1. Let N be large and suppose that $A = \{1, \dots, N/10\}$, considered as a subset of $[N]$. Find (in terms of N) a value of $\theta \in \mathbb{R}/\mathbb{Z}$ such that $|\hat{f}_A(\theta)| \geqslant N/2$.

2. Show that if A is a finite subset of an abelian group G then $|A + A| = |A|$ if and only if A is a coset of a (finite) subgroup of G.

3. Show that if A is a finite set of integers then $|2A| \geqslant 2|A| - 1$. When does equality occur?

4. The van der Waerden number $W(r, k)$ is the least N_0 such that, if $N \geqslant N_0$ and $[N]$ is coloured with r colours, one of the colour classes contains k elements in arithmetic progression. Prove that $W(r, 3) \leqslant e^{e^{Cr}}$ for some constant C and all r. Prove also that $W(r, 3) \geqslant e^{c(\log r)^2}$ for some constant $c > 0$ and all r.

5. Suppose that A is a finite subset of an abelian group and that $|A+A| \leqslant K|A|$. Show that there is a polynomial p (which may depend on K, but not on A) such that $|kA| \leqslant p(k)|A|$ for all integers k.

6. The aim of this exercise is to show that there is an absolute constant $c > 0$ such that, for every K, there are arbitrarily large sets $A \subset \mathbb{Z}$ with $|2A| \leqslant K|A|$ and $|3A| \geqslant cK^3|A|$.

(i) Show that it is enough to construct a set $A \subset \mathbb{Z}^3$ with these properties. We may also assume that $K > 100$, the result being trivial (for sufficiently small c) otherwise.

(ii) Let x be a large integer, and set

$$A = [x]^3 \cup \{(j, 0, 0) : j \in [Kx/4]\} \cup \{(0, j, 0) : j \in [Kx/4]\} \cup \{(0, 0, j) : j \in [Kx/4]\}.$$

Show that $|A + A| \leqslant K|A|$ if x is sufficiently large.

(iii) Show, however, that $|3A| \geqslant cK^3|A|$.

Can you generalise this construction to get a lower bound for $|kA|$?

7. Suppose that A is a set of integers and that $|2A| \geqslant 100|A|$. Is it true that $|3A| \geqslant 1000|A|$?

8. Suppose that A is a bounded open subset of $\mathbb{R}^d$. Show that $\mathrm{vol}(A+A) \geqslant 2^d \mathrm{vol}(A)$.

9. Construct a set $A \subset \{1, \dots, 10^5\}$ with no three elements in arithmetic progression, and with $|A| > 2012$.

10. Give examples of each of the following:

(i) a set A of integers with $|A+A| < |A-A|$. Is it true that there is an absolute constant K such that $|A+A| \leqslant K|A-A|$ for all sets of integers?

(ii) arbitrarily large sets of integers A, B such that $|A+B| \leqslant 1.01|A|$ and $|A-B| > |A|^{1.01}$;

(iii) arbitrarily large sets of integers A, B such that $|A+B| \leqslant 1.01|A|$ but $|A+2B| > |A|^{1.01}$;

(iv) arbitrarily large sets of integers A, B such that $|A+B| > |A|^{1.01}$ but $|A+2B| \leqslant 1.01|A+B|$;

(v) a set A of integers with $|A-A| < |A+A|$.

11. Let $0 < \alpha < 1$. Show that for every C there is $N_0 = N_0(\alpha, C)$ such that the following is true. If $N \geqslant N_0$, there exists a set $A \subset [N]$ with $|A| = \alpha N$ containing fewer than $\alpha^C N^2$ three-term arithmetic progressions.

12. Let $A \subset \mathbb{R}^d$ be a finite set which is symmetric (that is, $-x \in A$ if $x \in A$), contains 0, and is not contained in any proper subspace of $\mathbb{R}^d$.

(i) Explain why there is a nested sequence of subspaces $0 < V_1 < V_2 < \cdots < V_{d-1} < V_d = \mathbb{R}^d$ such that $A \cap V_{i+1} \neq A \cap V_i$ for all $i < d$.

(ii) Write $A_i := 2A \cap V_i$. Show that $2A_{i+1} \subsetneq A_{i+1} + V_i$.

(iii) For each i, choose $h \in 2A_{i+1} \setminus (A_{i+1} + V_i)$. Show that the sets $A + h_1, \dots, A + h_d$ are disjoint and conclude that $|5A| \geqslant d|A|$.

13. The aim of this exercise is to show that random functions $f : [N] \to \{-1, 1\}$ have no large Fourier coefficients. To begin with, fix $\theta \in \mathbb{R}/\mathbb{Z}$. Show that $\mathbb{E}\hat{f}(\theta) = 0$. Show also that $\mathbb{E}|\hat{f}(\theta)|^6 \ll N^3$. Conclude that

$$\mathbb{P}(|\hat{f}(\theta)| \geqslant \delta N) \ll 1/\delta^6 N^3.$$

Show that if $|\hat{f}(\theta)| \leqslant \delta N$ then $|\hat{f}(\theta')| \leqslant 2\delta N$ whenever $|\theta' - \theta| \leqslant \delta / N^2$. Conclude that $\sup_\theta |\hat{f}(\theta)| \leqslant N^{0.9}$ with probability tending to 1 as $N \to \infty$.

*By looking up the literature on Hoeffding's inequality and related matters, find a function ϕ such that $\phi(N) \leqslant \sup_\theta |\hat{f}(\theta)| \leqslant C\phi(N)$ almost surely as $N \to \infty$.

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