A STOCHASTIC PARTIAL DIFFERENTIAL EQUATION MODEL FOR THE PRICING OF MORTGAGE-BACKED SECURITIES

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Abstract. We develop a relatively simple dynamic structural model for the behaviour of a mortgage pool. By considering the wealth of individual mortgagors in the pool we model the process of default and prepayment and, by taking a limit as the pool size goes to infinity, derive a stochastic partial differential equation (SPDE) to describe the evolution of the pool. We show the existence and uniqueness of solutions to the SPDE and give a probabilistic representation of the solution. Mortgage-backed securities (MBS) are functions of the solution to this SPDE and we show how our model is able to capture in a flexible way the prices of credit risky tranches of MBS under different market conditions.

1. Introduction

The market for mortgage-backed securities (MBS) was one of the fastest growing and most important markets in the US financial industry from its launch in the early 1980s until the financial crisis of 2008. The securitization of mortgages enabled institutions exposed to mortgage risk to convert these risky, non-rated individual loans into securities that became liquid and most of which had (supposedly) low credit risk. This process accelerated after the dot com bubble of 2000 when the US Federal Reserve lowered interest rates on treasury bills and investment bankers found better investment opportunities in the housing market, which was booming. Through buying thousands of mortgage loans, combining them into an MBS and selling the tranches of the MBS to other investors, many investment banks obtained much better returns than treasury bills would have provided. The complexities of these products, the use of overly simplistic models and the rapid development of the market in MBS played a major role in the subprime mortgage crisis in 2008. Subsequently the market for agency MBS, those with a government backed guarantee, has rebounded with the market returning to pre-crisis levels, while for MBS from private financial institutions, the market remains very small. In the light of the crisis and the recent regrowth of this market, there remains a need to improve the mathematical models for mortgage-backed securities.

The fundamental underlying structure in an MBS is a pool of mortgages. The particular type of MBS is determined by how the interest and principal repayments of the loans in this pool are repackaged for sale to investors. The 'vanilla' version is the pass-through MBS, in which the interest and principal payments from the pool are passed on to investors with the issuer taking a service charge. A more complicated version is the Collateralized Mortgage Obligation (CMO) in which the pool is tranched, that is the income streams are packaged to enable investors with different risk preferences to invest in the MBS. In the pass-through case the risk of default by a mortgagor in the pool is passed on to all investors, whereas in the CMO case this risk is borne differently by the different tranches, with the lower tranches taking the highest risk. It may also be the case that any early principal repayments are given

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preferentially to the senior, low risk, tranches. Our focus in this paper will be on building a model for a mortgage pool and we will illustrate its performance by pricing CMOs.

When considering an MBS pool, a portfolio of mortgages, it is essential to model not only the defaults of the mortgagors, which leads to a reduction in the income stream from the portfolio, but also their prepayment behaviour. A prepayment occurs when the mortgage is repaid in full before its termination. This event may happen for a number of reasons but the consequence is that, although the principal is repaid, the corresponding income stream making up the MBS is removed. There is no accepted standard way to price an MBS due to the combination of uncertainties in the cash flows due to defaults and prepayments. As in other credit settings there are reduced form and structural approaches and our aim will be to develop a flexible dynamic structural approach to the modelling for an MBS.

For an individual mortgage Kau et al. in [30] and [29] discussed the prepayment and default as the underlying source of uncertainty for the first time. They described the mortgage rate as a solution to a partial differential equation using option pricing ideas with the underlying variables being the interest rate and the house price. Other papers that develop structural models are [28, 13]. The second approach to modelling residential mortgages is intensity or hazard rate based. Papers following this approach are [52], [29], [20], [21], [45], [22] .

In the early modelling of mortgage-backed securities, researchers considered either of the two risks involved with these securities separately. They either modelled the right of prepayment by ruling out the possibility of default or considered default only and overlooked the prepayment right of the borrower. In the setting of agency pass through MBS the government backing enables default risk to be ignored and Dunn and McConnell [14] were the first to use option pricing techniques to model these securities. They considered an agency pass through MBS as equivalent to a single mortgage consisting of a portfolio of a non-callable mortgage loan and an American style option.

In their models, Dunn and McConnell (and Kau et al), assume that the borrower's prepayment behaviour is optimal and borrowers refinance whenever it is optimal which may not be the case in reality, especially when dealing with residential mortgages. An alternative assumption, discussed in [26], [44], [27], and [43], is that the borrowers make a decision by looking at the economy and prepay when the mortgage and interest rates reach a certain level or threshold. We will follow this alternative assumption in our model.

The first model for MBS using hazard rates is due to Schwartz and Torous [49]. This approach models prepayment and default as a random time that is governed by some hazard rate process estimated from the actual prepayment and default data in large mortgage pools. Another model in this category is that of [48].

Our approach will be a dynamic structural one. We will assume that individuals in the pool have a wealth process which determines if they will default or make an early prepayment. We will assume that all individuals in the pool are subject to macroeconomic factors as well as their own idiosyncratic risks. The whole pool can then be modelled as the empirical measure of the wealth of individuals. By taking a large portfolio limit in a similar way to [3] we obtain a stochastic partial differential equation (SPDE) which describes the evolution of the limiting empirical measure. The loss of income from the pool is captured by the behaviour of the SPDE at the boundary of the domain as well as through a killing term which captures unexpected termination of the mortgage due to life events as well as the refinancing behaviour of individuals in the pool. Although this remains a relatively simple model our aim in this paper is to establish the mathematical framework, showing existence and uniqueness of the model and extending the results and approach of [3]. We will illustrate our model by giving examples to show that even with a small number of parameters it has the flexibility to capture different types of mortgage pools. A recent alternative dynamic large portfolio model for loans can be found in [50].

An outline of the paper is as follows. We will begin with a justification of the model that we set up and then state our main mathematical results in Theorems 2.3 and 2.4. In Sections 3 to 8 we will provide the necessary results and methods in order to prove these theorems. In Section 9 we will discuss the implementation of the model and illustrate the performance when pricing CMO tranches.

2. The Model

To understand the evolution of an MBS pool, we must consider the risks that affect the individual mortgages, as well as the economic factors that affect the whole pool of mortgages. Any model should capture the default and prepayment risks that can affect MBS cash flows and comprise a dynamic model of mortgage payers that is flexible enough to capture both regular and subprime mortgage pools.

We will summarize the significant factors that contribute to the risk of default and prepayment by an individual's 'wealth' (or financial health) and by adding appropriate conditions for default and prepayment. We therefore consider a wealth process for each individual, which we take to be a Brownian motion with drift (as in models for distance-to-default in credit settings [3]). When combining the individual mortgages into a pool, there are macroeconomic factors that will affect an individual's wealth, such as an overall decrease in the prices of houses or baseline interest rates. Thus we will have two Brownian motions driving the individual's wealth process, one for macroscopic factors, common to the whole pool, as well as an idiosyncratic one for the factors that affect a particular individual.

We now examine the key factors that affect a mortgage pool in order to build our model. At the time of initialization of an individual mortgage contract important quantities are an individual's wealth or credit score, the size of the loan or Loan-to-Value ratio (LTV ratio), the individual's income stream, and current interest rates. This will be incorporated into the initial conditions for the wealth process.

Once a contract is initialized, the events that can cause early termination are defaults and prepayments. During the life of a mortgage contract, default occurs when there is a shortfall in the borrower's income and they are unable to make the scheduled monthly payments for more than three months. Prepayment, on the other hand, occurs for reasons such as accumulating enough money to pay all the remaining mortgage principal or because of refinancing due to changes in interest rates or due to moving house.

The individual's wealth, the LTV ratio and the income stream at initialization ensure that the individual has the capability to repay his loan in monthly instalments provided his financial situation remains the same or improves over time. However, if the individual has taken more than 95% mortgage on their house, then they are more likely to have difficulties in repaying the loan over the life of the contract. Also if there is a shortfall in the individual's income stream due to unemployment or bankruptcy, for example, then a default may occur. Other events that typically terminate the contract on default are reduced working hours, reduced pay, illness, separation or the death of a partner. Empirical studies show that all these events play a role in default on mortgage contracts [4]. We will summarize all these effects in the idiosyncratic noise driving the individual's wealth process and regard default as occurring when this wealth process hits 0.

Empirical and theoretical studies such as [23], [8] suggest that the interest rates, LTV, borrower credit worthiness, loan size and other variables have an impact on the prepayment behaviour. An important example is the case where the mortgage interest rates decrease after the initialization of the mortgage giving mortgagors the incentive to prepay their current mortgages in order to refinance. Prepayments can also occur when the individual has excess money and is able to prepay all the remaining loan or when they sell the house due to unforeseen life changing events such as separation, death or unemployment. Thus we will capture these effects by an individual's wealth reaching an upper boundary, or through a random hazard process, that can depend on interest rates, in which the individual is removed from the pool.

We are now in a position to describe the model formally.

The mathematical model. We begin by considering a mortgage pool of N mortgages. We assume that all mortgages are initiated at the same time and have the same maturity, which we take to be 30 years. The individual mortgage holder then makes payments until they pay off the mortgage at maturity or drop out of the pool. For our model the mortgage holder may drop out of this pool under any of the following three conditions:

(i) The mortgage holder defaults on their repayments (the default time of the i^{th} mortgage is denoted τ_i^{def}),
(ii) The mortgage holder fully repays their mortgage (the repayment time of the i^{th} mortgage is denoted τ_i^{pay}),
(iii) The mortgage holder refinances their mortgage or is forced to sell their house due to changes in circumstances (the refinancing or house sale time of the i^{th} mortgage is denoted τ_i^{ref}),

The time the i^{th} mortgage exits the pool is then defined to be

$$\tau_i := \tau_i^{\text{def}} \wedge \tau_i^{\text{pay}} \wedge \tau_i^{\text{ref}}.$$

Note that for simplicity we assume that default occurs at the first time that a mortgage payment cannot be made and that we do not consider the possibility of partial prepayments where only a part of the mortgage principal is repaid at a given time.

To model default and repayment, assign the i^{th} mortgage an exogenous risk process, X^i, which will be referred to as the *wealth process*. This process is modelled as

$$X_t^i := X_0^i + \mu t + \sigma \rho W_t^M + \sigma \sqrt{1-\rho^2} W_t^i \tag{2.1}$$

where:

- $W^1, W^2, W^3, \ldots$ are independent Brownian motions representing idiosyncratic risk factors,
- W^M is another independent Brownian motion capturing exposure to market effects,
- The *correlation*, ρ, is a constant taking values in $[0,1)$,
- The *drift*, $\mu \in \mathbf{R}$, and *volatility*, $\sigma > 0$, are constants,
- $X_0^1, X_0^2, X_0^3, \ldots$ are i.i.d. $(0,1)$-valued random variables with density $V_0 : (0,1) \to (0,\infty)$, which is assumed to be bounded and compactly supported in $(0,1)$. These random variables are determined by factors such as the LTV ratio, creditworthiness and the prevailing interest rates.

