A FORWARD EQUATION FOR BARRIER OPTIONS UNDER THE BRUNICK&SHREVE MARKOVIAN PROJECTION

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Abstract. We derive a forward equation for arbitrage-free barrier option prices, in terms of Markovian projections of the stochastic volatility process, in continuous semi-martingale models. This provides a Dupire-type formula for the coefficient derived by Brunick and Shreve for their mimicking diffusion and can be interpreted as the canonical extension of local volatility for barrier options. Alternatively, a forward partial-integro differential equation (PIDE) is introduced which provides up-and-out call prices, under a Brunick-Shreve model, for the complete set of strikes, barriers and maturities in one solution step. Similar to the vanilla forward PDE, the above-named forward PIDE can serve as a building block for an efficient calibration routine including barrier option quotes. We provide a discretisation scheme for the PIDE as well as a numerical validation.

1. Introduction

Efficient pricing and hedging of exotic derivatives requires a model which is able to re-price accurately a range of liquidly traded market products. The case of calibration to vanillas is now widely documented and has been considered extensively in the literature since the work of Dupire [13] in the context of local volatility models (also known as the Gyöngy formula [23]). Nowadays, various sophisticated calibration techniques are in use in the financial industry; for example the work of Guyon and Labordère [22] as well as Ren, Madan and Qian [30]. The exact re-pricing of call options is then a must-have standard. However, practitioners are increasingly interested in taking into account the quotes of touch and barrier options as well; the extra information they embed can be valuable in obtaining arbitrage-free values of exotic products with barrier features.

A few published works already address this question from different angles and under different assumptions. For example, in Crosby and Carr [6], a particular class of models gives a calibration to both vanillas and barriers. Model-independent bounds on the price of double no-touch options where inferred from vanilla and digital option quotes in Cox and Obloj [8]. Pironneau [27] proves that the Dupire

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equation is still valid for a given barrier level, in a local volatility setting. The direct generalisation of this result to general stochastic volatility models appears not to be straightforward.

In our work, we approach the problem from the Brunick-Shreve mimicking point of view in the general framework of continuous stochastic volatility models. We derive a condition to be verified by the expectation of the stochastic variance conditional on the spot and its running maximum, (S_t, M_t), in order to reproduce barrier prices. This conditional expectation is often referred to as a Markovian projection. For simplicity, we focus on up-and-out call options with no rebate and continuous monitoring of the barrier, but the analysis extends to other payoff types. The derivation is inspired by the work of Derman, Kani [11] and Dupire [13, 14].

Gyöngy's mimicking result [23] provides the financial engineer with a recipe to build a low-dimensional Markovian process which reproduces exactly any marginal density of the spot price process for all times t. Brunick and Shreve [4] extend the result to the joint density of the spot and its running maximum. In particular they prove, for any stochastic volatility process, the existence of a mimicking low-dimensional Markovian process with the same joint-density for (S_t, M_t). A very recent paper by Guyon [21] explains how path-dependant volatility models, like the Brunick-Shreve one, may be very useful to replicate a market's spot-volatility dynamics, in particular highlighting the running maximum. In [16], Forde presents a way to retrieve the mimicking coefficient by deriving a forward equation for the characteristic function of (S_t, M_t); computation of the coefficient is possible via an inverse two-dimensional Fourier-Laplace transform. Here, we present an alternative method to retrieve the mimicking coefficient, which lies closer to the well-known Dupire formula and can hence benefit from the earlier work in the field of vanilla calibration.

We derive a partial-integro differential equation (PIDE) in strike, barrier level and maturity for up-and-out calls priced under the Brunick-Shreve model. This forward differential equation has the same useful properties (and drawbacks) as the Dupire forward PDE. As a consequence, it can be used to price a set of up-and-out calls in one single resolution. In this regard, our method shares some similarities with the forward equations derived by Carr and Hirsa [7]. We highlight a few key differences though.

First, we consider the class of stochastic volatility models rather than local volatility models with a jump term. In working directly with Brunick&Shreve's Markovian projection onto (S_t, M_t), we need to consider the running maximum explicitly in our derivation. This complicates the proof and makes the idea used in [7] – employing stopping times – not applicable in our case. For the above type of Markovian projection, one gets an unusual "Integro" term in the PIDE involving a second derivative, which requires particular care when solved numerically. In contrast, the "Integro" term in [7] comes from the jump process and is not treated in the same way as ours.

It is important to note that the diffusion of interest can be a fairly general stochastic process. More specifically, the variance process does not need to contain the running maximum in its parametrisation, and we do not particularly advocate the dynamic use of such a model here. Indeed, by doing so, one may find oneself with the logical conundrum that the model for the underlying depends on the time of inception of the *option*, i.e., the time the clock starts for the running maximum. The view we take here is that the Brunick-Shreve projection is a "code book" (to borrow a term from [5]) for barrier option prices, to which other models may be calibrated. Additionally, the forward PIDE enables in principle the pricing over a wide set of up-and-out call deals, creating a possible efficient direct solver for the inverse problem, i.e., to retrieve model parameters from any desired model class via the projected volatility $\sigma(S_t, M_t, t)$.

The remainder of this paper is organised as follows. In Section 2, we introduce the modelling setup and hypotheses and derive, as our first main result, a forward equation (in terms of the maturity) for barrier option prices, where the strike and barrier levels are spatial variables. Next, in Section 3, we derive a Dupire-type formula for barrier options, leading to a known setup for the reader familiar with vanilla calibration. In order to build the first step of a calibration routine to reprice up-and-out call options, we deduce a forward PIDE in Section 4 with better stability properties and develop a numerical solution scheme. Finally, in Section 5, we show that the forward PIDE and the "classical" backward pricing PDE agree on the price of barrier options, which validates our approach. Section 6 concludes.

2. Setup and Main Result

We consider a market with a risk-free, deterministic and possibly time-dependent short rate, $r(t)$ at time t, and continuously compounded deterministic dividend $q(t)$. Then $D(t) = \exp(-\int_0^t r(u)\,du)$ is the discount factor, and $M^q(t) = \exp(\int_0^t q(u)\,du)$ is the dividend capitalisation.

We assume the existence of a filtered probability space $\left(U, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q}\right)$ with a (not necessarily unique) risk-neutral measure $\mathbb{Q}$, under which the price process of a risky asset follows

$$\frac{dS_t}{S_t} = (r(t) - q(t))\,dt + \alpha_t\,dW_t, \tag{2.1}$$

where W is a standard Brownian motion and α_t is a continuous and positive $\mathcal{F}_t$-adapted semi-martingale such that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \alpha_u^2 S_u^2 du\right] < \infty. \tag{2.2}$$

A practically important example is $\alpha_t = f(S_t, t)\sqrt{V_t}$ with a *local volatility* function f and a CIR process V, which is from a class of models often referred to as *local stochastic volatility* (LSV) models.

In this paper, we consider an up-and-out call option on S with continuous barrier monitoring, with strike K, barrier B and maturity T. The arbitrage-free price under $\mathbb{Q}$ is

$$C(K, B, T) = \mathbb{E}^{\mathbb{Q}}\left[D(T)\left(S_T - K^+\right)\mathbf{1}_{M_T<B}\right],$$

where $\mathbf{1}_\omega$ is the indicator function of event ω and

$$M_t = \max_{0\leq u\leq t} S_u$$

the running maximum process of S. Adaptations of our results to other types of barrier options (such as put payoffs, down-and-out barriers etc) are easily possible using a similar derivation to below, while any generalisation to discretely monitored barriers would look substantially different.

We now derive the first main result which links the barrier option price to the Markovian projection of the stochastic volatility (2.1) onto the spot and its running maximum $(S_t, M_t)_{t\geq 0}$, which we define as

$$\sigma^2_{|S,M}(K, B, T) = \mathbb{E}^{\mathbb{Q}}\left[\alpha_T^2 \mid S_T = K, M_T = B\right]. \tag{2.3}$$

A recent result by Brunick&Shreve [4] shows that under growth conditions (2.2) on α, the function $\sigma_{|S,M}$ exists and is measurable. It also gives existence of a one-dimensional "mimicking" process $\widehat{S}$ on a suitable probability space, with running maximum $\widehat{M}$, such that for all t

$$(S_t, M_t) \overset{law}{=} (\widehat{S}_t, \widehat{M}_t), \tag{2.4}$$

where $\widehat{S}$ is the weak solution of

$$\frac{d\widehat{S}_t}{\widehat{S}_t} = (r(t) - q(t))\, dt + \sigma\left(\widehat{S}_t, \widehat{M}_t, t\right) d\widehat{W}_t, \tag{2.5}$$

with a standard Brownian motion $\widehat{W}$ and

$$\sigma(K, B, T) = \sigma_{|S,M}(K, B, T).$$

Remark. We can take two viewpoints of pricing under (2.5). First, we can think of σ in (2.5) as a model in its own right, with a local volatility which additionally depends on the running maximum – see [21] for a discussion of "path-dependent" volatility models. This extra flexibility of matching the path-dependence of the local volatility allows calibration to barrier contracts as well as vanillas, yet keeps the market complete. Second, and more common, a different volatility model may be used, for instance the aforementioned LSV model. In that case, if $\sigma^2(K, B, T)$ equals $\sigma^2_{|S,M}(K, B, T)$ for all K, B and T, then the mimicking Brunick-Shreve model will give the same barrier option prices as the higher-dimensional diffusion (as barrier option values are characterised precisely by the joint distribution of spot and running maximum). This behaviour is again of importance for calibration purposes. In either case, σ plays a similar role for barrier options as the Dupire local volatility does for vanillas. In this spirit, our main result below, Theorem 1, extends the Dupire formula to the barrier case.

