in[[N]]}\mathbb{E}_{n\in[N]}b(h)f(n+h)\overline{f(n)}\chi(h,n)\otimes\chi_h(n)\mathbf{c}_h(n)| \gg 1.$$

We may absorb b into $\mathbf{c}_h$. Applying Cauchy–Schwarz, we conclude

$$|\mathbb{E}_{h,h'\in[[N]]}\mathbb{E}_{n\in[N]}f(n+h)\overline{f(n+h')}\chi(h,n)\otimes\overline{\chi(h',n)}\otimes\chi_h(n)\otimes\overline{\chi_{h'}(n)}\mathbf{c}_{h,h'}(n)| \gg 1$$

Setting $h' = h+k$ (and approximating the cutoff $1_{h,h+k\in[[N]]}$ by a 1-step nilsequence, which may be absorbed into the $\mathbf{c}()$ term since $s \geqslant 3$) we obtain

$$\begin{aligned}|\mathbb{E}_{h,k\in[[2N]];n\in[N]}f(n+h)\overline{f(n+h+k)}(\chi(h,n)\otimes\overline{\chi(h+k,n)})\\ \otimes\chi_h(n)\otimes\overline{\chi_{h+k}(n)}\mathbf{c}_{h,k}(n)| \gg 1.\end{aligned}$$

Therefore there is some component $(\chi(h,n)\otimes\overline{\chi(h+k,n)})_i$ such that

$$\begin{aligned}|\mathbb{E}_{h,k\in[[2N]];n\in[N]}f(n+h)\overline{f(n+h+k)}(\chi(h,n)\otimes\overline{\chi(h+k,n)})_i\\ \chi_h(n)\otimes\overline{\chi_{h+k}(n)}\mathbf{c}_{h,k}(n)| \gg 1.\end{aligned} \tag{3.2}$$

Now we decompose the term $(\chi(h,n)\otimes\overline{\chi(h+k,n)})_i$ as in Proposition 3.1. From this decomposition, (3.2), and the pigeonhole principle there are scalar $\psi_k \in \Xi^{s-1}({}^*\mathbb{Z})$ and $\psi'_k \in \mathrm{Nil}^{(1,s-2)}({}^*\mathbb{Z}^2)$, varying in a limit fashion in k, such that

$$|\mathbb{E}_{h,k\in[[2N]];n\in[N]}f(n+h)\overline{f(n+h+k)}\psi_k(n)\psi'_k(h,n)\chi_h(n)\otimes\overline{\chi_{h+k}(n)}\mathbf{c}_{h,k}(n)| \gg 1.$$

Since, for fixed h, $\psi'_k(h,n)$ is degree $\leqslant (s-2)$ in n, the $\psi'_k(h,n)$ factor can be absorbed into the $\mathbf{c}()$ term, thus

$$|\mathbb{E}_{h,k\in[[2N]];n\in[N]}f(n+h)\overline{f(n+h+k)}\psi_k(n)\chi_h(n)\otimes\overline{\chi_{h+k}(n)}\mathbf{c}_{h,k}(n)| \gg 1.$$

Now we have $\psi_k(n) = \overline{\Delta_h\psi_k(n)}\psi_k(n+h)$, and by [2, Lemma E. 8 (iv)], $\Delta_h\psi_k(n)$ is a degree $\leqslant (s-2)$ nilsequence in n for each fixed h,k, and so may be absorbed into the $\mathbf{c}()$-term. Thus we obtain

$$|\mathbb{E}_{h,k\in[[2N]];n\in[N]}f(n+h)\overline{f(n+h+k)}\psi_k(n+h)\chi_h(n)\otimes\overline{\chi_{h+k}(n)}\mathbf{c}_{h,k}(n)| \gg 1$$

Making the change of variables $m = n+h$ (and extending f by zero outside of $[N]$, we obtain)

$$|\mathbb{E}_{h,k\in[[2N]];m\in[[3N]]}f(m)\overline{f(m+k)}\psi_k(m)\otimes\chi_h(m-h)\otimes\overline{\chi_{h+k}(m-h)}\mathbf{c}_{h,k}(m-h)| \gg 1.$$

Applying Cauchy–Schwarz again to eliminate the f and ψ_k factors, we conclude

$$|\mathbb{E}_{h,h',k\in[[2N]];m\in[[3N]]}\chi_h(m-h)\otimes\overline{\chi_{h+k}(m-h)} \otimes\overline{\chi_{h'}(m-h')}\otimes\chi_{h'+k}(m-h')\mathbf{c}_{h,h',k}(m)|\gg 1.$$

Making the substitution $h_1 := h$, $n := m-h$, $h_2 := h'+k$, $h_3 := h+k$, $h_4 := h'$ and rearranging the tensor product slightly, we obtain

$$|\mathbb{E}_{h_1,h_2,h_3,h_4\in[[4N]]:h_1+h_2=h_3+h_4}\mathbb{E}_{n\in[[5N]]} \chi_{h_1}(n)\otimes\chi_{h_2}(n+h_1-h_4)\otimes\overline{\chi_{h_3}(n)}\otimes\overline{\chi_{h_4}(n+h_1-h_4)}\mathbf{c}_{h_1,h_2,h_3,h_4}(n)|\gg 1,$$

where again we have approximated a cutoff (in this instance $1_{n+h\in[[3N]]}$) by a 1-step nilsequence and absorbed this into the $\mathbf{c}()$-term.