With these processes, the default time of the i^{th} entity is modelled as the first time X^i hits level zero,

$$\tau_i^{\text{def}} := \inf \left\{ t > 0 : X_t^i \leq 0 \right\},$$

and the repayment time as the first time X^i hits level one,

$$\tau_i^{\text{pay}} := \inf \left\{ t > 0 : X_t^i \geq 1 \right\}.$$

(See Figure 2.1.)

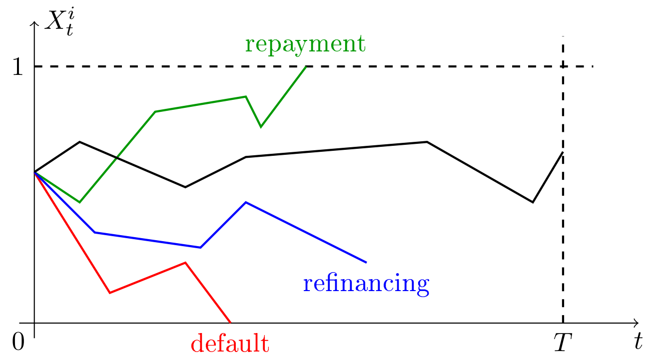

FIGURE 2.1. Four possible evolutions of X^i and the corresponding time τ_i. For the red curve $\tau_i = \tau_i^{\text{def}}$, for the blue curve $\tau_i = \tau_i^{\text{ref}}$, for the green curve $\tau_i = \tau_i^{\text{pay}}$ and for the black curve $\tau_i > T$.

To model the refinancing time, τ_i^{ref}, we specify that $\{\tau_i^{\text{ref}}\}_{i\geq 1}$ are conditionally i.i.d. with common law

$$\Lambda_t := \mathbf{P}(t < \tau_1^{\text{ref}} | \mathcal{F}^M)$$

given some market factors, and that τ_i^{ref} and X^i are also conditionally independent. Here, $\{\mathcal{F}_t^M\}_{t\geq 0}$ denotes the filtration generated by the market factors together with the market Brownian motion, hence Λ is a $\mathcal{F}^M$-random process, which will be referred to as the *refinancing process*. It will be further assumed that Λ is continuous, $\Lambda_0 = 1$ and that there exists a constant $\delta > 0$ such that

$$\mathbf{E}\big[\big(\Lambda_t - \Lambda_s\big)^2\big] = O\big(\big|t-s\big|^{1+\delta}\big), \quad \text{uniformly in } 0 \leq s < t \leq T, \tag{2.2}$$

as $|t-s| \to 0$. (This assumption is helpful in the proof of Corollary 3.2, which is then used to establish Theorem 2.2 in Section 4.) Our approach leaves the user relatively free to specify a dynamic model for Λ.

Example 2.1 (A possible model for refinancing)**.** A natural choice is to model refinancing times through an $\mathcal{F}^M$-random hazard rate process λ:

$$\Lambda := \exp\left\{-\int_0^t \lambda_s ds\right\}$$

We can then assign dynamics to λ. One choice is to assume that there is a fixed hazard rate $\bar{\lambda}$ (though it will typically be an $\mathcal{F}^M$-measurable random variable) to capture the sale of the house due to job loss or marital breakdown. There is also an underlying market interest rate, r, evolving according to a $\mathcal{F}^M$-measurable CIR process,

$$dr_t = \alpha(\beta - r_t)dt + \sigma_I \sqrt{r_t} dW_t^I, \tag{2.3}$$

where α, β and σ_I are constants and W^I is a standard Brownian motion that is typically correlated with W^M and that mortgage holders prefer to refinance to a lower rate when the current rate has dropped below a threshold K, depending on the initial rate, which could be captured by setting

$$\lambda_t = \bar{\lambda} + \text{constant} \times (K - r_t)^+ .$$

A CMO is a mortgage-backed security that is an option on the loss from the pool. With our model established, we are interested in the evolution of the *loss process*:

$$L_t^N := \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{\tau_i \le t}$$

To study the evolution of the spatial distribution of the wealth processes, $\{X^i\}_{i\ge 1}$, introduce the empirical process:

$$\nu_t^N := \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{t<\tau_i}\delta_{X_t^i}$$

where δ_x is the usual Dirac delta mass of the point $x \in [0,1]$. The empirical process takes random values in the space of sub-probability measures on $[0,1]$. For time $t>0$ and subset $S \subseteq (0,1)$, $\nu_t^N(S)$ represents the proportion of mortgages that have not left the pool by time t and have wealth processes taking values in S. Therefore the proportion of mortgages lost from the pool at time t is

$$1-\nu_t^N(0,\infty) = L_t^N,$$

which connects ν^N and L^N.

The limiting system. Observing that $\{\mathbf{1}_{t<\tau_i}\delta_{X_t^i}\}_{i\ge 1}$ is a family of conditionally i.i.d. random variables (given $\mathcal{F}^M$), Birkoff's Ergodic Theorem [15] implies that

$$\nu_t^N(S) \to \mathbf{P}\big(X_t^1 \in S; t<\tau_1 \,\big|\, \mathcal{F}^M\big), \quad \text{with probability } 1,$$

as $N\to\infty$. This leads us to define the *limit empirical process* to be

$$\nu_t\,(S) := \mathbf{P}\big(X_t^1 \in S; t<\tau_1 \,\big|\, \mathcal{F}^M\big) \tag{2.4}$$

and the *limiting loss process*

$$L_t := \mathbf{P}\big(\tau_1 \le t \,\big|\, \mathcal{F}^M\big) = 1-\nu_t\,(0,\infty)\,. \tag{2.5}$$

By construction, the marginals of ν^N and L^N converge to those of ν and L, however the following result establishes weak convergence at the process level. (See Section 4 for a full description of the relevant modes of convergence.)

Theorem 2.2 (Law of large numbers)**.** *Let $\mathcal{M}$ denote the space of finite-measures on* $[0,1]$ *equipped with the toplogy of weak convergence. Then $(\nu^N)_{N\ge 1}$ converges in law to ν as a sequence of càdlàg processes taking values in $\mathcal{M}$, with respect to the Skorokhod J_1 topology. Hence the sequence of loss processes, $(L^N)_{N\ge 1}$, converges in law to L on the space of càdlàg real-valued paths, with respect to the Skorokhod J_1 topology.*

This is a typical result for interacting particle systems. There are numerous examples of applications of similar models: the modelling of large collections of neurons and threshold hitting times for membrane potential levels in mathematical neuroscience [19, 11, 41, 12], the modelling of a large number of non-cooperative agents in mean-field games [24, 37, 5, 6], filtering theory [1, 7], mathematical genetics [10] and portfolio credit modelling [9, 51], to give a non-exhaustive list. Here, the interaction term is simple, so Theorem 2.2 is established in Section 4 through standard estimates together with some specific probabilistic estimates (see Section 3).

This result justifies the large population approximation

$$\mathbf{E}\Psi(L^N) \approx \mathbf{E}\Psi(L).$$

Thus when pricing an MBS, that is a tranche from the pool, we can use the limiting loss process L as an approximation to the loss from the pool.

The limit SPDE. Using the notation

$$\zeta(\phi) := \int_{(0,1)} \phi(x)\,\zeta(dx)$$

for a general measure ζ and measurable function ϕ on the unit interval, the following theorem states that the limit empirical process is characterised as the unique solution to a SPDE on the unit interval, which we call the limit SPDE. To avoid potential ambiguity caused by assigning mass to the boundary, solutions will be considered as processes taking values in the space

$$\mathcal{M}_0 := \{\zeta \in \mathcal{M} : \zeta(S) = \zeta(S \cap (0,1)),\ \text{for all measurable } S\}. \tag{2.6}$$

To encode the Dirichlet boundary conditions, the following test function space is used

$$C^{\text{test}} = \left\{\phi \in C^2([0,1]) : \phi(0) = 0 = \phi(1)\right\}.$$

(We will see in Section 5 why this is the natural space of test functions.)

Theorem 2.3 (The limit SPDE)**.** *The limit empirical process, ν, satisfies the* limit SPDE*: with probability 1*

$$\nu_t(\phi) = \nu_0(\phi) + \mu\int_0^t \nu_s(\partial_x\phi)\,ds + \frac{\sigma^2}{2}\int_0^t \nu_s(\partial_{xx}\phi)\,ds + \sigma\rho\int_0^t \nu_s(\partial_x\phi)\,dW_s^M + \int_0^t \nu_s(\phi)\,\Lambda_s^{-1}d\Lambda_s \tag{2.7}$$

for every $\phi \in C^{\text{test}}$ and $t \in [0,T]$. Furthermore, ν is the unique such solution: if $\nu' \in \mathcal{M}_0$ satisfies (2.7) and $\nu'_0 = \nu_0$,

$$\mathbf{P}(\nu'_t = \nu_t \text{ for every } t \in [0,T]) = 1.$$

The existing literature on stochastic PDEs is extensive. Most relevant to our setting are [32, 34, 35]. The approach we take is to convolve a candidate solution to produce a smooth approximate solution, which can then be manipulated classically. This technique is outlined in Section 6, and applying the method to the limit SPDE leads naturally to the energy estimates in Section 7. Uniqueness then follows as an immediate corollary in Section 8, however the following result is also a by-product. It is possible to push the method further than we do here and to obtain information about higher-order regularity of ν [38]. We will write $H^1(0,1)$ for the classical Sobolev space of functions in $L^2(0,1)$ with their derivatives also in $L^2(0,1)$.

Theorem 2.4 (Regularity)**.** *The limit empirical process, ν, has a density process, V, such that, with probability 1:*

(i) For every $t \in [0,T]$, $V_t \in L^2(0,1)$,
(ii) For almost all $t \in [0,T]$, $V_t \in H^1(0,1)$,
(iii) For every $t \in [0,T]$ and bounded and measurable ϕ, $\nu_t(\phi) = \int_0^1 V_t(x)\,\phi(x)\,dx$,
(iv) There exists a constant $c > 0$ depending only on the model parameters such that

$$\mathbf{E}\sup_{t\in[0,T]} \|V_t\|_{L^2}^2 + \mathbf{E}\int_0^T \|\partial_x V_t\|_{L^2}^2\,dt \le c\,\|V_0\|_{L^2}^2.$$

Writing (2.7) in terms of the density process V and using integration by parts we can write the limit SPDE for V as

$$dV_t = (-\mu\partial_x V_t + \frac{\sigma^2}{2}\partial_{xx}V_t)dt + \Lambda^{-1}V_t d\Lambda_t - \sigma\rho\partial_x V_t dW_t^M,$$

with V_0 a given initial density. A concrete version of this process is given in (9.2) and will be used for the numerical work in Section 9.