Assumption. *We assume that the Markovian process (S_t, M_t) – or, equivalently, $(\widehat{S}_t, \widehat{M}_t)$ – started at (S_0, M_0) at time 0, has a differentiable transition density ϕ under $\mathbb{Q}$ in the region $\Omega = \{(x, y) : 0 < x < y, S_0 < y\}$, such that it is the weak solution of the Kolmogorov forward equation*

$$\frac{\partial \phi}{\partial t} + (r(t) - q(t))\frac{\partial}{\partial x}(x\phi) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2 x^2 \phi\right) = 0, \qquad (x, y) \in \Omega,\ t > 0, \tag{2.6}$$

$$2\frac{\partial}{\partial x}\left(\sigma^2 x^2 \phi\right) + \frac{\partial}{\partial y}\left(\sigma^2 x^2 \phi\right) = (r(t) - q(t))\ x\ \phi, \qquad x = y,\ t > 0, \tag{2.7}$$

$$\phi(S_0, S_0, t) = 0, \qquad t > 0,$$

$$\phi(x, y, 0) = \delta(x - S_0)\delta(y - S_0), \qquad (x, y) \in \Omega.$$

For completeness, we give a formal derivation of these equations under the assumption of smoothness in Appendix A. This initial-boundary value problem is the adjoint to the backward equation satisfied by the option value C as a function of S_t, M_t and t, which has a homogeneous Neumann boundary condition on the diagonal $S = M$.

Remark. The assumption of smoothness of the joint density is non-trivial. For the Black-Scholes model, with constant σ, the joint density function is known explicitly and smooth. In a recent work, [17] show the existence of the joint density – not necessarily differentiable – of an Itô process X and its running minimum $\underline{X}$, where the local volatility is a sufficiently smooth and suitably bounded function of X_t and $\underline{X}_t$. We also note that although the density features in our proof, it does not appear in the final result itself, and we conjecture that the regularity assumption may be weakened. Nonetheless, forward equations involving the transition density are widely used for calibration in practice and we envisage that (2.6) can also be a useful building block for calibration in the present framework. Therefore, it seems reasonable to require that the model has enough regularity for the density to satisfy such an equation.

The time zero value of barrier options can be written as

$$\begin{aligned} C(K,B,T) &= D(T) \int_{S_0}^{\infty} \int_0^{\infty} (x-K)^+ \mathbf{1}_{y<B}\, \phi(x,y,T)\, \mathrm{d}x\, \mathrm{d}y \\ &= D(T) \int_{K\vee S_0}^{B} \int_K^{y} (x-K)\, \phi(x,y,T)\, \mathrm{d}x\, \mathrm{d}y. \end{aligned} \tag{2.8}$$

We can now extend the argument from Dupire [13] for European calls to derive a forward equation for barriers.

Theorem 1. *Under the above assumptions on the process given by (2.1), the value $C(K,B,T)$ of an up-and-out barrier call satisfies, for all $0<K<B$ and $T>0$,*

$$\begin{aligned} \frac{\partial^2 C(K,B,T)}{\partial B \partial T} + (r(T)-q(T))\,K \frac{\partial^2 C(K,B,T)}{\partial K \partial B} &= \frac{1}{2}\sigma^2_{|S,M}(K,B,T)\,K^2 \frac{\partial^3 C(K,B,T)}{\partial K^2 \partial B} - q(T)\frac{\partial C(K,B,T)}{\partial B} \\ &\quad -\frac{1}{2}\frac{\partial \left((B-K)\,B^2 \sigma^2_{|S,M}(B,B,T) \left.\frac{\partial^3 C(K,b,T)}{\partial K^2 \partial b}\right]_{K=B,b=B} \right)}{\partial B}, \end{aligned} \tag{2.9}$$

where $\sigma_{|S,M}$ is given by (2.3).

Proof. We assume $r=q=0$ for the time being and return to the general case at the end. Because of Brunick&Shreve's mimicking result, (2.4) and (2.5), and the smoothness assumption, we can work with a density satisfying the forward equation (2.6) and (2.7). Differentiating (2.8) with respect to B and T in the first line, using (2.6) in the second, and integrating by parts in the third,

$$\begin{aligned} \frac{\partial^2 C}{\partial T \partial B} &= \int_K^B (x-K)\frac{\partial \phi}{\partial t}(x,B,T)\,\mathrm{d}x \\ &= \int_K^B (x-K)\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x,B,T)x^2\phi(x,B,T)\right)\,\mathrm{d}x \\ &= \frac{1}{2}\left(\sigma^2(K,B,T)K^2\phi(K,B,T) - \sigma^2(B,B,T)B^2\phi(B,B,T)\right) + \frac{1}{2}(B-K)\left[\frac{\partial}{\partial x}\left(\sigma^2(x,B,T)x^2\phi(x,B,T)\right)\right]_{x=B}. \end{aligned}$$

Using the boundary condition (2.7) in the second line of

$$\begin{aligned} \frac{\partial}{\partial B}\left(\sigma^2(B,B,T)B^2\phi(B,B,T)\right) &= \left[\frac{\partial}{\partial x}\left(\sigma^2(x,B,T)x^2\phi(x,B,T)\right)\right]_{x=B} + \left[\frac{\partial}{\partial B}\left(\sigma^2(x,B,T)x^2\phi(x,B,T)\right)\right]_{x=B} \\ &= -\left[\frac{\partial}{\partial x}\left(\sigma^2(x,B,T)x^2\phi(x,B,T)\right)\right]_{x=B}, \end{aligned}$$

we get, by another application of the product rule,

$$\frac{\partial}{\partial B}\left((B-K)\sigma^2(B,B,T)B^2\phi(B,B,T)\right) = -(B-K)\left[\frac{\partial}{\partial x}\left(\sigma^2(x,B,T)x^2\phi(x,B,T)\right)\right]_{x=B} + \sigma^2(B,B,T)B^2\phi(B,B,T).$$

The result now follows by differentiating (2.8) once more to obtain

$$\frac{\partial^3 C}{\partial K^2 \partial B} = \phi, \tag{2.10}$$

and substituting everything above.

The case for general r and q follows similar calculations keeping track of first and zero order derivatives in K, or by considering a change of measure and the value capitalised by dividends. □

Remark. We can think of (2.9) in two ways. Given a model, either via σ directly or via a specification that allows computation of the Markovian projection, the forward equation allows computation of barrier option prices across all strikes, maturities and barriers as the solution of a single PDE. Conversely, if barrier call prices are observed on the market, (2.9) allows inferences on the Markovian projection (2.3) of the volatility. Similar to the Dupire formula for European calls, as a continuum of prices is not available, some sort of interpolation is required and can be notoriously unstable. We will return to these points in depth in the following sections.

For future reference, we define

$$\widetilde{C}\left(K,B,T\right) = \mathbb{E}^{\mathbb{Q}}\left[D\left(T\right)M^q\left(T\right)\left(S_T - K\right)^+ \mathbf{1}_{M_T<B}\right],$$

the capitalisation of the market price with dividends. By doing so, we eliminate the term $q\left(T\right)\frac{\partial C(K,B,T)}{\partial B}$ if we replace $C\left(K,B,T\right)$ by $\widetilde{C}\left(K,B,T\right)$ in (2.9). We will work with $\widetilde{C}$ in the remainder of the article.

3. A Dupire-Type Formula for Barrier Options

Equation (2.9) cannot be solved for the Markovian projection $\sigma_{|S,M}$ directly. Furthermore, it suffers from similar issues to those one finds with the evaluation of Dupire's formula, in that derivatives of the value function over a continuum of strikes and maturities are required. Practical approximations using the sparsely available data are necessarily sensitive to the method of interpolation. This is exacerbated here as (2.9) involves derivatives up to order four. In order to reduce the impact of this issue to some extent, equation (2.9) is first rewritten in a different form.