Note that the inner average vanishes unless h_1, h_2, h_3, h_4 lie in H. We thus conclude that for many additive quadruples (h_1, h_2, h_3, h_4) in H, that $\chi_{h_1}(n)\otimes\chi_{h_2}(n+h_1-h_4)\otimes\overline{\chi_{h_3}(n)}\otimes\overline{\chi_{h_4}(n+h_1-h_4)}$ is $(s-2)$-biased, which is what we were required to prove.

4. Minor issues in Sections 9–12

References in this section are all to [2]. The most serious issue here concerns the group G^* in Section 12.

The second paragraph after the bullets on page 1309 states

> Let G^* be the subgroup of $G^{\vec{D}_*+\vec{D}'}$ generated by $(m-1)$-fold iterated commutators $\tilde{e}_{i_1,j_1},\dots,\tilde{e}_{i_m,j_m}$ with $i_1+\cdots+i_m = s-1$ in which $j_\ell > D_{*,i_\ell}$ for at least two values of ℓ, or $j_\ell > D_{*,i_\ell} + D'_{\mathrm{lin}_{i_\ell}}$ for at least one value of ℓ. Then G^* is a subgroup of the central group $G^{\vec{D}_*+\vec{D}'}_{(s-1,r_*)}$ of $G^{\vec{D}_*+\vec{D}'}$ and $\tilde{G}$ is isomorphic to the quotient of $G^{\vec{D}_*+\vec{D}'}$ by G^*. We let $\tilde{\phi} : G^{\vec{D}_*+\vec{D}'}\to\tilde{G}$ denote the quotient map. From Theorem 11.1, the character $\eta : G^{\vec{D}_*+\vec{D}'}_{(s-1,r_*)}\to\mathbb{R}$ annihilates G^* and thus descends to a vertical character $\tilde{\eta} : \tilde{G}_{(s-1,r_*)}\to\mathbb{R}$.

This is mildly incorrect in a rather confusing way, and should read

> Let G^* be the subgroup of $G^{\vec{D}_*+\vec{D}'}$ generated by $(m-1)$-fold iterated commutators $\tilde{e}_{i_1,j_1},\dots,\tilde{e}_{i_m,j_m}$ in which $j_\ell > D_{*,i_\ell}$ for at least two values of ℓ, or $j_\ell > D_{*,i_\ell} + D'_{\mathrm{lin}_{i_\ell}}$ for at least one value of ℓ. Let $\tilde{\phi} : G^{\vec{D}_*+\vec{D}'}\to\tilde{G}$ denote the natural map with $\tilde{\phi}(e_{i,j}) := \tilde{e}_{i,j}$. Then $\ker\tilde{\phi} = G^*$, and so G^* is normal in $G^{\vec{D}_*+\vec{D}'}$ and $\tilde{G}$ is isomorphic to the quotient of $G^{\vec{D}_*+\vec{D}'}$ by G^*. From Theorem 11.1, the character $\eta : G^{\vec{D}_*+\vec{D}'}_{(s-1,r_*)}\to\mathbb{R}$ is trivial on $G^*\cap G^{\vec{D}_*+\vec{D}'}_{(s-1,r_*)}$ and thus, since $\tilde{G}_{(s-1,r_*)} = G^{\vec{D}_*+\vec{D}'}_{(s-1,r)}/(G^*\cap G^{\vec{D}_*+\vec{D}'}_{(s-1,r_*)})$, descends to a vertical character $\tilde{\eta} : \tilde{G}_{(s-1,r_*)}\to\mathbb{R}$.

Note that we have used the letter m as a dummy variable here, instead of r as in [2], which may have been the source of the confusion.

Finally, we record the following further minor but potentially confusing typos in Sections 9–12 and Appendix E of the paper.

- p1279, Definition 9.1, second bullet: r should be r_*.

- p1279, line 14: n should be m.
- Theorem 11.1 options (iii) and (iv) are (somewhat strangely!) redundant, being simply repeats of (i) and (ii), and not subsequently referenced.
- p1297, Lemma 10.8: $G_{s-1}^{\vec{D}'}$ here should be $G_{(s-1,r_*)}^{\vec{D}'}$.
- In Definition 9.11 and the proof of Lemma 9.12, $G_i = G_{(i,0)}$.
- In the definition of the degree-rank filtration on $\tilde{G}$ on page 1309, we should include all commutators with $i_1 + \cdots + i_{r'} > d$, without the additional restriction $r' \geqslant r$ in this case. Then the definition is compatible with the degree-rank filtration on $G^{\vec{D}+\vec{D}'}$ (see page 1279) in the sense that

$$\tilde{G}_{(d,r)} = G_{(d,r)}/(G_{(d,r)} \cap G^*).$$

- p1359, two instances after the fourth displayed equation: $F \otimes \overline{\tilde{F}}$ should be $F^{\otimes d!} \otimes \overline{\tilde{F}}$. Similarly in equation (E.3) on the same page, $\chi(n) \otimes \overline{\tilde{\chi}(n,\ldots,n)}$ should read $\chi(n)^{\otimes d!} \otimes \overline{\tilde{\chi}(n,\ldots,n)}$.

References

[1] B. J. Green and T. Tao, *The quantitative behaviour of polynomial orbits on nilmanifolds,* Ann. Math. **175** (2012), no. 2, 465–540.

[2] B. J. Green, T. Tao and T. Ziegler, *An inverse theorem for the Gowers $U^{s+1}[N]$-norm*, ANn. Math. **176** (2012), 1231–1372.

[3] J. Leng, A. Sah and M. Sawhney, *Quasipolynomial bounds on the inverse theorem for the Gowers $U^{s+1}[N]$-norm,* manuscript.

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