3. Boundary estimate

The main probabilistic argument in this paper provides control on the decay of ν near the boundaries at zero and one:

Proposition 3.1 (Boundary estimate)**.** *There exists a constant $\beta > 0$ such that*

$$\mathbf{E}\big[\nu_t\,(0,\varepsilon)^2 + \nu_t\,(1-\varepsilon,1)^2\big] = O\big(\varepsilon^{3+\beta}\big), \quad \textit{uniformly in } t \in [0,T]\,,$$

as $\varepsilon \to 0$.

The following proof is a slight extension of the argument in [3, Lemma 3.5], which is applicable to the case of a single absorbing boundary on the half-line. The key to this argument is the fact that the second moment of the mass near the boundary can be written in terms of the probability that a two-dimensional correlated Brownian motion is close to the apex of the positive quadrant, without having exited that domain. Exact formulae are then available for the law of this process [25, 42]. Similar applications also appear in CVA adjustments [39, 40].

Proof of Proposition 3.1. Begin by defining two new processes, ν^0 and ν^1

$$\nu_t^0(S) := \mathbf{P}(X_t^1 \in S; t < \tau_0^{\mathrm{def}} | \mathcal{F}^M), \quad \nu_t^1(S) := \mathbf{P}(X_t^1 \in S; t < \tau_1^{\mathrm{pay}} | \mathcal{F}^M),$$

where $S \subseteq \mathbf{R}$ and $t \in [0,1]$. From the definition of ν in (2.4), it is clear that $\nu \leq \nu^0$ and $\nu \leq \nu^1$. Now, ν^0 is of the form of the corresponding process from [3, Lemma 3.5], hence it follows that there exists a constant $\beta > 0$ such that

$$\mathbf{E}\big[\nu_t\,(0,\varepsilon)^2\big] = O(\varepsilon^{3+\beta}), \quad \text{uniformly in } t > 0 \text{ and } \varepsilon > 0. \tag{3.1}$$

The process ν^1 is not of the required form to immediately apply [3, Lemma 3.5], so define the reflected process

$$\nu_t^{1,\mathrm{r}}(S) := \nu_t^1(1-S), \quad \text{where } 1-S := \{1-x : x \in S\}\,.$$

Then $\nu_t^{1,\mathrm{r}}\,(0,\varepsilon) = \nu_t^1\,(1-\varepsilon,1)$ and

$$\nu_t^{1,\mathrm{r}}(S) = \mathbf{P}(X_t^{1,\mathrm{r}} \in S; t < \tau^{1,\mathrm{r}} | \mathcal{F}^M),$$

where $X^{1,\mathrm{r}} = 1 - X^1$ and $\tau^{1,\mathrm{r}} = \tau_1^{\mathrm{pay}}$, so $\nu^{1,\mathrm{r}}$ is the conditional law of a killed linear Brownian motion, and therefore is of the form required to apply [3, Lemma 3.5] and so

$$\mathbf{E}\big[\nu_t\,(1-\varepsilon,1)^2\big] = O(\varepsilon^{3+\beta}), \quad \text{uniformly in } t > 0 \text{ and } \varepsilon > 0. \tag{3.2}$$

By applying (3.1) and (3.2) it follows that for $\varepsilon < 1$

$$\mathbf{E}\big[\nu_t\,(0,\varepsilon)^2 + \nu_t\,(1-\varepsilon,1)^2\big] = O(\varepsilon^{3+\beta}),$$

which completes the proof. □

A useful corollary of this result, which is used in the next section, is that the boundary estimate controls the probability of a pair of wealth processes hitting zero or one in a small time interval:

Corollary 3.2 (Small time killing estimate)**.** *There exists a constant $\eta > 0$ such that*

$$\mathbf{P}\big(s < \tau_1, \tau_2 \leq t\big) = O\big(|t-s|^{1+\eta}\big), \quad \textit{uniformly in } 0 \leq s < t \leq T,$$

as $|t-s| \to 0$.

Proof. Conditioning on the values of X^1 and X^2 and the filtration $\mathcal{F}^M$ gives

$$\mathbf{P}\big(s<\tau_1,\tau_2\le t\big)=\mathbf{E}\int_0^1\!\!\int_0^1\mathbf{P}\big(s<\tau_1,\tau_2\le t\big|\mathcal{F}^M,X_t^1\!=\!x_1,X_t^2\!=\!x_2\big)\nu_s\big(dx^1\big)\nu_s\big(dx^2\big).$$

For a free parameter $\varepsilon>0$, this integral can be bounded above by considering the range of integration over the regions $(0,\varepsilon)$, $(1-\varepsilon,1)$ and their complement, which gives

$$\mathbf{P}\big(s<\tau_1,\tau_2\le t\big)\le\mathbf{E}\big[\nu_s\big(0,\varepsilon\big)^2\big]+\mathbf{E}\big[\nu_s\big(1-\varepsilon,1\big)^2\big]+\mathbf{P}\big(\sup_{u\in[s,t]}\big|X_u^1-X_s^1\big|\ge\varepsilon\big)$$
$$+\,\mathbf{P}\big(s<\tau_1^{\text{ref}},\tau_2^{\text{ref}}\le t\big),$$

where the third term is implied by the fact if $X_s^1\in(\varepsilon,1-\varepsilon)$, then X^1 must be displaced by at least ε in order to reach the boundary. Using Proposition 3.1, the definition in (2.1), the conditional independence of τ_i^{ref} and the assumption in (2.2)

$$\mathbf{P}\big(s<\tau_1,\tau_2\le t\big)\le c\varepsilon^{3+\beta}+\mathbf{P}\big(|\mu|\,|t-s|+\sigma\sup_{u\in[0,t-s]}|B_u|\ge\varepsilon\big)+\mathbf{E}\big[\big(\Lambda_t-\Lambda_s\big)^2\big]$$
$$\le c\varepsilon^{3+\beta}+2\Phi\big(-\sigma^{-1}\varepsilon|t-s|^{-1/2}+\sigma^{-1}|\mu||t-s|^{1/2}\big)+O\big(\big|t-s\big|^{1+\delta}\big),$$

provided $\varepsilon>|\mu||t-s|$, where Φ is the c.d.f. for the standard normal distribution. Setting $\varepsilon=|t-s|^\gamma$ with $\gamma<1/2$ and using the exponential decay of Φ at $-\infty$ gives

$$\mathbf{P}\big(s<\tau_1,\tau_2\le t\big)=O\big(|t-s|^{(3+\beta)\gamma}+|t-s|^{1+\delta}\big),\quad\text{as }\ |t-s|\to0.$$

Taking any η satisfying

$$\frac{1}{3+\beta}<\eta<\frac{1}{2}$$

completes the proof. □

4. Convergence of the system; Proof of Theorem 2.2

This section addresses the convergence of the sequence $\big(\nu^N\big)_{N\ge1}$ to the limit empirical process, ν. The finite empirical processes, ν^N, will be considered as measure-valued càdlàg processes. The space of finite-measures on $[0,1]$ will be denoted $\mathcal{M}$ and the space of càdlàg functions from $[0,T]$ to $\mathcal{M}$ denoted $D_{\mathcal{M}}$.

Here, $\mathcal{M}$ is equipped with the topology of weak convergence, which is metrised by

$$\text{dist}_{\text{BL}}\left(\zeta_1,\zeta_2\right):=\sup_{\phi\in\text{BL}}\left|\zeta_1\left(\phi\right)-\zeta_2\left(\phi\right)\right|,$$

[16, Problem 3.11.2], where BL denotes the space

$$\text{BL}:=\{\phi:[0,1]\to\mathbf{R}\text{ s.t. }\ |\phi\left(x\right)|\le1\ \text{and}\ \ |\phi\left(x\right)-\phi\left(y\right)|\le|x-y|\ \text{ for all }x,y\in[0,1]\}.$$

The space $D_{\mathcal{M}}$ will be equipped with the Skorokhod J_1 topology, and the reader can find a full description in [16, Chapter 3]. A useful characterisation of convergence in this topology is given by [16, Theorem 3.8.8], and so for our problem it suffices to check:

Lemma 4.1. *Both the following hold:*

(i) For all $k\ge1$ and $0\le t_1<t_2<\cdots<t_k$ the law of the marginal

$$\big(\nu_{t_1}^N,\nu_{t_2}^N,\ldots,\nu_{t_k}^N\big)\in\mathcal{M}^k$$

converges to the law of

$$\big(\nu_{t_1},\nu_{t_2},\ldots,\nu_{t_k}\big)\in\mathcal{M}^k$$

as $N\to\infty$,

(ii) With the constant $\eta > 0$ from Corollary 3.2

$$\mathbf{E}\big[\|\nu^N_{t+h} - \nu^N_t\|^4_{\mathrm{BL}}\,\|\nu^N_t - \nu^N_{t-h}\|^4_{\mathrm{BL}}\big] = O\big(h^{1+\eta}\big), \quad \textit{uniformly in } N \textit{ and } t,$$

as $h \to 0$.

Before proving this result, the following lemma is helpful.

Lemma 4.2 (Moment calculation)**.** *For $N \geq 1$ and $0 \leq s < t \leq T$ define*

$$A^N_{s,t} := \frac{1}{N}\sum_{i=1}^N \big|X^i_{t\wedge\tau^i} - X^i_{s\wedge\tau^i}\big|.$$

Then as $|t-s| \to 0$

$$\mathbf{E}\big[\big|A^N_{s,t}\big|^4\big] = O\big(|t-s|^2\big), \quad \textit{uniformly in } s,t \textit{ and } N.$$

Proof. By Hölder's inequality and the fact that the wealth processes are identically distributed

$$\mathbf{E}\big[\big|A^N_{s,t}\big|^4\big] \leq \mathbf{E}\frac{1}{N}\sum_{i=1}^N \big|X^i_{t\wedge\tau^i} - X^i_{s\wedge\tau^i}\big|^4 \leq \mathbf{E}\big[\big|X^1_{t\wedge\tau^1} - X^1_{s\wedge\tau^1}\big|^4\big].$$

The result then follows immediately from the definition in (2.1) by applying Doob's maximal inequality. □

Proof of Lemma 4.1. (i) The Arzela–Ascoli Theorem [2, Theorem 7.2] gives that BL is a compact subset of $C([0,1], \|\cdot\|_\infty)$. It therefore follows that the collection of spatial projections

$$\Big\{\pi^\phi : D_{\mathcal{M}} \to D_{\mathbf{R}}\Big\}_{\phi\in\mathrm{BL}}, \quad \pi^\phi(\zeta) := (\zeta_t(\phi))_{t\in[0,T]}$$

strongly separates points in $\mathcal{M}$, and hence is convergence-determining [16, Theorem 3.4.5]. Hence it suffices to show that

$$\big(\nu^N_{t_1}(\phi_1), \nu^N_{t_2}(\phi_2), \dots, \nu^N_{t_k}(\phi_k)\big) \xrightarrow{\mathrm{d}} \big(\nu_{t_1}(\phi_1), \nu_{t_2}(\phi_2), \dots, \nu_{t_k}(\phi_k)\big),$$

as $N \to \infty$, and this follows from the conditional independence of the wealth processes, as in (2.4).

