CHAPTER 8

The sum-product phenomenon in $\mathbb{C}$

In this section we give some extremely elegant and (in retrospect!) simple arguments of Solymosi establishing sum-product phenomena in fields with a metric structure. We focus on $\mathbb{R}$ and $\mathbb{C}$.

THEOREM 8.1 (Solymosi). *Suppose that A is a finite set of complex numbers. Then $|A + A| + |A \cdot A| \geqslant c|A|^{5/4}$.*

Essentially the only property of the field $\mathbb{C}$ that is relevant for Solymosi's argument is the so-called *Besicovitch property.*

DEFINITION 8.1 (Besicovitch constant). Suppose that (X, d) is a metric space. The Besicovitch constant of X (if it is defined) is the largest k such that there exist balls $B_i = B(x_i, r_i)$, $i = 1, \dots, k$ with the property that x_i is never in the interior of B_j if $i \neq j$, and such that $\bigcap_{i=1}^k B_i$ is nonempty.

LEMMA 8.1 (Besicovitch constant of $\mathbb{C}$). *The Besicovitch constant of $\mathbb{C}$ is 6.*

Proof. This is simple Euclidean geometry exercise. Suppose that $B_i = B(x_i, r_i)$, $i = 1, \dots, 7$, are balls intersecting in some point z. Suppose that the centres $x_1, \dots, x_7$ are arranged in order, radially about z. The angles $x_1 z x_2, x_2 z x_3, \dots, x_7 z x_1$ must be at least $\pi/3$ since the distance $|x_i - x_{i+1}|$ is greater than or equal to both $|x_i - z|$ and $|x_{i+1} - z|$. This is obviously a contradiction. $\square$

Proof of Theorem 8.1. Suppose that $A \subseteq \mathbb{C}$ is a finite set and that the additive doubling $\sigma_+[A]$ and the multiplicative doubling $\sigma_\times[A]$ are both at most K. Our aim is to show that $K \geqslant c|A|^{1/4}$.

To each point $a \in A$ associate the nearest neighbour a^* of a in $A \setminus \{a\}$, making an arbitrary choice if there are ties to be broken. To motivate the proof, suppose that the following (false) assumption held: for any triple $(a_1, a_2, a_3) \in A \times A \times A$ the unique nearest neighbour to $a_1 + a_2$ in $A + A$ is $a_1^* + a_2$, and the unique nearest neighbour to $a_1 a_3$ in $A \cdot A$ is $a_1^* a_3$. We could then consider the map

$$\psi : A \times A \times A \to (A + A) \times (A + A) \times (A \cdot A) \times (A \cdot A)$$

defined by $\phi(a_1, a_2, a_3) = (a_1 + a_2, a_1^* + a_2, a_1 a_3, a_1^* a_3)$. Now it is an easy algebraic exercise to see that this map is injective. Furthermore, by our false assumption, knowledge of $a_1 + a_2$ and $a_1 a_3$ tells us the values of $a_1^* + a_2$ and $a_1^* a_3$, which means that $\mathrm{im}(\phi) \leqslant |A + A||A \cdot A|$. We would then have $|A + A||A \cdot A| \geqslant |A|^3$, a much stronger result than the one we have claimed.

The problem, of course, is our false assumption. It turns out that something a little weaker *is* true: for many triples (a_1, a_2, a_3) there are not many points of $A + A$ closer to $a_1 + a_2$ than $a_1^* + a_2$, and not many points of $A \cdot A$ closer to $a_1 a_3$ than $a_1^* a_3$. More precisely we will examine *well-behaved* triples (a_1, a_2, a_3) for which $a_1^* + a_2$ is "almost " the nearest neighbour of $a_1 + a_2$ in $A + A$ in the sense that

$$U_{a_1,a_2} := |\{u \in A + A : |a_1^* + a_2 - u| \leqslant |(a_1^* + a_2) - (a_1 + a_2)|\}| \leqslant 100K \tag{8.1}$$

and for which $a_1^* a_3$ is "almost" the nearest neighbour of $a_1 a_3$ in the sense that

$$V(a_1, a_3) := |\{v \in A \cdot A : |a_1^* a_3 - v| \leqslant |a_1^* a_3 - a_1 a_3|\}| \leqslant 100K. \tag{8.2}$$

It is not obvious that there *are* such triples, but we claim that this good behaviour is quite generic: there are at least $|A|^3/2$ such triples.

Examining (8.1) in the first instance, fix a_2. Then the balls $B_{|a_1^* - a_1|}(a_1 + a_2)$, $a_1 \in A$, have Besicovitch's intersection property. It follows that no u can lie in 7 of them. It follows that

$$\sum_{a_1} U_{a_1,a_2} \leqslant 6|A + A| \leqslant 6K|A|.$$

An essentially identical argument using (8.2) implies that

$$\sum_{a_1} V_{a_1,a_3} \leqslant 6K|A|.$$

The number of pairs (a_1, a_2) for which $U_{a_1,a_2} \geqslant 100K$ is thus at most $|A|^2/10$, as is the number of pairs (a_1, a_3) for which $V_{a_1,a_3} \geqslant 100K$. The claim follows.

Now suppose that $x = a_1 + a_2$, $y = a_1^* + a_2$, $z = a_1 a_3$ and $w = a_1^* a_3$ are known. The same simple algebraic exercise as before confirms that a_1, a_1^*, a_2 and a_3 may be recovered from knowledge of x, y, z and w, and hence by the claim just proved the number of choices for the quadruple (x, y, z, w) such that (a_1, a_2, a_3) constitute a well-behaved triple is at least $|A|^3/2$. Now there are $|A + A|$ ways to specify x and $|A \cdot A|$ ways to specify z. Suppose these have been chosen, and consider the possible choices of y. Single out one $\overline{y}$ corresponding to a well-behaved triple $(\overline{a}_1, \overline{a}_2, \overline{a}_3)$ with $|x - \overline{y}| = |\overline{a}_1^* - \overline{a}_1|$ maximal. Then for all permissible y we have

$$|\overline{a}_1 + \overline{a}_2 - y| = |x - y| \leqslant |x - \overline{y}| = |\overline{a}_1^* - \overline{a}_1|.$$

By the definition of well-behaved triple, and specifically in view of (8.1), there are at most $100K$ choices for y. Similarly there are at most $100K$ choices for w. It follows that

$$|A|^3/2 \leqslant |A + A| \cdot |A \cdot A| \cdot (100K)^2 \leqslant 10^4 K^4 |A|^2,$$

from which the result follows immediately. □