At $K = 0$, equation (2.9) still holds and gives

$$\frac{\partial \widetilde{C}\left(0,B,T\right)}{\partial T} = -\frac{1}{2}\sigma^2_{|S,M}\left(B,B,T\right)B^3 \left.\frac{\partial^3 \widetilde{C}\left(K,b,T\right)}{\partial K^2 \partial b}\right\rfloor_{K=B,b=B} \qquad \forall\left(B,T\right) \in \left]S_0,+\infty\right[\times\mathbb{R}^+_*. \tag{3.1}$$

It links the price of the foreign no-touch option,

$$ForNT\left(B,T\right) = D\left(T\right)\mathbb{E}^{\mathbb{Q}}\left[S_T\mathbf{1}_{M_T<B}\right] = \frac{\widetilde{C}\left(0,B,T\right)}{M^q\left(T\right)}, \tag{3.2}$$

to the market-implied joint-density of (S_t, M_t) at $K = B$; see (2.10). This equation also suggests that it is probably not possible to find a forward equation for no-touch options alone without involving the joint density of spot and running maximum. If we substitute (3.1) into the last term of (2.9), then we get

$$\frac{\partial^2 \widetilde{C}\left(K,B,T\right)}{\partial B \partial T} + \left(r\left(T\right) - q\left(T\right)\right)K\frac{\partial^2 \widetilde{C}\left(K,B,T\right)}{\partial K \partial B} = \frac{1}{2}\sigma^2_{|S,M}\left(K,B,T\right)K^2\frac{\partial^3 \widetilde{C}\left(K,B,T\right)}{\partial K^2 \partial B} + \frac{\partial\left(\frac{(B-K)^+}{B}\frac{\partial \widetilde{C}(0,B,T)}{\partial T}\right)}{\partial B}.$$

Therefore, as $B \geq K$ in the case of interest,

$$\frac{\partial^2 \widetilde{C}(K,B,T)}{\partial B \partial T} - \frac{K}{B^2}\frac{\partial \widetilde{C}(0,B,T)}{\partial T} - \frac{(B-K)}{B}\frac{\partial^2 \widetilde{C}(0,B,T)}{\partial T \partial B} + (r(T) - q(T))\, K \frac{\partial^2 \widetilde{C}(K,B,T)}{\partial K \partial B} = \frac{1}{2}\sigma^2_{|S,M}(K,B,T)\, K^2 \frac{\partial^3 \widetilde{C}(K,B,T)}{\partial K^2 \partial B}. \tag{3.3}$$

Corollary 2. *The unique Brunick-Shreve mimicking volatility for up-and-out call options is*

$$\sigma_{|S,M}(K,B,T) = \sqrt{\frac{\frac{\partial^2 \widetilde{C}(K,B,T)}{\partial B \partial T} - \frac{K}{B^2}\frac{\partial \widetilde{C}(0,B,T)}{\partial T} - \frac{(B-K)}{B}\frac{\partial^2 \widetilde{C}(0,B,T)}{\partial T \partial B} + (r(T) - q(T))\, K \frac{\partial^2 \widetilde{C}(K,B,T)}{\partial K \partial B}}{\frac{1}{2}K^2 \frac{\partial^3 \widetilde{C}(K,B,T)}{\partial K^2 \partial B}}} \qquad 0 \leq K \leq B,\ T \geq 0. \tag{3.4}$$

This formula provides a somewhat more stable way of extracting the Brunick-Shreve volatility since we replaced the higher order derivative with respect to strike at barrier level with first and second order derivatives at zero strike. In addition to the numerical improvement, it is also easier in practice to retrieve barrier prices at zero strike with foreign no-touch prices, recalling (3.2), for which quotes are often available (e.g., in the FX markets).

As mentioned at the start of this section, the issue remains that one needs to interpolate between option prices to build a function of (K, B, T) which is sufficiently smooth; in practice, this can lead to quite different Brunick-Shreve volatilities for various interpolation methods or small changes in input data. It is well known that ill-posed problems of this kind can be regularised through a penalised optimisation routine, a good example of which is the calibration of local volatility through Tikhonov regularisation as presented, e.g., by Crépey in [9], Egger and Engl in [15], as well as Achdou and Pironneau in [1]. However, in order to achieve this, a suitable forward partial differential equation is required. We discuss the construction and discretisation of such a forward equation in the following section.

4. A Forward Partial-Integro Differential Equation For Barrier Option Prices

We now propose a further rearrangement of (2.9), which is more suited to the numerical computation of $C(K, B, T)$ taking the Markovian projection $\sigma_{|S,M}$ as input.

4.1. **Formulation as PIDE.** Equation (2.9) can also be expressed in a PIDE form by integrating with respect to B. We start by integrating the diffusive term of (2.9),

$$\int_{S_0 \vee K}^{B} \tfrac{1}{2}\sigma^2_{|S,M}(K,b,T)\, K^2 \tfrac{\partial^3 \widetilde{C}(K,b,T)}{\partial K^2 \partial b}\, \mathrm{d}b = \tfrac{1}{2}\sigma^2_{|S,M}(K,B,T)\, K^2 \tfrac{\partial^2 \widetilde{C}(K,B,T)}{\partial K^2} - \int_{S_0 \vee K}^{B} \tfrac{1}{2} K^2 \tfrac{\partial^2 \widetilde{C}(K,b,T)}{\partial K^2}\tfrac{\partial \sigma^2_{|S,M}(K,b,T)}{\partial b}\, \mathrm{d}b,$$

since $\widetilde{C}(K, S_0, T) = 0$ for all K and T. The other terms can all be directly integrated with respect to B taking into account that, similarly, no integration constant will appear.

This allows us to define the following initial boundary value problem.

Corollary 3. *The up-and-out call price under the stochastic volatility model (2.1) follows a Volterra type PIDE expressed as an initial boundary value problem,*

$$\frac{\partial \widetilde{C}(K,B,T)}{\partial T} + (r(T) - q(T))K\frac{\partial \widetilde{C}(K,B,T)}{\partial K} - \frac{1}{2}\sigma^2_{|S,M}(K,B,T)K^2\frac{\partial^2 \widetilde{C}(K,B,T)}{\partial K^2} = -\frac{1}{2}\sigma^2_{|S,M}(B,B,T)B^2(B-K)\frac{\partial^3 \widetilde{C}(B,B,T)}{\partial K^2 \partial B} - \int_{S_0 \vee K}^{B} \frac{1}{2}K^2 \frac{\partial^2 \widetilde{C}(K,b,T)}{\partial K^2}\frac{\partial \sigma^2_{|S,M}(K,b,T)}{\partial b}\,\mathrm{d}b \quad \forall (K,B,T) \in [0,+\infty[\times]S_0,+\infty] \times \mathbb{R}^+_* . \tag{4.1}$$

$$\widetilde{C}(K,B,0) = (S_0 - K)^+ \mathbf{1}_{S_0<B} \qquad T = 0 \tag{4.2}$$

$$\widetilde{C}(B,B,T) = 0 \qquad K = B \tag{4.3}$$

$$\widetilde{C}(K,S_0,T) = 0 \qquad B = S_0 \tag{4.4}$$

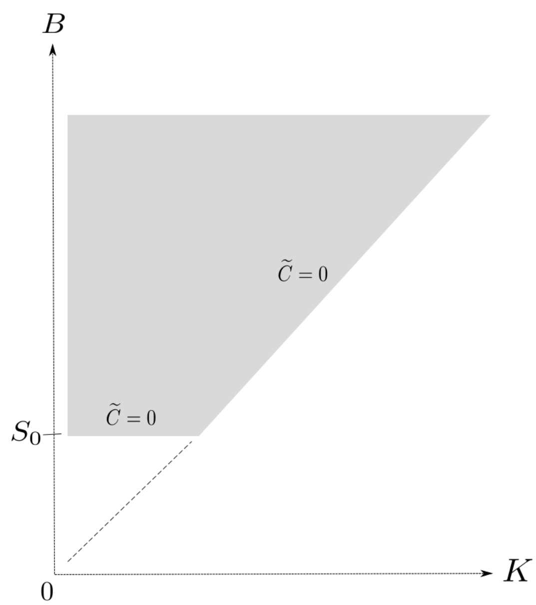


FIGURE 4.1. PIDE Domain and Boundaries

The initial condition (4.2) is the payoff obtained at maturity. Condition (4.3) expresses that the option gets knocked out before getting in-the-money if $K = B$ while (4.4) says the option gets knocked out immediately at inception if $S_0 = B$.