(ii) For any $\phi \in \mathrm{BL}$ and $t, t+h \in [0,T]$

$$\begin{aligned}
\big|\nu^N_{t+h}(\phi) - \nu^N_t(\phi)\big| &\leq \frac{1}{N}\sum_{i=1}^N \big|\phi\big(X^i_{t+h}\big)\mathbf{1}_{t+h<\tau^i} - \phi\big(X^i_t\big)\mathbf{1}_{t<\tau^i}\big| \\
&\leq \frac{1}{N}\sum_{i=1}^N \big|\phi\big(X^i_{(t+h)\wedge\tau^i}\big) - \phi\big(X^i_{t\wedge\tau^i}\big)\big| + \frac{1}{N}\sum_{i=1}^N \mathbf{1}_{t\leq\tau^i<t+h} \\
&\leq \frac{1}{N}\sum_{i=1}^N \big|X^i_{(t+h)\wedge\tau^i} - X^i_{t\wedge\tau^i}\big| + \big|L^N_{t+h} - L^N_t\big|,
\end{aligned}$$

where the second line follows from the triangle inequality and the equation

$$\phi\left(X^i_t\right)\mathbf{1}_{t<\tau_i} = \phi\left(X^i_{t\wedge\tau_i}\right) - \phi(0)\,\mathbf{1}_{\tau_i\leq t}.$$

This implies

$$\big\|\nu^N_{t+h} - \nu^N_t\big\|_{\mathrm{BL}} \leq A^N_{t,t+h} + B^N_{t,t+h}, \quad \text{where } B^N_{t,t+h} := \big|L^N_{t+h} - L^N_t\big|,$$

with the notation A^N from Lemma 4.1. Therefore, using the simple inequality $(a+b)^4 \leq 8\left(a^4+b^4\right)$ and the fact that B^N is bounded by 1 gives

$$\begin{aligned}
\mathbf{E}\big[\|\nu^N_{t+h}-\nu^N_t\|^4_{\mathrm{BL}}\|\nu^N_t-\nu^N_{t-h}\|^4_{\mathrm{BL}}\big] &\leq \mathbf{E}\big[\big(A^N_{t,t+h}+B^N_{t,t+h}\big)^4\big(A^N_{t-h,t}+B^N_{t-h,t}\big)^4\big] \\
&\leq 64\mathbf{E}\big[\big(\big(A^N_{t,t+h}\big)^4+\big(B^N_{t,t+h}\big)^4\big)\big(\big(A^N_{t-h,t}\big)^4+\big(B^N_{t-h,t}\big)^4\big)\big] \\
&\leq 64\mathbf{E}\big[\big(A^N_{t,t+h}\big)^4\big(A^N_{t-h,t}\big)^4\big]+64\mathbf{E}\big[\big(A^N_{t,t+h}\big)^4\big] \\
&\quad +64\mathbf{E}\big[\big(A^N_{t-h,t}\big)^4\big]+64\mathbf{E}\big[\big(B^N_{t-h,t}\big)^2\big(B^N_{t-h,t}\big)^2\big],
\end{aligned}$$

and Cauchy–Schwarz and Lemma 4.1 reduces this to

$$\mathbf{E}\big[\|\nu^N_{t+h}-\nu^N_t\|^4_{\mathrm{BL}}\|\nu^N_t-\nu^N_{t-h}\|^4_{\mathrm{BL}}\big] \leq 64\mathbf{E}\big[\big(B^N_{t-h,t}\big)^2\big(B^N_{t-h,t}\big)^2\big]+O\big(h^2\big). \tag{4.1}$$

The first term on the right-hand side of (4.1) can be expanded as

$$\mathbf{E}\big[\big(B^N_{t-h,t}\big)^2\big(B^N_{t-h,t}\big)^2\big] = \frac{1}{N^4}\sum_{1\leq i,j,k,l\leq N}\mathbf{E}\left[X_iX_jY_kY_l\right],$$

with the random variables

$$X_i = \mathbf{1}_{t<\tau_i\leq t+h} \quad \text{and} \quad Y_i = \mathbf{1}_{t-h<\tau_i\leq t}.$$

Noticing that $X_iX_jY_kY_l = 0$ whenever $i=k,l$ or $j=k,l$, it is always the case that $X_iX_jY_kY_l \leq X_iY_k$, therefore

$$\mathbf{E}\big[\big(B^N_{t-h,t}\big)^2\big(B^N_{t-h,t}\big)^2\big] \leq \mathbf{E}\big[X_1Y_2\big] \leq \mathbf{P}\big(t-h<\tau_1,\tau_2\leq t+h\big).$$

Hence the result is proved by (4.1) and Corollary 3.2. □

The first part of Theorem 2.2 now follows by [16, Theorem 3.8.8]. The second part of Theorem 2.2 then follows from the fact that the map

$$\zeta \in \mathcal{M} \mapsto \zeta\left(0,1\right) \in \mathbf{R}$$

is continuous (since $\mathbf{1}_{[0,1]} \in \mathrm{BL}$) and that $L^N_t = 1-\nu^N_t\left(0,1\right)$. □

5. The limit empirical process solves the limit SPDE

In this short section the first half of the statement of Theorem 2.3 is proved:

Proposition 5.1 (Existence)**.** *The limit empirical process, ν, satisfies the limit SPDE.*

Our strategy is to consider the dynamics of a single wealth process under the action of a test function and to calculate its conditional law. The following lemma is useful as it allows us to interchange conditional expectation and stochastic integration:

Lemma 5.2. *Let H be a real-valued adapted process with*

$$\mathbf{E}\int_0^T H_s^2 ds < \infty.$$

Then, with probability 1,

$$\mathbf{E}\Big[\int_0^t H_s dW^M_s\big|\mathcal{F}^M_t\Big] = \int_0^t \mathbf{E}\big[H_s\big|\mathcal{F}^M_s\big]dW^M_s \quad \textit{and} \quad \mathbf{E}\Big[\int_0^t H_s dW^1_s\big|\mathcal{F}^M_t\Big] = 0.$$

for every $t \in [0,T]$.

Proof. By multiplying H_s by $\mathbf{1}_{s<t}$, it suffices to take $t = T$. First, suppose that H is a basic process, that is

$$H_u = Z\mathbf{1}_{s_1<u\leq s_2},$$

where $s_1 < s_2 \leq T$ are real numbers and Z is $\mathcal{F}_{s_1}$-measurable. Then

$$\begin{aligned}\mathbf{E}\big[\int_0^T H_s dW_s \big| \mathcal{F}_T^M\big] = \mathbf{E}\big[Z\big(W_{s_2}^M - W_{s_1}^M\big)\big|\mathcal{F}_T^M\big] &= \mathbf{E}\big[Z\big|\mathcal{F}_{s_1}^M\big]\big(W_{s_2}^M - W_{s_1}^M\big)\\ &= \int_0^T \mathbf{E}\big[Z\big|\mathcal{F}_s^M\big]\mathbf{1}_{s_1<s\leq s_2} dW_s^M\\ &= \int_0^T \mathbf{E}\big[H_s\big|\mathcal{F}_s^M\big] dW_s^M\end{aligned}$$

and, using the fact that $W_{s_2}^1 - W_{s_1}^1$ is independent of $\sigma\big(\mathcal{F}_T^W, \mathcal{F}_{s_1}\big)$ because W^1 and W^M are independent and W^1 has independent increments,

$$\begin{aligned}\mathbf{E}\big[\int_0^T H_s dW_s^1 \big| \mathcal{F}_t^M\big] = \mathbf{E}\big[Z\big(W_{s_2}^1 - W_{s_1}^1\big)\big|\mathcal{F}_t^M\big] &= \mathbf{E}\big[\mathbf{E}\big[Z\big(W_{s_2}^1 - W_{s_1}^1\big)\big|\sigma\big(\mathcal{F}_T^M, \mathcal{F}_{s_1}\big)\big]\big|\mathcal{F}_t^M\big]\\ &= \mathbf{E}\big[Z\mathbf{E}\big[\big(W_{s_2}^1 - W_{s_1}^1\big)\big|\sigma\big(\mathcal{F}_T^M, \mathcal{F}_{s_1}\big)\big]\big|\mathcal{F}_t^M\big]\\ &= \mathbf{E}\big[Z\mathbf{E}\big[W_{s_2}^1 - W_{s_1}^1\big]\big|\mathcal{F}_T^M\big] = 0.\end{aligned}$$

So the result holds in this case and immediately extends to linear combinations of basic processes. The usual density argument extends the result to all required H. □

Proof of Proposition 5.1. Begin by considering the dynamics of $\phi\left(X_t^i\right)$ for any $\phi \in C^{\text{test}}$. Since ϕ is smooth, Itô's formula is applicable:

$$\begin{aligned}\phi\left(X_t^1\right) = \phi\left(X_0^1\right) + \mu\int_0^t \partial_x\phi\left(X_s^1\right) ds + \frac{\sigma^2}{2}\int_0^t \partial_{xx}\phi\left(X_s^1\right) ds + \sigma\rho\int_0^t \partial_x\phi\left(X_s^1\right) dW_s^M\\ + \sigma\sqrt{1-\rho^2}\int_0^t \partial_x\phi\left(X_s^1\right) dW_s^1.\end{aligned}$$

As $\phi(0) = 0 = \phi(1)$, stopping the equation at the first exit time from the unit interval, $\tau_1^{\text{exit}} := \tau_1^{\text{def}} \wedge \tau_1^{\text{pay}}$, gives

$$\begin{aligned}\phi\big(X_t^1\big)\mathbf{1}_{t<\tau_1^{\text{exit}}} &= \phi\big(X_{t\wedge\tau_1^{\text{exit}}}^1\big)\\ &= \phi\big(X_0^1\big) + \mu\int_0^t \partial_x\phi\big(X_s^1\big)\mathbf{1}_{s<\tau_1^{\text{exit}}} ds + \frac{\sigma^2}{2}\int_0^t \partial_{xx}\phi\big(X_s^1\big)\mathbf{1}_{s<\tau_1^{\text{exit}}} ds \qquad (5.1)\\ &\quad + \sigma\rho\int_0^t \partial_x\phi\big(X_s^1\big)\mathbf{1}_{s<\tau_1^{\text{exit}}} dW_s^M + \sigma\sqrt{1-\rho^2}\int_0^t \partial_x\phi\big(X_s^1\big)\mathbf{1}_{s<\tau_1^{\text{exit}}} dW_s^1.\end{aligned}$$

By introducing the process

$$\bar{\nu}_t(\phi) := \mathbf{E}\left[\phi\left(X_t^1\right)\mathbf{1}_{t<\tau^{\text{exit}}}\middle|\mathcal{F}^M\right]$$

and taking a conditional expectation over (5.1) with respect to $\mathcal{F}^M$ (using Lemma 5.2) the following SPDE for $\bar{\nu}$ is obtained:

$$d\bar{\nu}_t(\phi) = \mu\bar{\nu}_t(\partial_x\phi)\, dt + \frac{\sigma^2}{2}\bar{\nu}_t(\partial_{xx}\phi)\, dt + \sigma\rho\bar{\nu}_t(\partial_x\phi)\, dW_t^M. \qquad (5.2)$$

