Two slightly tricky statements about continued fractions

Let α be a real number, and let $[a_0, a_1, a_2, \dots]$ be its continued fraction expansion. Write $\frac{p_k}{q_k}$ for the convergents. To avoid annoyances, I'll assume that this expansion is infinite. This is so if and only if α is irrational. The less obvious direction is the *only if* direction, which is the statement that the continued fraction expansion of a rational number is finite. To see this, suppose that $\alpha = \frac{p}{q}$ is rational and has an infinite continued fraction expansion. Certainly, for each k, we have $\frac{p}{q} \neq \frac{p_k}{q_k}$, and so

$$|\alpha - \frac{p_k}{q_k}| = |\frac{p}{q} - \frac{p_k}{q_k}| \geqslant \frac{1}{qq_k}.$$

However we also know that $|\alpha - \frac{p_k}{q_k}| \leqslant \frac{1}{q_k^2}$. Hence $q_k \leqslant q$ for all k, a contradiction.

We proved in lectures that

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < \alpha < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Also, remember, we have

$$|\alpha - \frac{p_k}{q_k}| < \frac{1}{q_k q_{k+1}}$$

for all k.

We will make very frequent use of the following observation: if $\frac{a}{b}$ and $\frac{a'}{b'}$ are distinct fractions in lowest terms, then

$$|\frac{a}{b} - \frac{a'}{b'}| = \frac{|ab' - a'b|}{bb'} \geqslant \frac{1}{bb'}.$$

Now down to business. We first prove that convergents to α are record approximants in a certain sense.

Theorem 1. *Let α be an irrational real number and let $\frac{p_i}{q_i}$ be the convergents to α. Suppose that k is odd. Then any rational in the interval $(\frac{p_{k-1}}{q_{k-1}}, \frac{p_k}{q_k})$ has denominator greater than q_k. A similar statement holds when k is even.*

Proof. Suppose that some rational $\frac{p}{q}$ in lowest terms lies in the interval stated and that $q \leqslant q_k$. We split into two cases.

Case 1. $\alpha < \frac{p}{q} < \frac{p_k}{q_k}$. Then

$$|\frac{p_k}{q_k} - \frac{p}{q}| \geqslant \frac{1}{qq_k} > \frac{1}{q_k q_{k+1}} > |\alpha - \frac{p_k}{q_k}|,$$

contradiction.

Case 2. $\frac{p_{k-1}}{q_{k-1}} < \frac{p}{q} < \alpha$. Then

$$|\frac{p_{k-1}}{q_{k-1}} - \frac{p}{q}| \geqslant \frac{1}{qq_{k-1}} \geqslant \frac{1}{q_k q_{k-1}} > |\alpha - \frac{p_{k-1}}{q_{k-1}}|,$$

contradiction. □

Lemma 1. *For all k we have $|\alpha - \frac{p_{k-1}}{q_{k-1}}| > |\alpha - \frac{p_k}{q_k}|$.*

Proof. Since $\frac{p_{k-1}}{q_{k-1}}$ and $\frac{p_k}{q_k}$ lie on opposite sides of α, it is enough to show that $|\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k}| > 2|\alpha - \frac{p_k}{q_k}|$. However we have $|\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k}| \geqslant \frac{1}{q_{k-1}q_k}$ and $|\alpha - \frac{p_k}{q_k}| < \frac{1}{q_k q_{k+1}}$, and so it suffices to show that $q_{k+1} \geqslant 2q_{k-1}$. This follows from the fact that $q_{k+1} = a_{k+1}q_k + q_{k-1}$. □

Putting these facts together, we immediately obtain the following result.

Theorem 2. *Suppose that α is an irrational number and that $\frac{p_k}{q_k}$ is a convergent to α. Then the only rational $\frac{p}{q}$ with $q \leqslant q_k$ for which $|\alpha - \frac{p}{q}| \leqslant |\alpha - \frac{p_k}{q_k}|$ is $\frac{p_k}{q_k}$ itself.*

Secondly, we prove that all good approximants to α are convergents.

Theorem 3. *Suppose that $\frac{p}{q}$ is a fraction in lowest terms and that $|\alpha - \frac{p}{q}| \leqslant \frac{1}{2q^2}$. Then $\frac{p}{q} = \frac{p_k}{q_k}$ for some k.*

Proof. Since the denominators q_k are increasing, we may select a unique k such that $q_k \leqslant q < q_{k+1}$. Suppose that $\frac{p_k}{q_k} < \alpha < \frac{p_{k+1}}{q_{k+1}}$ (the other case is very similar).

First note that if $\frac{p}{q} < \frac{p_k}{q_k}$ then

$$|\alpha - \frac{p}{q}| > |\frac{p}{q} - \frac{p_k}{q_k}| \geqslant \frac{1}{qq_k} \geqslant \frac{1}{q^2},$$

contrary to assumption.

If $\frac{p}{q} = \frac{p_k}{q_k}$ then we are done. Suppose, then, that $\frac{p_k}{q_k} < \frac{p}{q}$. We have

$$|\frac{p}{q} - \frac{p_k}{q_k}| \geqslant \frac{1}{qq_k} > \frac{1}{q_k q_{k+1}} > |\alpha - \frac{p_k}{q_k}|,$$

so in fact $\frac{p_k}{q_k} < \alpha < \frac{p}{q}$. By the previous theorem, we cannot have $\frac{p}{q} \leqslant \frac{p_{k+1}}{q_{k+1}}$, and hence $\frac{p_k}{q_k} < \alpha < \frac{p_{k+1}}{q_{k+1}} < \frac{p}{q}$. We now divide into two cases.

Case 1. (q large). Suppose that $q \geqslant \frac{1}{2}q_{k+1}$. Then

$$|\alpha - \frac{p}{q}| > |\frac{p}{q} - \frac{p_{k+1}}{q_{k+1}}| \geqslant \frac{1}{qq_{k+1}} \geqslant \frac{1}{2q^2},$$

contrary to assumption.

Case 2. (q small). Suppose that $q < \frac{1}{2}q_{k+1}$. Then

$$|\alpha - \frac{p}{q}| = |\frac{p}{q} - \frac{p_k}{q_k}| - |\alpha - \frac{p_k}{q_k}| \geqslant \frac{1}{qq_k} - \frac{1}{q_k q_{k+1}} = \frac{1}{q_k}(\frac{1}{q} - \frac{1}{q_{k+1}}) > \frac{1}{2q^2},$$

also a contradiction.

□