Remark. We note that in the models we will consider, no boundary condition needs to be specified at $K = 0$, since the coefficients of the K-derivatives vanish sufficiently fast as $K \to 0$. This is the case, e.g., if $\sigma_{|S,M}$ is bounded when $K \to 0$.

In fact, a boundary condition at $K = 0$ could be derived by differentiating (2.8) twice,

$$\frac{\partial^2 \widetilde{C}(0,B,T)}{\partial K^2} = D(T)M^q(T)\int_{S_0}^{B} \phi(0,y,T)\,\mathrm{d}y = D(T)M^q(T)\mathbb{Q}(S_0 < M_T < B \mid S_T = 0)\,\psi(0,T),$$

where $\psi(0,T)$ is the density of S_T at 0. The latter is equal to zero since the density of the spot at zero is zero under the log-spot model in (2.1). Linearity conditions, such as (2), also known as “Zero Gamma” for spot PDEs, are commonly used in finance and particularly useful for exotic contracts. They usually lead to stable numerical schemes (see Windcliff [37]).

We also note that

$$\frac{\partial^2 \widetilde{C}\,(B,B,T)}{\partial K^2} = D\,(T)\,M^q\,(T)\,\mathbb{Q}\,(S_0 < M_T < B \mid S_T = B)\,\psi(B,T) = 0, \tag{4.5}$$

because if the spot is equal to B at T, then the probability that the running maximum is smaller than B is zero. While we will use the Dirichlet condition on the boundary $K = B$ for computations, (4.5) allows us to express the third order cross derivative in (4.1) by a third order derivative only with respect to K. Indeed,

$$0 = \frac{\partial}{\partial B}\left(\frac{\partial^2 \widetilde{C}\,(B,B,T)}{\partial K^2}\right) = \frac{\partial^3 \widetilde{C}\,(B,B,T)}{\partial K^3} + \frac{\partial^3 \widetilde{C}\,(B,B,T)}{\partial K^2 \partial B},$$

leading to

$$\frac{\partial^3 \widetilde{C}\,(B,B,T)}{\partial K^3} = -\frac{\partial^3 \widetilde{C}\,(B,B,T)}{\partial K^2 \partial B}. \tag{4.6}$$

The domain is unbounded for large B. We denote B_{Max} the maximum value of barrier levels we will consider in the numerical solution. Note that because of the structure of (4.1), no boundary condition is needed for $B = B_{Max}$.

4.2. **Finite Difference Approximation.** We define a solution mesh that contains $M+1$ time points, $N+1$ space points in strike and $P+1$ space points in barrier levels. The grid is assumed uniform in both time and space leading to the following definition of the step sizes: $\Delta_T = \frac{T_{Max}}{M+1}$, $\Delta_B = \frac{B_{Max}-S_0}{P+1}$, $\Delta_K = \frac{B_{Max}-K_0}{N+1}$. Because of the algorithm that follows, we impose $\Delta_K = \Delta_B$. This will ensure that for any B_j, the corresponding strike grid will contain at least all B_u for all u smaller than j. This leads to:

- $K_i = i\Delta_K$, with $i \in [\![0, N]\!]$
- $B_j = S_0 + j\Delta_B$, with $j \in [\![0, P]\!]$
- $T_m = m\Delta_T$, with $m \in [\![0, M]\!]$

We can identify an interesting property of (4.1), especially visible with the substitution (4.6), in order to approximate the solution inductively on a discrete lattice. At $B = S_0$, the solution is known to be uniformly zero over the strike and time variable. Assume now an approximate solution is known up to a certain B. Moving from B to $B + \Delta_B$, and approximating the integral by a quadrature rule, gives a PDE in time and strike at the $B + \Delta_B$ level which can be solved by finite differences. From now on, we call the PDE at a given barrier level B_j a PDE layer. We can then solve the PIDE for each layer at points B_j from S_0 to B_{Max}.

We denote the discrete solution vector in such a layer by

$$\mathbf{u}^m_{\cdot,j} = \left[\widetilde{C}\,(K_0, B_j, T_m)\,, \ldots, \widetilde{C}\,(B_j, B_j, T_m)\right]'$$

of size $n_j = \frac{B_j - K_0}{\Delta_K} + 1$, where $'$ denotes the transpose. We also denote by $\mathbf{I}_n$ the identity matrix of size $n \times n$.

Derivatives. Derivatives are approximated by centred finite differences at each space point except at $K=0$ and $K=B_j$, where they are computed respectively forward and backward. For the time being, the boundary conditions are not taken into account. We assume equally spaced points and define two operators as follows:

$$\begin{aligned} \delta_K \mathbf{u}_{i,j}^m &= \frac{\mathbf{u}_{i+1,j}^m - \mathbf{u}_{i-1,j}^m}{2\Delta_K}, \\ \delta_{KK} \mathbf{u}_{i,j}^m &= \frac{\mathbf{u}_{i+1,j}^m - 2\mathbf{u}_{i,j}^m + \mathbf{u}_{i-1,j}^m}{\Delta_K^2}. \end{aligned} \tag{4.7}$$

The usual matrix derivative operator can be defined for both the first and second order derivative:

$$\mathbf{D} = \frac{1}{\Delta_K}\begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{D_2} = \frac{1}{\Delta_K^2}\begin{bmatrix} 1 & -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & (0) & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & (0) & 0 & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 & 1 \end{bmatrix}.$$

We also define the forward time difference operator

$$\delta_T \mathbf{u}_{i,j}^m = \frac{\mathbf{u}_{i,j}^{m+1} - \mathbf{u}_{i,j}^m}{\Delta T}.$$

Integral. The integral term will be computed using the trapezoidal quadrature rule. This yields an order two consistent approximation. The term of interest is

$$F(K_i, B_j, T_m) = \int_{S_0 \vee K}^{B_j} -\frac{1}{2} K_i^2 \frac{\partial^2 \widetilde{C}(K_i, b, T_m)}{\partial K^2} \frac{\partial \sigma^2_{|S,M}(K_i, b, T_m)}{\partial b} \, \mathrm{d}b.$$

Let us assume that we know the solution of the PIDE for the discrete set of barriers $(B_n)_{n<j}$. We also know that, when the barrier is at the spot level, the integrand is zero. Hence,

$$\begin{aligned} F(K_i, B_j, T_m) &= -\sum_{n=1}^{j-1} \left(\frac{1}{2} K_i^2 \frac{\partial^2 \widetilde{C}(K_i, B_n, T_m)}{\partial K^2} \frac{\partial \sigma^2_{|S,M}(K_i, B_n, T_m)}{\partial B} \Delta_B \right) \\ &\quad - \frac{1}{4} K_i^2 \frac{\partial^2 \widetilde{C}(K_i, B_j, T_m)}{\partial K^2} \frac{\partial \sigma^2_{|S,M}(K_i, B_j, T_m)}{\partial B} \Delta_B \\ &\quad + \mathcal{O}\left(\Delta_B^2\right). \end{aligned} \tag{4.8}$$

In a forward induction over j, the sum in the first line can be computed for all K_i and T_m as the solution is known for all the barrier levels involved. This sum is then handled as a source function for the PDE layer of level B_j. We define a vector

$$\mathbf{f}_{\cdot,j}^m = \left[\; -\sum_{n=1}^{j-1} \left(\tfrac{1}{2} K_i^2 \delta_{KK} \mathbf{u}_{i,n}^m \tfrac{\partial \sigma^2_{|S,M}(K_i, B_n, T_m)}{\partial B} \Delta_B \right) \; \right]_{i=0,1,\dots,n_j}.$$

The remaining term in (4.8) gives a small correction to the diffusion at B_j and we can incorporate it in the discretisation of the corresponding diffusive term of (4.1).

Boundary Derivative Term. To approximate the "boundary derivative" at (B, B, T) in (4.1), we use (4.6) and a first order approximation to the third derivative with discretisation matrix written as

$$\mathbf{\Phi} = \frac{1}{\Delta_K^3}\begin{bmatrix} 0 & \dots & 0 & 0 & 1 & 3 & -3 & 0 \\ \vdots & (0) & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 1 & 3 & -3 & 0 \end{bmatrix}$$

As this term is present in the discretised equation for all interior mesh points, this reduces the consistency order in Δ_K of the overall scheme to one. We found higher order finite differences to be unstable.

Remark. A second order accurate stable scheme can be obtained by using (2.10) to replace the third order derivative at the boundary by the density, which can be found by solving the forward Kolmogorov equation numerically.