Since $\tau_1 = \tau_1^{\mathrm{exit}} \wedge \tau_1^{\mathrm{ref}}$ and τ_1^{exit} and τ_1^{ref} are conditionally independent given $\mathcal{F}^M$, it follows that

$$\nu_t(\phi) = \mathbf{E}\big[\phi(X_t^1)\mathbf{1}_{t<\tau_1^{\mathrm{exit}}}\mathbf{1}_{t<\tau_1^{\mathrm{ref}}}\big|\mathcal{F}^M\big] = \mathbf{E}\big[\phi(X_t^1)\mathbf{1}_{t<\tau_1^{\mathrm{exit}}}\big|\mathcal{F}^M\big]\mathbf{P}\big(t < \tau_1^{\mathrm{ref}}\big|\mathcal{F}^M\big) = \Lambda_t\bar{\nu}_t(\phi),$$

and, because Λ is of finite variation, combining this result with the product rule gives

$$d\nu_t(\phi) = \Lambda_t d\bar{\nu}_t(\phi) + \bar{\nu}_t(\phi)\, d\Lambda_t = \Lambda_t d\bar{\nu}_t(\phi) + \nu_t(\phi)\, \Lambda_t^{-1} d\Lambda_t.$$

Substituting the SPDE for $\bar{\nu}$ from (5.2) completes the result. □

The final calculation in the above proof will be used again, so it is helpful to summarise the result:

Lemma 5.3. *If $\bar{\nu}_t(\phi) := \Lambda_t^{-1}\nu_t(\phi)$, then $\bar{\nu}$ satisfies the SPDE*

$$\bar{\nu}_t(\phi) = \nu_0(\phi) + \mu\int_0^t \bar{\nu}_s(\partial_x\phi)\, ds + \frac{\sigma^2}{2}\int_0^t \bar{\nu}_s(\partial_{xx}\phi)\, ds + \sigma\rho\int_0^t \bar{\nu}_s(\partial_x\phi)\, dW_s^M,$$

for every $t \in [0, T]$ and $\phi \in C^{\mathrm{test}}$, with probability 1. (Note, $\Lambda_0 = 1$.)

6. The kernel smoothing method

This section serves as an overview of the kernel smoothing method which is used in Section 7 to prove Theorem 2.3. The method originated from [36] and [31], where stochastic evolution equations are considered on the whole space, and was adapted in [3] to incorporate systems on the half-line with Dirichlet boundary conditions and [38] for weighted spaces. Here, our approach is to work on the unit interval, hence the relevant smoothing kernel is defined as:

Definition 6.1 (Smoothing kernel)**.** For $x, y \in [0, 1]$ and $\varepsilon > 0$, the *smoothing kernel* is defined to be

$$G_\varepsilon(x, y) := \frac{1}{\sqrt{2\pi\varepsilon}}\sum_{n=-\infty}^{\infty}\Big[\exp\Big\{-\frac{(x-y+2n)^2}{2\varepsilon}\Big\} - \exp\Big\{-\frac{(x+y+2n)^2}{2\varepsilon}\Big\}\Big],$$

which is the Green's function for the heat equation on the unit interval with Dirichlet boundary conditions. Importantly, $G_\varepsilon(x, 0), G_\varepsilon(x, 1) = 0$, so that $G_\varepsilon(x, \cdot) \in C^{\mathrm{test}}$, for every fixed $x \in (0, 1)$.

If ζ is a finite measure on the unit interval, then integrating the smoothing kernel in one of its variables yields a smooth function (recall the definition of $\mathcal{M}_0$ from (2.6)):

Definition 6.2 (T_ε)**.** For $\zeta \in \mathcal{M}_0$, define $T_\varepsilon\zeta \in C^\infty$ to be the function

$$T_\varepsilon\zeta(x) := \zeta(G_\varepsilon(x, \cdot)) = \int_{(0,1)} G_\varepsilon(x, y)\,\zeta(dy), \quad \text{for } x \in [0, 1].$$

It is a standard result that $T_\varepsilon\zeta$ approximates ζ in a distributional sense: for any smooth and compactly supported $\phi \in C_0^\infty(0, 1)$

$$\int_{(0,1)} T_\varepsilon\zeta(x)\,\phi(x)\, dx \to \zeta(\phi), \quad \text{as } \varepsilon \to 0. \tag{6.1}$$

This fact allows us to prove the following key result. Here, $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^1}$ denote the usual norms on the unit interval,

$$\|\phi\|_{L^2} := \Big(\int_0^1 \phi(x)^2\, dx\Big)^{1/2} \quad \text{and} \quad \|\phi\|_{H^1} := \Big(\int_0^1 \phi(x)^2 + \partial_x\phi(x)^2\, dx\Big)^{1/2},$$

and H_0^1 denotes the closure of $C_0^\infty(0, 1)$ under $\|\cdot\|_{H^1}$.

Proposition 6.3 (The lim inf Proposition)**.** *Let $\zeta \in \mathcal{M}_0$ and let $\|\cdot\|_2$ denote the L^2 norm on $[0,1]$. If*

$$\liminf_{\varepsilon \to 0} \|T_\varepsilon \zeta\|_{H^1} < \infty,$$

then ζ has a density $Z \in H_0^1(0,1)$, that is

$$\zeta(\phi) = \int_0^1 Z(x)\,\phi(x)\,dx, \quad \text{for all } \phi \in C_0^\infty(0,1),$$

and Z satisfies

$$\|Z\|_{H^1} \le \liminf_{\varepsilon \to 0} \|T_\varepsilon \zeta\|_{H^1}.$$

Proof. It follows from (6.1) and the Cauchy–Schwarz inequality that

$$\big|\zeta(\phi)\big| = \liminf_{\varepsilon \to 0} \Big|\int_{(0,1)} T_\varepsilon \zeta(x)\,\phi(x)\,dx\Big| \le \liminf_{\varepsilon \to 0} \big\|T_\varepsilon \zeta\big\|_{L^2} \big\|\phi\big\|_{L^2},$$

for any $\phi \in C_0^\infty(0,1)$. Therefore the map $\phi \mapsto \zeta(\phi)$ extends uniquely to a map in the dual of L^2 with norm not greater than the above limit infimum, by the Hahn–Banach Theorem [46, Theorem III.5]. Also, by the Riesz Representation Theorem [17, D.3 Theorem 2], we can conclude that this map is realised as an integral over the unit interval against some $Z \in L^2$. Combining these results gives that

$$\zeta(\phi) = \int_0^1 Z(x)\,\phi(x)\,dx, \text{ for every } \phi \in C_0^\infty(0,1) \quad \text{and} \quad \|Z\|_{L^2} \le \liminf_{\varepsilon \to 0} \|T_\varepsilon \zeta\|_{L^2}.$$

Similarly, but now also integrating by parts,

$$\big|\zeta(\partial_x \phi)\big| = \liminf_{\varepsilon \to 0} \Big|\int_{[0,1]} \partial_x T_\varepsilon \zeta(x)\,\phi(x)\,dx\Big| \le \liminf_{\varepsilon \to 0} \big\|\partial_x T_\varepsilon \zeta\big\|_{L^2} \big\|\phi\big\|_{L^2},$$

so by the same reasoning there exists $Y \in L^2$ such that

$$\zeta(\partial_x \phi) = \int_0^1 Y(x)\,\phi(x)\,dx, \text{ for every } \phi \in C_0^\infty(0,1) \quad \text{and} \quad \|Y\|_{L^2} \le \liminf_{\varepsilon \to 0} \|\partial_x T_\varepsilon \zeta\|_{L^2}.$$

However, it is the case that

$$\int_0^1 Z(x)\,\partial_x \phi(x)\,dx = \zeta(\partial_x \phi) = \int_0^1 Y(x)\,\phi(x)\,dx, \quad \text{for every } \phi \in C_0^\infty(0,1)$$

so $Y = -\partial_x Z$, which suffices to complete the proof. □

This result generalises easily to measure-valued stochastic processes, and the following result is a simple corollary of Fatou's lemma, but is in a form that will be convenient in the forthcoming sections.

Proposition 6.4 (The lim inf Proposition for processes)**.** *Suppose that ξ is a càdlàg stochastic process taking values in $\mathcal{M}_0$. If there exists a constant $K > 0$ such that*

$$\liminf_{\varepsilon \to 0} \Big\{ \mathbf{E}\big[\sup_{t \in [0,T]} \|T_\varepsilon \xi_t\|_{L^2}^2\big] + \mathbf{E} \int_0^T \|\partial_x T_\varepsilon \xi_t\|_{L^2}^2\,dt \Big\} \le K < \infty,$$

then there exists a process Ξ such that:

(i) For every $t \in [0,T]$, $\Xi_t \in L^2(0,1)$,
(ii) For almost all $t \in [0,T]$, $\Xi_t \in H_0^1(0,1)$,

(iii) *For every* $t \in [0,T]$

$$\xi_t(\phi) = \int_0^1 \Xi_t(x)\,\phi(x)\,dx,$$

(iv) $\mathbf{E}\big[\sup_{t\in[0,T]} \|\Xi_t\|_{L^2}^2\big] + \mathbf{E}\int_0^T \|\partial_x \Xi_t\|_{L^2}^2\,dt \le K.$

Remark 6.5. Although Proposition 6.4 is stated for measure-valued proposes, it is also valid for processes taking values in the space of signed measures, or even the space of distributions, but with the appropriate modification of (2.6) for test functions, rather than sets.

It is often useful to note that T_ε is a contraction on L^2:

Proposition 6.6 (Contractivity). *Let* $f \in L^2(0,1)$*, then for every* $\varepsilon > 0$

$$\|T_\varepsilon f\|_{L^2} \le \|f\|_{L^2},$$

with the notation $T_\varepsilon f(x) = \int_0^1 G_\varepsilon(x,y)\,f(y)\,dy$.

Proof. After noticing that $\int_0^1 G_\varepsilon(x,y)\,dy \le 1$ ($G_\varepsilon(x,\cdot)$ is a defective density), the proof is a simple application of Cauchy–Schwarz:

$$\|T_\varepsilon f\|_{L^2}^2 \le \int_0^1\!\!\int_0^1 G_\varepsilon(x,y)\,dy \cdot \int_0^1 G_\varepsilon(x,y)\,f(y)^2\,dydx \le \int_0^1\!\!\int_0^1 G_\varepsilon(x,y)\,f(y)^2\,dxdy \le \|f\|_{L^2}^2.$$

□

7. Energy estimation; proof of Theorem 2.4

In this section the energy estimate in Theorem 2.4 is derived. This result will be used to show uniqueness in the next section.

