1. Let A be a finite set of n integers. Show that $E(A, A) \geqslant \binom{n+1}{2}$, and that equality can occur. What is the largest possible value of $E(A, A)$?

2. Let $B(R, \varepsilon)$ be a Bohr set in $\mathbb{Z}/N\mathbb{Z}$, where $R = \{r_1, \dots, r_d\}$. Show, without using the geometry of numbers, that $B(R, \varepsilon)$ contains a (one-dimensional) progression P of length at least $\varepsilon N^{1/d}$. Convince yourself that by using this result one may establish the (local) inverse theorem for the U^3-norm without appealing to the geometry of numbers.

3. Define a function $f : [N] \to \mathbb{C}$ as follows. Set $M := \lfloor \sqrt{N} \rfloor$. Define $f(x + My) = e^{2\pi i (x^2+y^2)\sqrt{2}}$ if $1 \leqslant x, y \leqslant M/4$, and 0 otherwise. Show that $\|f\|_{U^3(N)} \geqslant c$, for some constant $c > 0$ not depending on N.

4. Suppose that a set A of integers contains at least $c|A|^2$ 3-term arithmetic progressions. Show that $E(A, A) \geqslant c'|A|^3$, where $c' > 0$ depends only on c. Using results from the course, conclude that A contains a 4-term progression.

5. Suppose that a set A has size n and additive energy cn^3. Show that A contains at least $c^3 n^7$ octuples $(a_1, a_2, \dots, a_8)$ with $a_1 + a_2 + a_3 + a_4 = a_5 + a_6 + a_7 + a_8$.

6. Let $c > 0$, and suppose that $N > N_0(c)$ is a large positive integer. Give an example of a function $\phi : [N] \to \mathbb{R}$ with the property that there are at least $N^3/100$ quadruples $h_1, h_2, h_3, h_4 \in [N]$ with $h_1 + h_2 = h_3 + h_4$ and $\phi(h_1) + \phi(h_2) = \phi(h_3) + \phi(h_4)$, but ϕ does not agree with a linear function $\alpha h + \beta$ on any set of size cN.

7. Consider the function $\chi_+(x)$ defined by $\chi_+(x) = 1$ if $x \in [\frac{1}{3}, \frac{2}{3}]$, $\chi_+(x) = 0$ if $x \notin [\frac{1}{3} - \varepsilon, \frac{2}{3} + \varepsilon]$, and χ_+ is linear on the intervals $[\frac{1}{3} - \varepsilon, \frac{1}{3}]$ and $[\frac{2}{3}, \frac{2}{3} + \varepsilon]$. Show that the Fourier coefficients $\hat{\chi}_+(m)$ decay like $1/|m|^2$ as $|m| \to \infty$. Construct a similar function $\chi_-(x)$, and convince yourself that these functions may be used to give an alternative proof of the equidistribution theorem.

8. Using Roth's theorem on 3-term progressions, conclude that there is a sequence of positive integers n such that $\|n^2\sqrt{2}\|_{\mathbb{R}/\mathbb{Z}} \to 0$.

9. Consider the following bipartite graph. Take $V = W = \mathcal{P}([n])$, the set of all subsets of $[n]$, and join $v \in V$ to $w \in W$ if $|v \triangle w| \leqslant n/2$. Suppose that V' is such that between any pair of points $v_1, v_2 \in V$ there are at least $\varepsilon 2^n$ paths $v_1 \to w \to v_2$ of length 2. Show that $\varepsilon \to 0$ as $n \to \infty$.

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