PIDE in Terms of Matrix Operations. If we take into account the finite difference approximations and quadrature rule for the integral, it is now possible to rewrite the PIDE in a discretised form, for a given triplet $(i, j, m) \in [\![0, N]\!] \times [\![0, M]\!] \times [\![0, P]\!]$:

$$\begin{aligned} \delta_T \mathbf{u}_{i,j}^m + \left(r\left(T_m\right) - q\left(T_m\right)\right) K_i \delta_K \mathbf{u}_{i,j}^m - \frac{1}{2}\left(\sigma_{|S,M}^2\left(K_i, B_j, T_m\right) - \frac{1}{2}\frac{\partial \sigma_{|S,M}^2\left(K_i, B_j, T_m\right)}{\partial B}\Delta_B\right) K_i^2 \delta_{KK} \mathbf{u}_{i,j}^m & \\ + \frac{1}{2}\sigma_{|S,M}^2\left(B_j, B_j, T_m\right) B_j^2 \left(B_j - K\right)^+ \delta_{KKB}^- \mathbf{u}_{n_j,j}^m & = \\ - \sum_{n=1}^{j-1} \frac{1}{2} K_i^2 \delta_{KK} \mathbf{u}_{i,j}^m \frac{\partial \sigma_{|S,M}^2\left(K_i, B_n, T_m\right)}{\partial B}\Delta_B & \end{aligned} \tag{4.9}$$

and specify the coefficient matrices

$$\begin{aligned} \mathbf{A}_{\cdot,j}^m &= \left(r\left(T_m\right) - q\left(T_m\right)\right) \operatorname{diag}\left(K_0, ..., K_{n_j}\right) \\ \mathbf{B}_{\cdot,j}^m &= -\frac{1}{2}\operatorname{diag}\left(\left(\sigma_{|S,M}^2\left(K_i, B_j, T_m\right) K_i^2 - \frac{1}{2} K_i^2 \frac{\partial \sigma_{|S,M}^2\left(K_i, B_j, T_m\right)}{\partial B}\Delta_B\right)_{i \in [\![0, n_j]\!]}\right) \\ \mathbf{C}_{\cdot,j}^m &= -\frac{1}{2}\operatorname{diag}\left(\left(\sigma_{|S,M}^2\left(B_j, B_j, T_m\right) B_j^2 \left(B_j - K_i\right)^+\right)_{i \in [\![0, n_j]\!]}\right). \end{aligned}$$

Remark. We can also approximate $\mathbf{B}_{\cdot,j}^m$ further by Taylor expansion,

$$\mathbf{B}_{\cdot,j}^m = -\frac{1}{2}\operatorname{diag}\left(\left(\sigma_{|S,M}^2\left(K_i, B_j - \frac{\Delta_B}{2}, T_m\right) K_i^2\right)_{i \in [\![0, n_j]\!]}\right) + \mathcal{O}\left(\Delta_B^2\right),$$

which has a negative sign irrespective of the mesh size and does not alter the convergence order.

Under forward Euler time stepping, the complete scheme can be more compactly written as

$$\begin{cases} \frac{\mathbf{u}_{\cdot,j}^{m+1} - \mathbf{u}_{\cdot,j}^m}{\Delta T} + \mathbf{L}_{\cdot,j}^m \mathbf{u}_{\cdot,j}^m = \mathbf{f}_{\cdot,j}^m, \\ \mathbf{L}_{\cdot,j}^m = \mathbf{A}_{\cdot,j}^m \mathbf{D} + \mathbf{B}_{\cdot,j}^m \mathbf{D_2} + \mathbf{C}_{\cdot,j}^m \mathbf{\Phi}. \end{cases} \tag{4.10}$$

Under θ-time-stepping, the scheme becomes

$$\left(\mathbf{I}_{n_j} + \theta\Delta_T \mathbf{L}_{\cdot,j}^{m+1}\right)\mathbf{u}_{\cdot,j}^{m+1} = \left(\mathbf{I}_{n_j} - (1-\theta)\,\Delta_T \mathbf{L}_{\cdot,j}^{m}\right)\mathbf{u}_{\cdot,j}^{m} + \theta\Delta_T \mathbf{f}_{\cdot,j}^{m+1} + (1-\theta)\,\Delta_T \mathbf{f}_{\cdot,j}^{m}.$$

This includes the second-order accurate Crank-Nicolson scheme for $\theta = 0.5$, which is used for our numerical computations.

This section shows that we can adapt the classical tools of finite difference methods in order to solve this PIDE. Each of the layers being a one-dimensional PDE, it is solved by successive roll-forward performed by a Gaussian elimination. Although the Thomas algorithm cannot be applied directly since $\mathbf{L}_{\cdot,j}^{m}$ is not tri-diagonal (even after applying the necessary boundary conditions), a sparse Gaussian elimination will still be $\mathcal{O}(n)$ as the dense sub-matrix has a fixed number of columns.

Solution Algorithm. We conclude with a possible algorithm to solve numerically the PIDE:

Algorithm 1 PIDE Discretisation

$\mathbf{u}_{\cdot,j}^{0} = \left((S_0 - K_i)^+ \mathbf{1}_{S_0 < B_j}\right)_{i\in[\![0,N]\!]}$
$\mathbf{u}_{\cdot,0}^{0} = 0$
$\mathbf{f}_{\cdot,0}^{m} = 0$
for $(\ j = 0\,;\ j \le P\,;\ j++)$ **do**
  solve B_j-layer PDE for $\left(\mathbf{u}_{\cdot,j}^{m}\right)_{m\in[\![0,M]\!]}$:

$$\left(\mathbf{I}_{n_j} + \theta\Delta_T \mathbf{L}_{\cdot,j}^{m+1}\right)\mathbf{u}_{\cdot,j}^{m+1} = \left(\mathbf{I}_{n_j} - (1-\theta)\,\Delta_T \mathbf{L}_{\cdot,j}^{m}\right)\mathbf{u}_{\cdot,j}^{m} + \theta\Delta_T \mathbf{f}_{\cdot,j}^{m+1} + (1-\theta)\,\Delta_T \mathbf{f}_{\cdot,j}^{m}$$

  compute $\mathbf{f}_{\cdot,j+1}^{m}$ from $\mathbf{f}_{\cdot,j}^{m}$ and $\mathbf{u}_{\cdot,j}^{m}$ (for $j < P$)
end for

5. Numerical Results

5.1. **Pricing Under the Mimicking Brunick-Shreve Model.** For validation purposes of the forward equation, we will use a numerical solution of the backward pricing PDE in the Brunick-Shreve model. The backward PDE gives the price over a range of S and M and t, for a fixed K, B and T. We briefly explain the numerical solution by handling a Neumann boundary condition on a classical Black-Scholes PDE (where the volatility is a function of S, a parameter M, and t). Under $\mathbb{Q}$, we recall that the spot diffusion of the underlying can be written as

$$\frac{dS_t}{S_t} = (r(t) - q(t))\,dt + \sigma(S_t, M_t, t)\,dW_t.$$

Augmented State Feynman-Kac PDE. Assume that strike K, barrier B and maturity T are all fixed. Then, following identical steps to the derivation by Shreve [35] in the Black-Scholes case (see also Appendix A), we get the following.

Proposition 4. *The up-and-out call price denoted C and priced under a Brunick-Shreve model is the solution to the following initial boundary value problem:*

(5.1)
$$\begin{cases} \frac{\partial C}{\partial t} + (r(t) - q(t))\, x \frac{\partial C}{\partial x} + \frac{1}{2} x^2 \sigma^2(x, y, t) \frac{\partial^2 C}{\partial x^2} - r(t)\, C = 0, & 0 < x < y,\ S_0 < y < B,\ 0 < t < T, \\ & \\ C(0, y, t) = 0 & t \leq T \\ C(x, y, T) = (x - K)^+ \mathbf{1}_{y < B} & y > S_0 \\ \left.\frac{\partial C(x,y,t)}{\partial y}\right\rfloor_{x=y} = 0 & y > S_0 \\ C(x, B, t) = 0 & x > 0,\ t \leq T \end{cases}$$

Finite Difference Approximation. We notice that the PDE is not "genuinely" two-dimensional as derivatives with respect to y only enter via the Neumann boundary condition (on the diagonal where x is equal to y). This means that for a fixed y, we have to solve a Black-Scholes type PDE on the space domain $[0, y]$. From now on, the one-dimensional PDE for a given level of y will be called a PDE layer.

If $y = B$, then $C(x, y, t) = 0$. We can then use the Neumann boundary condition to bootstrap backwards in y from B to S_0. To illustrate the idea by a backward finite difference (although we will use higher order interpolation later),

$$\left.\frac{\partial C(x, y, t)}{\partial y}\right\rfloor_{x=y} = \frac{C(y, y, t) - C(y, y + \Delta_y, t)}{\mathrm{d}y} + \mathcal{O}(\Delta_y) = 0 \implies C(y, y, t) = C(y, y + \Delta_y\, t) + \mathcal{O}\left(\Delta_y^2\right).$$

Since $C(B, B, t) = 0$, we can build all the PDE layers by backward reasoning, using an approximate Dirichlet boundary condition at $x = y$ for each layer, $C(y, y, t) \approx C(y, y + \Delta_y, t)$. Every PDE layer will then depend on layers with a greater y level. The premium value is retrieved from $C(S_0, S_0, 0)$.