It is simpler to consider the evolution equation for the process $\bar\nu$ from Lemma 5.3:

$$\bar\nu_t(\phi) = \nu_0(\phi) + \mu\int_0^t \bar\nu_s(\partial_x\phi)\,ds + \frac{\sigma^2}{2}\int_0^t \bar\nu_s(\partial_{xx}\phi)\,ds + \sigma\rho\int_0^t \bar\nu_s(\partial_x\phi)\,dW_s^M.$$

Our strategy is to show that the smooth approximation, $T_\varepsilon\bar\nu$, satisfies an approximate version of this SPDE.

For any $x \in [0,1]$ and $\varepsilon > 0$, $G_\varepsilon(x,\cdot)$ is a element of C^{test} (see Definition 6.1) and so can be set in the SPDE for $\bar\nu$, which gives

$$\begin{aligned} T_\varepsilon\bar\nu_t(x) = \bar\nu_t(G_\varepsilon(x,\cdot)) &= \nu_0(G_\varepsilon(x,\cdot)) + \mu\int_0^t \bar\nu_s(\partial_y G_\varepsilon(x,\cdot))\,ds \\ &\quad + \frac{\sigma^2}{2}\int_0^t \bar\nu_s(\partial_{yy}G_\varepsilon(x,\cdot))\,ds + \sigma\rho\int_0^t \bar\nu_s(\partial_y G_\varepsilon(x,\cdot))\,dW_s^M \end{aligned} \tag{7.1}$$

The following result is a straightforward calculation, but allows the order of differentiation to be switched in the above equation:

Lemma 7.1 (Switching derivatives). *Let* $\zeta \in \mathcal{M}$ *and define*

$$r_\varepsilon(z) := \frac{2}{\sqrt{2\pi\varepsilon}}\sum_{n=-\infty}^{\infty}\exp\Big\{-\frac{(z+2n)^2}{2\varepsilon}\Big\}, \quad \text{for } z \in \mathbf{R}$$

and

$$R_\varepsilon\zeta(x) := \zeta(r_\varepsilon(x+\cdot)) = \int_{(0,1)} r_\varepsilon(x+y)\,\zeta(dy), \quad \text{for } x \in [0,1].$$

The following hold for all $x, y \in \mathbf{R}$ *and* $\varepsilon > 0$*:*

(i) $\partial_y G_\varepsilon(x,y) = -\partial_x G_\varepsilon(x,y) - \partial_x r_\varepsilon(x+y)$,
(ii) $\partial_{yy} G_\varepsilon(x,y) = \partial_{xx} G_\varepsilon(x,y)$,
(iii) $\zeta(\partial_y G_\varepsilon(x,\cdot)) = -\partial_x T_\varepsilon \zeta(x) - \partial_x R_\varepsilon \zeta(x)$,
(iv) $\zeta(\partial_{yy} G_\varepsilon(x,\cdot)) = \partial_{xx} T_\varepsilon \zeta(x)$.

Applying these results to (7.1) gives the smoothed version of the limit SPDE for $T_\varepsilon \bar{\nu}$:

$$\begin{aligned} T_\varepsilon \bar{\nu}_t(x) &= T_\varepsilon \nu_0(x) - \mu \int_0^t \partial_x T_\varepsilon \bar{\nu}_s(x)\,ds + \frac{\sigma^2}{2}\int_0^t \partial_{xx} T_\varepsilon \bar{\nu}_s(x)\,ds \\ &\quad - \sigma\rho \int_0^t \partial_x T_\varepsilon \bar{\nu}_s(x)\,dW_s^M - (\mu+\sigma\rho)\int_0^t \partial_x R_\varepsilon \bar{\nu}_s(x)\,dW_s^M, \end{aligned} \tag{7.2}$$

and then Itô's formula for the square $(T_\varepsilon \bar{\nu})^2$ yields the equation

$$\begin{aligned} d(T_\varepsilon \bar{\nu}_t)^2 &= 2T_\varepsilon \bar{\nu}_t dT_\varepsilon \bar{\nu}_t + d[T_\varepsilon \bar{\nu}]_t \\ &= -2\mu T_\varepsilon \bar{\nu}_t \partial_x T_\varepsilon \bar{\nu}_t dt + \sigma^2 T_\varepsilon \bar{\nu}_t \partial_{xx} T_\varepsilon \bar{\nu}_t dt - 2\sigma\rho T_\varepsilon \bar{\nu}_t \partial_x T_\varepsilon \bar{\nu}_t dW_t^M \\ &\quad - 2(\mu+\sigma\rho) T_\varepsilon \bar{\nu}_t \partial_x R_\varepsilon \bar{\nu}_t dW_t^M + \sigma^2\rho^2 (\partial_x T_\varepsilon \bar{\nu}_t)^2 dt + (\mu+\sigma\rho)^2 (\partial_x R_\varepsilon \bar{\nu}_t)^2 dt, \end{aligned} \tag{7.3}$$

where the dependency on x has been omitted. The aim is now to integrate over the spatial dimension. Since $T_\varepsilon \bar{\nu}_t$ is a smooth bounded function (for fixed ε)

$$\int_0^1 \Big(\mathbf{E}\int_0^T |T_\varepsilon \bar{\nu}_t \partial_x T_\varepsilon \bar{\nu}_t|^2\,dt\Big)^{1/2} dx < \infty,$$

hence the stochastic Fubini Theorem [53, (1.4)] implies

$$\int_0^1 \int_0^t T_\varepsilon \bar{\nu}_t(x)\,\partial_x T_\varepsilon \bar{\nu}_t(x)\,dW_s^M dx = \int_0^t \int_0^1 T_\varepsilon \bar{\nu}_t(x)\,\partial_x T_\varepsilon \bar{\nu}_t(x)\,dx dW_s^M = 0,$$

which vanishes by integrating by parts and using $T_\varepsilon \bar{\nu}_t(0) = 0 = T_\varepsilon \bar{\nu}_t(1)$. Therefore, integrating equation (7.3) over $x \in (0,1)$ gives

$$\begin{aligned} \|T_\varepsilon \bar{\nu}_t\|_{L^2}^2 &= \|T_\varepsilon \nu_0\|_{L^2}^2 - \sigma^2(1-\rho^2)\int_0^t \|\partial_x T_\varepsilon \bar{\nu}_s\|_{L^2}^2\,ds + (\mu+\sigma\rho)^2 \int_0^t \|\partial_x R_\varepsilon \bar{\nu}_s\|_{L^2}^2\,ds \\ &\quad - 2(\mu+\sigma\rho)\int_0^t\int_0^1 T_\varepsilon \bar{\nu}_s \partial_x R_\varepsilon \bar{\nu}_s dx dW_s^M, \end{aligned}$$

where the stochastic Fubini theorem has been used on the final term.

By taking a supremum over $t \in [0,T]$ and an expectation in the above equation, and using the estimate

$$\begin{aligned} \mathbf{E}\sup_{t\in[0,T]} \int_0^t\int_0^1 T_\varepsilon \bar{\nu}_s \partial_x R_\varepsilon \bar{\nu}_s dx dW_s^m &\le \mathbf{E}\Big(\int_0^T \Big(\int_0^1 T_\varepsilon \bar{\nu}_t \partial_x R_\varepsilon \bar{\nu}_t dx\Big)^2 dW_t^M\Big)^{1/2} \\ &\le \mathbf{E}\Big(\int_0^T \|T_\varepsilon \bar{\nu}_t\|_{L^2}^2 \|\partial_x R_\varepsilon \bar{\nu}_t\|_{L^2}^2\,dt\Big)^{1/2} \\ &\le \mathbf{E}\Big[\sup_{t\in[0,T]} \|T_\varepsilon \bar{\nu}_t\|_{L^2} \Big(\int_0^T \|\partial_x R_\varepsilon \bar{\nu}_t\|_{L^2}^2\,dt\Big)^{1/2}\Big] \\ &\le \delta \mathbf{E}\sup_{t\in[0,T]} \|T_\varepsilon \bar{\nu}_t\|_{L^2}^2 + (4\delta)^{-1}\mathbf{E}\int_0^T \|\partial_x R_\varepsilon \bar{\nu}_t\|_{L^2}^2\,dt, \end{aligned}$$

which follows from the Cauchy–Schwarz inequality, Young's inequality with parameter δ and the Burholder–Davis–Gundy inequality [47, Theorem IV.42.1], the following inequality is reached:

$$c_1\mathbf{E}\sup_{t\in[0,T]}\|T_\varepsilon\bar{\nu}_t\|_{L^2}^2+\sigma^2\left(1-\rho^2\right)\mathbf{E}\int_0^T\|\partial_xT_\varepsilon\bar{\nu}_t\|_{L^2}^2\,dt\le\|T_\varepsilon\nu_0\|_{L^2}^2+c_2\mathbf{E}\int_0^T\|\partial_xR_\varepsilon\bar{\nu}_t\|_{L^2}^2\,dt$$

where $c_i>0$ denotes some numerical constants depending only on the model parameters and the choice of δ. Since $\rho<1$ and $\Lambda_t\Lambda_s^{-1}<1$ whenever $s<t$, it is possible to multiply throughout by Λ_T^2 and normalise this inequality to obtain a constant $c>0$ such that

$$\mathbf{E}\sup_{t\in[0,T]}\|T_\varepsilon\nu_t\|_{L^2}^2+\mathbf{E}\int_0^T\|\partial_xT_\varepsilon\nu_s\|_{L^2}^2\,ds\le c\,\|T_\varepsilon\nu_0\|_{L^2}^2+c\mathbf{E}\int_0^T\|\partial_xR_\varepsilon\nu_t\|_{L^2}^2\,dt. \tag{7.4}$$

It is now clear that the energy estimate requires control of the remainder term. The following result connects this problem to the boundary estimate from Section 3.

Lemma 7.2 (Vanishing remainder)**.** *Suppose ξ is a càdlàg process taking values in $\mathcal{M}_0$ for which there exists a constant β such that*

$$\mathbf{E}\big[\xi_t\left(0,\varepsilon\right)^2+\xi_t\left(1-\varepsilon,1\right)^2\big]=O\big(\varepsilon^{3+\beta}\big),\quad \textit{uniformly in } t\in[0,T]\,.$$

Then

$$\mathbf{E}\int_0^T\|\partial_xR_\varepsilon\xi_t\|_{L^2}^2\,dt\to0,\quad \textit{as } \varepsilon\to0.$$

Proof. For fixed ε, direct calculation gives

$$\partial_xr_\varepsilon\left(z\right)=-\sum_{n=-\infty}^{\infty}a_{n,\varepsilon}\left(z\right),\quad \text{where } a_{n,\varepsilon}\left(z\right):=\frac{z+2n}{\varepsilon^{3/2}\sqrt{2\pi}}\exp\Big\{-\frac{\left(z+2n\right)^2}{2\varepsilon}\Big\}.$$

Therefore, if $n\neq-1,0$ and $z\in[0,2]$, then

$$|a_{n,\varepsilon}\left(z\right)|\le c_1\varepsilon^{-3/2}\left(n+1\right)e^{-n^2}e^{-1/\varepsilon},\quad \text{whenever } \varepsilon<1,$$

uniformly in ε, z and n, where $c_1>0$ is a numerical constant, and hence

$$|\partial_xr_\varepsilon\left(z\right)|\le|a_{0,\varepsilon}\left(z\right)|+|a_{-1,\varepsilon}\left(z\right)|+c_2\varepsilon^{-3/2}e^{-1/\varepsilon}.$$

This leads to the inequality

$$|\partial_xR_\varepsilon\xi_t\left(x\right)|\le\int_0^1|a_{0,\varepsilon}\left(x+y\right)|\,\xi_t\left(dy\right)+\int_0^1|a_{-1,\varepsilon}\left(x+y\right)|\,\xi_t\left(dy\right)+c_2\varepsilon^{-3/2}e^{-1/\varepsilon}.$$

It is now possible to split the range of integration in the two above integrals. For the first, consider the integral over $(0,\varepsilon^\eta)$ and its complement, where $\eta>0$ is a free parameter. This gives

$$\int_0^1|a_{0,\varepsilon}\left(x+y\right)|\,\xi_t\left(dy\right)\le c_3\varepsilon^{-3/2+\eta}\xi_t\left(0,\varepsilon^\eta\right)\exp\{-x^2/2\varepsilon\}+c_3\varepsilon^{-3/2}\exp\{-\varepsilon^{\eta-1/2}\},$$

where $c_3>0$ is a further constant. A similar expression is obtained for the other integral term, and so if η is chosen such that $\eta<1/2$ then

$$\mathbf{E}\big\|\partial_xR_\varepsilon\xi_t\big\|_{L^2}^2\le c_4\varepsilon^{-5/2+2\eta}\big(\mathbf{E}\big[\xi_t\big(0,\varepsilon^\eta\big)^2\big]+\mathbf{E}\big[\xi_t\big(1-\varepsilon^\eta,1\big)^2\big]\big)+o\big(1\big),\quad \text{as } \varepsilon\to0,$$

using the fact $\int_0^1e^{-x^2/2\varepsilon}dx=O\big(\varepsilon^{1/2}\big)$. Applying the hypothesis gives

$$\mathbf{E}\int_0^T\|\partial_xR_\varepsilon\xi_t\|_{L^2}^2\,ds\le c_4\varepsilon^{-5/2+(5+\beta)\eta}+o\left(1\right),$$

which vanishes if η is chosen to satisfy

$$\frac{1}{2(1+\beta/5)} < \eta < \frac{1}{2}.$$

Since $\beta > 0$, such a selection is possible and this completes the proof. □

Combining Lemma 7.2 with Proposition 3.1 and applying the result to (7.4) gives

$$\liminf_{\varepsilon\to 0}\left\{\mathbf{E}\sup_{t\in[0,T]}\|T_\varepsilon\nu_t\|_{L^2}^2+\mathbf{E}\int_0^T\|\partial_x T_\varepsilon\nu_s\|_{L^2}^2\,ds\right\}\le c\liminf_{\varepsilon\to 0}\|T_\varepsilon\nu_0\|_{L^2}^2\le c\,\|V_0\|_{L^2}^2\,,$$

where the superadditivity of the limit infimum has been used. Fatou's lemma and Proposition 6.4 then completes the result. □

8. Uniqueness; Proof of Theorem 2.3

Here, we complete the proof of the uniqueness result using the methods developed in the previous chapters. Begin by supposing that ν' is a càdlàg process that solves the limit SPDE and satisfies $\nu_0' = \nu_0$. The difference $\Delta := \nu - \nu'$, although not a positive measure, still satisfies the limit SPDE (by linearity), but with zero initial condition. Likewise, the work from the previous section applies, so (7.4) reduces to

$$\mathbf{E}\sup_{t\in[0,T]}\|T_\varepsilon\Delta_t\|_{L^2}^2\le c\mathbf{E}\int_0^T\|\partial_x R_\varepsilon\Delta_t\|_{L^2}^2\,dt, \tag{8.1}$$

by dropping the derivative term on the left-hand side.

Since

$$\|\partial_x R_\varepsilon\Delta_t\|_{L^2}^2\le 2\,\|\partial_x R_\varepsilon\nu_t\|_{L^2}^2+2\left\|\partial_x R_\varepsilon\nu_t'\right\|_{L^2}^2, \quad \text{for every } t\le T,$$

to deduce that the right-hand side of (8.1) vanishes, all that is required is to check that ν' satisfies the estimate from Proposition 3.1, since then the result follows by Lemma 7.2.

Proposition 8.1. *For the candidate solution, ν', there exists $\beta > 0$ such that*

$$\mathbf{E}[\nu_t'(0,\varepsilon)^2]+\mathbf{E}[\nu_t'(1-\varepsilon,1)^2]=O(\varepsilon^{3+\beta}), \quad \textit{uniformly in } t\in[0,T]\,,$$

as $\varepsilon \to 0$.

Proof. By applying the product rule to the process defined by

$$\bar{\nu}_t'(\phi):=\Lambda_t^{-1}\nu_t'(\phi)\,,$$

it follows that $\tilde{\nu}'$ satisfies

$$d\bar{\nu}_t'(\phi)=\mu\bar{\nu}_t'(\partial_x\phi)\,dt+\frac{\sigma^2}{2}\bar{\nu}_t'(\partial_{xx}\phi)\,dt+\sigma\rho\bar{\nu}_t'(\partial_x\phi)\,dW_t^M, \quad \text{for } \phi\in C^{\text{test}}.$$

Since $\Lambda \le 1$, $\nu' \le \bar{\nu}'$, so it suffices to show that $\bar{\nu}'$ satisfies the statement of the result.

Our strategy is to consider the following measure on $(0,1)\times(0,1)$:

$$m_t(\phi):=\mathbf{E}\int_0^1\!\!\int_0^1\phi(x,y)\,\bar{\nu}_t'(dx)\,\bar{\nu}_t'(dy)\,, \quad \text{for } \phi:(0,1)\times(0,1)\to\mathbf{R}.$$

First consider ϕ of the form $\phi(x,y) = \psi_1(x)\psi_2(y)$, where $\psi_1, \psi_2 \in C^{\text{test}}$. Applying the product rule and taking expectation gives

$$dm_t(\phi)=m_t(\mathcal{L}\phi)\,dt, \quad \text{where } \mathcal{L}=\mu\nabla+\frac{\sigma^2}{2}\Delta+\sigma\rho\partial_{xy}, \tag{8.2}$$

which then extends to all smooth $\phi : (0,1) \times (0,1) \to \mathbf{R}$ that satisfy

$$\phi\left(\partial\left((0,1) \times (0,1)\right)\right) = \{0\}.$$

Now let $G_t^{0,1} : (0,1)^2 \times (0,1)^2 \to \mathbf{R}$ be the Green's function associated to $\mathcal{L}$ on the unit square with absorbing boundary conditions. Define

$$H_t f(\mathbf{x}) := \int_0^1 \int_0^1 G_t^{0,1}(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) \, d\mathbf{y}, \quad \text{for } f : (0,1)^2 \to \mathbf{R},$$

and, for a fixed $\phi \in C_0^\infty((0,1)^2)$ and $t_0 \in [0,T]$, define

$$\psi_t := H_{t_0 - t}\phi, \quad \text{for } t \le t_0.$$

Then ψ_t satisfies $\partial_t \psi_t + \mathcal{L}\psi_t = 0$, so the Itô–Wentzell formula [33] applied to (8.2) gives

$$dm_t(\psi_t) = m_t(\partial_t \psi_t + \mathcal{L}\psi_t)\, dt = 0, \quad \text{hence } m_{t_0}(\phi) = m_0(H_{t_0}\phi).$$

By a density argument and the assumption that $\bar{\nu}_0'$ has no atoms at 0 or 1, we can extend to sets:

$$\begin{aligned}\mathbf{E}[\bar{\nu}_t'(S)^2] &= m_{t_0}(S \times S) = m_0\left(H_{t_0}\mathbf{1}_{S\times S}\right) \\ &= \int_0^1\int_0^1 \mathbf{P}\left(X_t^1, X_t^2 \in S; t < \tau_1 \wedge \tau_2 \,\middle|\, X_0^1 = x^1, X_0^2 = x^2\right) \nu_t\left(dx^1\right) \nu_t\left(dx^2\right),\end{aligned}$$

The argument in the proof of Proposition 3.1 can then be applied to this final expression to give the result. □

Proposition 8.1 and inequality (8.1) give

$$\liminf_{\varepsilon \to 0} \mathbf{E} \sup_{t \in [0,T]} \|T_\varepsilon \Delta_t\|_{L^2}^2 = 0,$$

and this completes the result by Proposition 6.4 and Remark 6.5. □

9. Simulation and pricing examples

In this section we illustrate the flexibility of our model by showing how it can be used to price options on mortgage pools. The two examples that follow explain the effects of varying the individual input parameters.

How to simulate the model and standard parameters. In our example, we will use a semi-analytic Monte Carlo scheme for simulating the processes L^{def}, L^{pay} and L^{ref}, which represent the proportional loss from the mortgage pool due to default, early repayment and refinancing. The total loss will be denoted $L^{\text{total}} = L^{\text{def}} + L^{\text{pay}} + L^{\text{ref}}$. We will fix a set of standard parameters:

- *Time horizon*: $T = 30$ (years),
- *Parameters for distance-to-default*: $\mu = 0.024$, $\sigma = 0.115$, $\rho = 0.35$,
- *Initial distance-to-default*: Beta distribution, $V_0(x) \propto x^{b_1}(1-x)^{b_2}$, where $b_1 = 2.7$, $b_2 = 3.05$,
- *Refinancing parameters*: $\Lambda_t = \exp\{-\int_0^t \lambda_s ds\}$, with

$$\lambda_t = \max\{K - r_t, 0\}, \qquad dr_t = a(b - r_t)dt + \sigma_I \sqrt{r_t} dW_t^I, \tag{9.1}$$

  where $r_0 = 0.06$ $K = 0.05$, $a = 0.6$, $b = 0.06$, $\sigma_I = 0.25$ and W^I a standard Brownian motion with $[W^M, W^I]_t = 0.35t$.

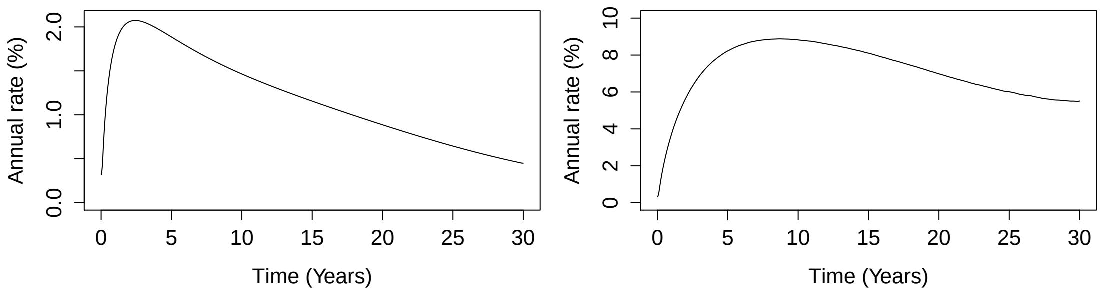

FIGURE 9.1. (Standard parameters) The expected annual default rate is presented on the left. The expected annual repayment rate plus refinancing rate is on the right.