Denote by Δ_x the desired *average* step size of the spot grid. The best accuracy was numerically achieved with a uniformly spaced spot grid (except for the last two nodes, as per below) and a refined running maximum grid close to S_0. More precisely, for the space discretisation of y for $N_y = \left\lceil \frac{B - S_0}{\Delta_x} \right\rceil$ points, we will use an exponential meshing as defined in [36] and set $\forall\, i \in [\![0, N_y]\!]$:

$$\begin{aligned} y_i &= (S_0 - \theta) + \theta \exp(\lambda z_i), \\ \lambda &= 2, \\ \theta &= \frac{B - S_0}{e^\lambda - 1}, \\ z_i &= \frac{i}{(N_y + 1)}. \end{aligned}$$

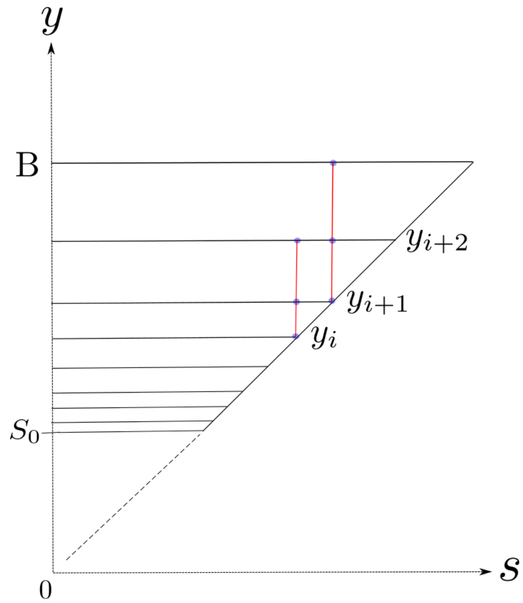


FIGURE 5.1. Mesh construction for backward equation

The spot grid is not kept constant and is recomputed for every PDE layer, where the number of grid points N_x is reduced so that we keep $N_x = \left\lceil \frac{y_i}{\Delta_x} \right\rceil$ fixed. Moreover, the mesh is chosen such that $y - \Delta_y$ lies on the grid of the PDE layer of level y and also on one mesh layer further up, which becomes relevant for the construction in the next paragraph. This construction, illustrated in Figure 5.1, ensures that there is no need to interpolate to retrieve the boundary condition for the next PDE layer. Apart from the last two points in each layer, the mesh points are spaced uniformly.

Remark. In order to increase accuracy, the boundary condition in the numerical tests was computed using a second order Taylor expansion instead of a simple backward finite difference. However, the bootstrap idea stays the same. Indeed, write $c(y) = c(y_{i+2}, y, t)$ where y_i is a certain given level of the y-discretisation and $c'(y) = \frac{\partial c(y_{i+2}, y, t)}{\partial y}$. We consider a Taylor expansion around $y = y_{i+2}$: $c(y) = c(y_{i+2}) + \frac{1}{2}(y - y_{i+2})^2 c''(y_{i+2}) + o(y^2)$ (as $c'(y_{i+2}) = 0$ by the boundary condition). Using this at points y_i and y_{i+1}, we get $c(y_i) = c(y_{i+2}) + \left(\frac{y_{i+2} - y_i}{y_{i+2} - y_{i+1}}\right)^2 (c(y_{i+1}) - c(y_{i+2}))$.

5.2. **Numerical Validation.** Our numerical validation consists in pricing a set of up-and-out call options for different strikes, maturities and barriers with:

(1) the Forward PIDE (4.1) and one numerical solution for the whole set of deal parameters;
(2) the Backward Feynman-Kac PDE (5.1) and as many solutions as sets of deal parameters.

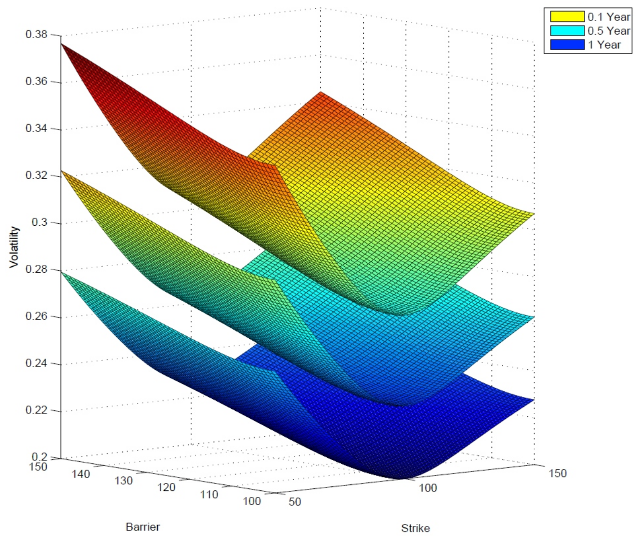
The goal is to make them match with about one basis point tolerance.

The Brunick-Shreve volatility is generated with an SVI parametrisation in both barrier and strike dimension (details can be found in [19]) defined as:

$$\begin{cases} \sigma\left(x,y,t\right) = \frac{1}{2}\left(\sigma_{SVI}\left(\log\left(\frac{x}{S_0}\right),t+1\right) + \sigma_{SVI}\left(\log\left(\frac{y}{S_0}\right),t+1\right)\right) \\ \sigma_{SVI}\left(k,t\right) = \sqrt{a+b\left(\rho\left(k-m\right)+\sqrt{\left(k-m\right)^2+\sigma^2}\right)} \end{cases}$$

The Brunick-Shreve volatility surface has a shape as in Figure 5.2.

FIGURE 5.2. Assumed Brunick-Shreve volatility surface; $a = 0.04, b = 0.2, \sigma = 0.2, \rho = m = 0$, and $T \in \{0.1, 0.5, 1\}$



The spot is $S_0 = 100$, the risk-free rate is $r = 0.1$ and the dividend yield is $q = 0.05$.

We use the numerical schemes as described in 4.2 and 5.1 for the PIDE and PDE solution, respectively, with Crank-Nicolson time stepping. The discretisation parameters are $N = P = N_y = 1000$ space steps (with N_x adjusted as described) and $M = 1000$ time steps.

We compared prices for (K, B, T) covering the set $[0, 120] \times [100, 120] \times \{1\}$ with 120 points in strike and 40 points in barrier levels, see Table 1. The error is computed as the relative error if the net present value (NPV) is above one and as absolute error otherwise.

Average Difference	Maximum Difference
4.6e-5	3.5e-4

TABLE 1. Difference between forward and backward solutions over strikes between 0 and 120, and up-and-out barriers between 100 and 120, all for maturity 1.

We can analyse more precisely the behaviour for a few barrier levels. For example, Figure 5.3 shows the difference as a function of strike. The associated values for a barrier fixed at 120 are in Table 2.

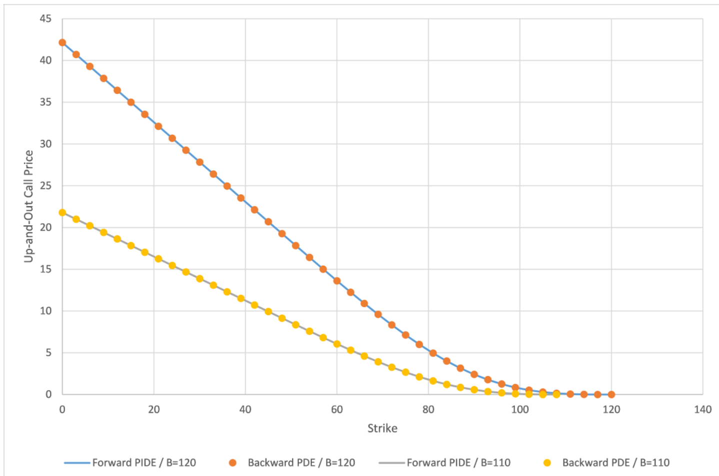


FIGURE 5.3. Forward PIDE vs Backward PDE:
up-and-out call option prices with Brunick-Shreve volatility, S=100 and T=1Y.

Strike	Forward PIDE	Backward PDE	Rel. Diff.
0	42.1486	42.1486	1e-06
9	37.8567	37.8568	1e-06
18	33.5649	33.5650	2e-06
27	29.2731	29.2732	2e-06
36	24.9815	24.9815	3e-06
45	20.6928	20.6929	3e-06
54	16.4263	16.4264	2e-06
63	12.2536	12.2535	5e-06
72	8.3438	8.3436	2e-05
81	4.9680	4.9677	6e-05
90	2.4170	2.4168	9e-05
99	0.8472	0.8472	9e-07
108	0.1546	0.1547	1e-04
117	0.0023	0.0023	8e-05
120	0.0000	0.0000	0

TABLE 2. Barrier = 120, Forward = 105.13

The results are conclusive and the Forward PIDE proves its usability and effectiveness to form part of a calibration routine.

6. Conclusions

In this paper, we provided a new and numerically effective way to work with the Brunick-Shreve Markovian projection of a stochastic variance of a semi-martingale S with running maximum M onto (S_t, M_t), under observation of barrier option prices. We emphasise the ability of this approach to constitute the backbone of a more sophisticated calibration routine. As for the vanilla case, it provides the same advantages (and disadvantages) as the Dupire approach leading to the derivation of a Dupire-type formula for the considered barrier contracts. The latter was then re-arranged to write a forward PIDE which is convenient to control the numerical stability of best fit algorithms [9]. There is a well-known literature on the calibration of vanilla options through Markovian projection for LSV models (see Guyon and Labordère [22] and Ren, Madan and Quian [30]). We believe that extending these methods combined with the Forward PIDE we presented in this paper will lead to novel calibration algorithms for barrier options.

References

[1] Yves Achdou and Olivier Pironneau. *Computational Methods for Option Pricing*. Frontiers in applied mathematics. Society for Industrial and Applied Mathematics, Philadelphia, 2005.

[2] Robert Aebi. Itô formula for non-smooth functions. *RIMS, Kyoto Univ*, 28:595–602, 1992.

[3] Murex Analytics. Local volatility model. *Murex*, Internal, 2007.

[4] Gerard Brunick and Steven Shreve. Mimicking an Itô process by a solution of a stochastic differential equation. *Ann. Appl. Probab.*, 23(4):1584–1628, 2013.

[5] René Carmona and Sergey Nadtochiy. Local volatility dynamic models. *Finance and Stochastics*, 13(1):1–48, 2009.

[6] Peter Carr and John Crosby. A class of Lévy process models with almost exact calibration to both barrier and vanilla FX options. *Quant. Finance*, 10(10):1115–1136, 2010.

[7] Peter Carr and Ali Hirsa. Forward evolution equations for knock-out options. In Michael C. Fu, Robert A. Jarrow, Ju-Yi J. Yen, and Robert J. Elliott, editors, *Advances in Mathematical Finance*, Applied and Numerical Harmonic Analysis, pages 195–217. Birkhäuser Boston, 2007.

[8] Alexander M.G. Cox and Jan Obłój. *Robust pricing and hedging of double no-touch options*, volume 15 of *Finance Stoch*, pages 573–605. Springer-Verlag, Sep 2011.

[9] Stéphane Crépey. Tikhonov regularization. *Encyclopedia of Quantitative Finance*, pages 1807–1812, 2010.

[10] Emanuel Derman and Iraj Kani. The volatility smile and its implied tree. *Goldman Sachs*, Quantitative Strategies Research Notes, 1994.

[11] Emanuel Derman, Iraj Kani, and Michael Kamal. Trading and hedging local volatility. *Goldman Sachs*, Quantitative Strategies Research Notes, 1996.

[12] Bruno Dupire. Pricing and hedging with smiles. In *Mathematics of derivative securities (Cambridge, 1995)*, volume 15 of *Publ. Newton Inst.*, pages 103–111. Cambridge Univ. Press, Cambridge, 1997.

[13] Bruno Dupire. A unified theory of volatility. In Peter Carr, editor, *Derivatives Pricing: The Classic Collection*. Risk Books, London, 2004.

[14] Bruno Dupire. Functional Itô calculus. *Bloomberg L.P.*, Bloomberg Portfolio Research Paper(2009-04-FRONTIERS), 2009.

[15] Herbert Egger and Heinz W Engl. Tikhonov regularization applied to the inverse problem of option pricing: convergence analysis and rates. *Inverse Problems*, 21(3):1027, 2005.

[16] Martin Forde. On the Markovian projection in the Brunick-Shreve mimicking result. *Statistics & Probability Letters*, 85:98–105, 2013.

[17] Martin Forde, Andrey Pogudin, and Hongzhong Zhang. Hitting times, occupation times, tri-variate laws and the forward Kolmogorov equation for one-dimensional diffusion with memory. *Adv. in Appl. Probab.*, 45(3):595–893, 2013.

[18] Bengt Fornberg. Generation of finite difference formulas on arbitrarily spaced grids. *Math. Comput.*, 51(51):699–706, 1988.

[19] Jim Gatheral. *The Volatility Surface: a Practitioner's Guide*. John Wiley & Sons, Hoboken, N.J, 2006.

[20] Mircea Grigoriu. *Stochastic Calculus: Applications in Science and Engineering*. Birkhäuser, New York, third edition, 2013.

[21] Julien Guyon. Path-dependent volatility. *Risk Magazine*, Sep. 2014, 2014.

[22] Julien Guyon and Pierre Henry-Labordere. The smile calibration problem solved. *Société Générale*, SSRN.1885032, 2011.

[23] István Gyöngy. Mimicking the one-dimensional marginal distributions of processes having an Itô differential. *Probab. Theory Relat. Fields*, 71(4):501–516, 1986.

[24] John C. Hull. *Options, Futures, and other Derivatives*. Pearson Prentice Hall, 6. ed., Pearson Internat. ed edition, 2006.

[25] Simon McNamara. Efficient calibration of FX smile models. *BNP Paribas*, Internal, 2013.

[26] Bernt Øksendal. *Stochastic Differential Equations: An Introduction with Applications (Universitext)*. Springer, Jan 2014.

[27] Olivier Pironneau. Dupire-like identities for complex options. *Compte Rendu de l'académie des sciences I*, 344:127–133, 2007.

[28] Vladimir Piterbarg. Markovian projection method for volatility calibration. *Barclays Capital*, SSRN.906473, 2007.

[29] Philip Protter. *Stochastic Integration and Differential Equation*. Springer-Verlag, Berlin, Heidelberg, second edition, 1992.

[30] Yong Ren, Dilip Madan, and Michael Qian Qian. Calibrating and pricing with embedded local volatility models. *Risk Magazine*, Sep. 2007, 2007.

[31] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[32] Chris L. C. G. Rogers and David Williams. *Diffusions, Markov processes, and Martingales. Vol. 1*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. Cambridge University Press, Cambridge, second edition, 2000.

[33] Chris L. C. G. Rogers and David Williams. *Diffusions, Markov processes, and Martingales. Vol. 2*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000.

[34] Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill Book Co., New York, third edition, 1976.

[35] Steven E. Shreve. *Stochastic Calculus for Finance II: Continuous-time Models*. Springer Finance. Springer-Verlag, New York, 2004.

[36] Richard White. Numerical solution to PDEs with financial applications. *OpenGamma Quantitative Research*, 2013.

[37] Heath Windcliff, Peter A. Forsyth, and Ken R.Vetzal. Analysis of the stability of the linear boundary condition for the Black-Scholes equation. *Journal of Computational Finance*, 8:65–92, 2004.

Appendix A. Derivation of the Kolmogorov Forward and Backward Equations

In this section, we formally derive the forward and backward Kolmogorov equations under a model of the type

$$\begin{cases} \dfrac{dS_t}{S_t} = (r\,(t) - q\,(t))\,dt + \sigma(S_t, M_t, t)\,dW_t, \\ M_t = \max\limits_{0\le u\le t} S_u. \end{cases}$$

The Backward Equation. Let h be a given càdlàg function. Since (S_t, M_t) is a Markovian vector, there exists a function v such that

$$D(t)v(x,y,t) = \mathbb{E}^{\mathbb{Q}}\left[D(T)h(S_T, M_T) \mid S_t = x,\, M_t = y\right] \qquad \forall (x,y)\in\Omega,\ 0\le t\le T.$$

We assume that v is smooth and belongs to $C^{2,1,1}(\Omega\times\mathbb{R}_+)$.

(A.1)
$$\begin{cases} \frac{\partial C}{\partial t} + \left(r\left(t\right) - q\left(t\right)\right) x \frac{\partial C}{\partial x} + \frac{1}{2} x^2 \sigma^2\left(x, y, t\right) \frac{\partial^2 C}{\partial x^2} - r\left(t\right) C = 0, & 0 < x < y,\ S_0 < y < B,\ 0 < t < T, \\ \\ C\left(0, y, t\right) = 0 & t \leq T \\ C\left(x, y, T\right) = \left(x - K\right)^+ \mathbf{1}_{y<B} & y > S_0 \\ \left.\frac{\partial C(x,y,t)}{\partial y}\right\rfloor_{x=y} = 0 & y > S_0 \\ C\left(x, B, t\right) = 0 & x > 0,\ t \leq T \end{cases}$$

Using iterated conditioning,

$$\mathbb{E}^{\mathbb{Q}}\left[D(T)h(S_T, M_T)\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[D(T)h(S_T, M_T) \mid S_t, M_t\right]\right] = \mathbb{E}^{\mathbb{Q}}\left[D(t)v(S_t, M_t, t)\right]. \tag{A.2}$$

Now write

$$C_t = v(S_t, M_t, t).$$

By Itô-Doeblin's lemma and recalling that the running maximum process M has finite variation, zero quadratic variation and zero cross-variation with S,

$$d(D(t)C_t) \quad = \quad -r\left(t\right) D(t) C_t \, dt + D(t) \left[\frac{\partial v}{\partial t} \, dt + \frac{\partial v}{\partial x} \, dS_t + \frac{\partial v}{\partial y} \, dM_t + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} \, d[S]_t\right].$$

The process C is a martingale if and only if

$$\begin{aligned} \frac{\partial v}{\partial t} - r\left(t\right) v + \left(r\left(t\right) - q\left(t\right)\right) \frac{\partial v}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} &= 0 \\ \left.\frac{\partial v}{\partial y}\right\rfloor_{x=y} &= 0. \end{aligned} \tag{A.3}$$

This gives the desired backward PDE. We note that the boundary condition is equivalent to

$$\frac{\partial v(y, y, t)}{\partial y} = \left.\frac{\partial v(x, y, t)}{\partial x}\right\rfloor_{x=y}. \tag{A.4}$$

The Forward Equation. We assume as before that the joint density function of the above process exists. This allows us to derive the Kolmogorov equation in a weak sense, which is the formulation that is usually useful in practice.

We denote the joint density function of (S_t, M_t) by ϕ . We will show that ϕ satisfies

$$\frac{\partial \phi}{\partial t} + (r(t) - q(t)) \frac{\partial}{\partial x}(x\phi) - \frac{1}{2} \frac{\partial^2}{\partial x^2} \left(\sigma^2 x^2 \phi\right) \quad = \quad 0, \qquad (x, y) \in \Omega,\ t > 0, \tag{A.5}$$

$$\begin{aligned} 2 \frac{\partial}{\partial x} \left(\sigma^2(x, y, t) x^2 \phi\right) + \frac{\partial}{\partial y} \left(\sigma^2(x, y, t) x^2 \phi\right) &= (r(t) - q(t))\, x\, \phi, & x = y,\ t > 0, \\ \phi(S_0, S_0, t) &= 0, & t > 0, \\ \phi(x, y, 0) &= \delta(x - S_0)\delta(y - S_0), & (x, y) \in \Omega. \end{aligned} \tag{A.6}$$

From (A.2),

$$\mathbb{E}^{\mathbb{Q}}\left[D(t)v(S_t,M_t,t)\right]=\int_{S_0}^{\infty}\mathrm{d}y\int_0^y \mathrm{d}x\,D(t)v(x,y,t)\phi(x,y,t)\quad\equiv\quad\text{const},\tag{A.7}$$

so differentiating (A.7) with respect to t we get

$$0=-\int_{S_0}^{\infty}\mathrm{d}y\int_0^y\mathrm{d}x\,r\left(t\right)D(t)v\phi+\int_{S_0}^{\infty}\mathrm{d}y\int_0^y\mathrm{d}x\,D(t)\frac{\partial v}{\partial t}\phi+\int_{S_0}^{\infty}\mathrm{d}y\int_0^y\mathrm{d}x\,D(t)v\frac{\partial\phi}{\partial t}$$

Hence, by inserting (A.3) and cancelling $r\left(t\right)D(t)v\phi$,

$$\begin{aligned}\int_{S_0}^{\infty}\mathrm{d}y\int_0^y\mathrm{d}x\,v\frac{\partial\phi}{\partial t}&=\int_{S_0}^{\infty}\mathrm{d}y\int_0^y\mathrm{d}x\,\left[\left(r\left(t\right)-q\left(t\right)\right)\frac{\partial v_t}{\partial x}+\frac{1}{2}\sigma_t^2x^2\frac{\partial^2v_t}{\partial x^2}\right]\phi\\&=\int_{S_0}^{\infty}I_\mu(y)\,\mathrm{d}y+\frac{1}{2}\int_{S_0}^{\infty}I_\sigma(y)\,\mathrm{d}y,\end{aligned}\tag{A.8}$$

where, explicitly with all arguments,

$$\begin{aligned}I_\sigma(y)&=\int_0^y\mathrm{d}x\left(\sigma^2\left(x,y,t\right)x^2\frac{\partial^2v\left(x,y,t\right)}{\partial x^2}\right)\phi(x,y,t)\\I_\mu(y)&=\int_0^y\mathrm{d}x\left(\left(r\left(t\right)-q\left(t\right)\right)x\frac{\partial v(x,y,t)}{\partial x}\right)\phi(x,y,t)\end{aligned}$$

For I_σ, we can perform integration by parts twice: to get

$$\begin{aligned}I_\sigma(y)&=\left.\frac{\partial v(x,y,t)}{\partial x}\right]_{x=y}\sigma^2(y,y,t)y^2\phi(y,y,t)-v(y,y,t)\left.\frac{\partial\sigma^2(x,y,t)x^2\phi(x,y,t)}{\partial x}\right]_{x=y}\\&\quad+\int_0^y\mathrm{d}x\,v(x,y,t)\frac{\partial^2\sigma^2(x,y,t)x^2\phi(x,y,t)}{\partial x^2}\end{aligned}$$

We note that "integration constants" do not vanish at $x=y$ because ϕ can not be assumed to reach zero at this point, neither does the test function v (i.e., the up-and-out call price is not zero when S_t is equal to M_t).

Using (A.4) and integrating by parts w.r.t y,

$$\begin{aligned}\int_{S_0}^{\infty}I_\sigma(y)\,\mathrm{d}y&=-\int_{S_0}^{\infty}\mathrm{d}y\,v(y,y,t)\left[\left.\frac{\partial\sigma^2(x,y,t)x^2\phi(x,y,t)}{\partial x}\right]_{x=y}+\frac{\partial\sigma^2(y,y,t)y^2\phi(y,y,t)}{\partial y}\right]\\&\quad+\int_{S_0}^{\infty}\mathrm{d}y\int_0^y v(x,y,t)\left[\frac{\partial^2\sigma^2(x,y,t)x^2\phi(x,y,t)}{\partial x^2}\right]\mathrm{d}x\\&\quad-v\left(S_0,S_0,t\right)\sigma^2\left(S_0,S_0,t\right)S_0^2\phi\left(S_0,S_0,t\right).\end{aligned}$$

For I_μ we integrate by parts once,

$$I_\mu(y)=\left(r\left(t\right)-q\left(t\right)\right)v(y,y,t)y\phi(y,y,t)-\int_0^y v(x,y,t)\left[\left(r\left(t\right)-q\left(t\right)\right)\frac{\partial x\phi(x,y,t)}{\partial x}\right]\mathrm{d}x.$$

Inserting in (A.8),

$$\int_{S_0}^{\infty} \mathrm{d}y \int_0^y \mathrm{d}x\, v(x,y,t) \left[\frac{\partial \phi(x,y,t)}{\partial t} + (r(t) - q(t)) \frac{\partial x \phi(x,y,t)}{\partial x} - \frac{1}{2} \left(\frac{\partial^2 \sigma^2(x,y,t) x^2 \phi(x,y,t)}{\partial x^2} \right) \right] =$$

$$\int_{S_0}^{\infty} \mathrm{d}y\, v(y,y,t) \left[(r(t) - q(t)) y \phi(y,y,t) - \frac{1}{2} \left(\frac{\partial \sigma^2(x,y,t) x^2 \phi(x,y,t)}{\partial y} \bigg\rfloor_{x=y} + 2 \frac{\partial \sigma^2(x,y,t) x^2 \phi(x,y,t)}{\partial x} \bigg\rfloor_{x=y} \right) \right]$$

$$- \frac{1}{2} v(S_0, S_0, t)\, \sigma^2(S_0, S_0, t)\, S_0^2 \phi(S_0, S_0, t)\,.$$

Since this equation holds for all functions h, the joint density function follows the required PDE and boundary conditions.

It is straightforward to generalise the approach to work directly under (2.1), say under a local-stochastic volatility model, to derive forward and backward equations for the joint density of S_t, M_t and the stochastic variance V_t. We find again the dual operator with respect to the L_2 inner product, now with three spatial dimensions.