The evolution of the system from an initial density V_0 and interest rate r_0 is then described by an SPDE and SDE pair

$$
\begin{aligned}
dV_t &= \left(-\mu\partial_x V + \frac{\sigma^2}{2}\partial_{xx}V_t - \max\{K - r_t, 0\}V_t\right)dt - \sigma\rho\partial_x V_t dW_t^M \\
dr_t &= a(b - r_t)dt + \sigma_I\sqrt{r_t}dW_t^I.
\end{aligned}
\tag{9.2}
$$

Our numerical scheme will regard the SPDE for V as a heat equation with a rough drift.

We have chosen this set of parameters as they produce annual conditional default and prepayment rates, see Figure 9.1, which approximate the standard shapes of the rate curves used in conventional analyses of MBS. The market convention in assessing prepayment is to take the Public Securities Association (PSA) prepayment benchmark, a monthly series of annual prepayment rates. The PSA curve is a piecewise linear function, with the conditional prepayment rate increasing linearly from 0 to 6% over the first 30 months and then constant at 6% for the remainder, see [18] for a discussion. Similarly there is a market convention in assessing default through the Standard Default Assumption (SDA) curve. This is again a piecewise linear function with an initial increase in annual conditional default rates to 0.6%, a plateau and then a decrease back to 0.03% at 120 months and then constant thereafter. For the precise SDA curve see [18]. Naturally our model will not reproduce exactly the PSA and SDA curves, but we choose a set of parameters so that our annual conditional prepayment and default curves are broadly similar and hence this choice serves as a reasonable benchmark.

The only source of randomness in each individual simulation is due to W^M and W^I. Once these sample paths have been simulated, the system can be generated by pushing the initial condition forward on the half-line by the heat equation with rough drift. Specifically, the algorithm we use is:

(i) Fix the number of time points, N_{time}. This fixes the corresponding mesh size, $\Delta t = T/N_{\text{time}}$,
(ii) Simulate W^M and W^I for N_{time} steps on the uniform grid. Use W^I and (9.1) to generate Λ,
(iii) For step $i \in \{1, 2, \dots, N_{\text{time}}\}$, let V^i denote the approximation to $V_{i\Delta t}$, the system's density process. (In practise, V^i will be stored as the interpolation of a fixed number of grid points.) Let $L^{i,\text{def}}$, $L^{i,\text{pay}}$ and $L^{i,\text{ref}}$ (all real numbers) denote the approximations to $L^{\text{def}}_{i\Delta t}$, $L^{\text{pay}}_{i\Delta t}$ and $L^{\text{ref}}_{i\Delta t}$,
(iv) At step $i = 0$, set $V^0 = V_0$ (ensure V_0 is normalised to be a p.d.f.) and $L^{0,\text{def}} = L^{0,\text{pay}} = L^{0,\text{ref}} = 0$,

(v) Fix the maps

$$\text{shift}(f;a)(x) := f(x-a), \qquad \text{scale}(f;a)(x) := af(x), \qquad \text{trunc}(f)(x) := f(x)\mathbf{1}_{x\in(0,1)},$$

and some quadrature routine (to a given level of precision)

$$\text{quad}(f;a,b) \approx \int_a^b f(x)dx, \qquad \text{gauss}(f;t)(x) = (2\pi)^{-\frac{1}{2}}\text{quad}(z \mapsto f(x+t^{\frac{1}{2}}z)e^{-z^2/2}; -\infty, +\infty).$$

The map "gauss" pushes f forward by a Normal$(0,t)$-density,

(vi) For $i \geq 1$, let increment $:= \sigma\rho(W^M_{i\Delta t} - W^M_{(i-1)\Delta t}) + \mu\Delta t$ and

$$f_{\text{temp}} := \text{shift}(\text{gauss}(V^{i-1}; \sigma\sqrt{1-\rho^2}\Delta t); \text{increment}),$$

which is the conditional distribution of a representative wealth process, without killing at the boundaries. Set the default and repayment increments to be the mass that has passed over the respective boundaries:

$$L^{i,\text{def}} = L^{i-1,\text{def}} + \text{quad}(f_{\text{temp}}; -\infty, 0), \qquad L^{i,\text{pay}} = L^{i-1,\text{pay}} + \text{quad}(f_{\text{temp}}; 1, +\infty).$$

Truncate the lost mass and scale the density for the proportion, $\Delta\Lambda_i = \Lambda_{(i-1)\Delta t} - \Lambda_{i\Delta t}$, that has refinanced:

$$V^i = \text{scale}(\text{trunc}(f_{\text{temp}}; 0, 1), 1 - \Delta\Lambda_i), \qquad L^{i,\text{ref}} = \Delta\Lambda_i \cdot \text{quad}(f_{\text{temp}}; 0, 1).$$

We will not demonstrate the validity of the algorithm in this paper.

A tranched product. (The figures in this subsection should be viewed in colour.) The MBS that we will price with our model will be a tranched product. Suppose that there is a recovery rate of $R = 40\%$ and consider tranche (a, d), with $a < d$ the attachment and detachment points. With the notation

$$\text{tranche}_{a,d}(x) := \begin{cases} 0, & \text{if } x < a \\ x - a, & \text{if } a \leq x \leq d \\ d - a, & \text{if } x > d, \end{cases}$$

the product is comprised of two parts:

- *Protection leg*: At monthly payment dates, T_i, the protection buyer receives the loss due to default in the tranche:
$$\text{protection}_i := \text{tranche}_{a,d}((1-R)L^{\text{def}}_{T_i}) - \text{tranche}_{a,d}((1-R)L^{\text{def}}_{T_{i-1}})$$
- *Fee leg*: The protection buyer pays a proportion (the *spread*), s, of the outstanding notational in the tranche in each period:
$$\text{fee}_i := d - a - \text{tranche}_{a,d}((1-R)L^{\text{def}}_{T_{i-1}}).$$

The spread is fixed to balance the expected cash-flows from the two legs:

$$s := \frac{\text{discounted protection}}{\text{discounted fee}} = \frac{\mathbf{E}\sum_{i:T_i<30 \text{ years}} \text{discount}(T_i) \cdot \text{protection}_i}{\mathbf{E}\sum_{i:T_i<30 \text{ years}} \text{discount}(T_i) \cdot \text{fee}_i}, \tag{9.3}$$

where discount(T_i) is an appropriate discounting factor, which for this paper we will fix as a constant compounded annual 1% rate.

To calculate the expectations in (9.3) we will average over a number, N_{sims}, of independent simulations:

$$s \approx \frac{\sum_{i=1}^{N_{\text{sims}}} \text{discounted protection for simulation } i}{\sum_{i=1}^{N_{\text{sims}}} \text{discounted fee for simulation } i}.$$

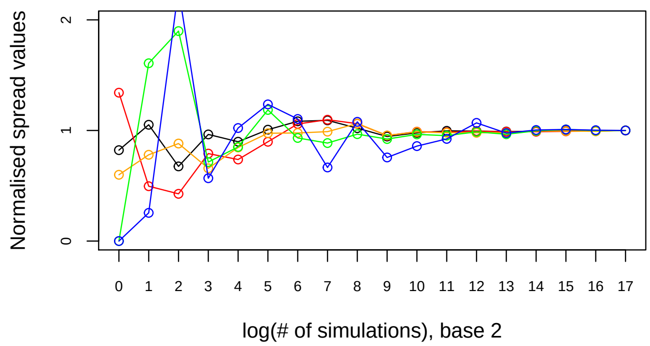

FIGURE 9.2. (Standard parameters) The normalised spread value is calculated by dividing the spread at each number of simulations by the spread at 2^{17} simulations. The tranches attachment/ detachment points are (0, 0.03), (0.03, 0.06), (0.06, 0.09), (0.09, 0.12), (0.12, 1). With the standard parameters, relatively few simulations assign loss to the high tranches, hence the lower stability for these tranches.

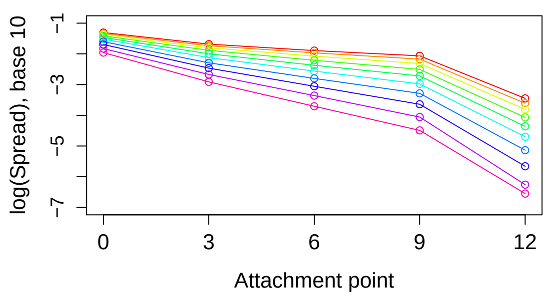

FIGURE 9.3. (Standard parameters) Each line represents the output for a specific choice of μ (red $= 0$, increasing in increments of 0.005 to pink $= 0.045$), with all other parameters equal to the standard parameters. Increasing μ decreases the spread for every tranche.

Figure 9.2 gives an indication of the convergence properties of this approximation. For the standard parameters, we see stable behaviour around $N_{\text{sims}} = 2^{13}$. The appropriate choice of N_{sims} depends on the model and tranche parameters — for example, if we only wish to price lower tranches then $N_{\text{sims}} = 2^9$ might be acceptable. For the remaining examples in this section we fix $N_{\text{sims}} = 2^{12}$.

Pricing example: Standard market conditions. In this subsection we price the tranched MBS with the standard parameters capturing the behaviour of prices in non-distressed market conditions.

In Figure 9.3, we see that increasing the value of μ decreases the spread for every tranche. This is as expected; increasing μ decreases the probability of a representative wealth process hitting the default boundary at zero, and hence reduces the probability that the tranche notional is lost.

The behaviour in Figure 9.4, which gives the change in spread as the correlation ρ is varied, is not monotone. For senior tranches, increasing ρ increases the spread. This is because larger values of ρ lead to a higher probability of systemic default, which is necessary to accumulate losses in these higher tranches. However, increasing ρ also increases the probability that wealth processes will experience a simultaneous positive move away from the boundary, thus reducing

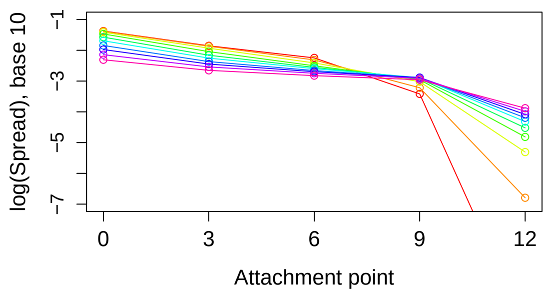

FIGURE 9.4. (Standard parameters) Each line represents the output for a specific choice of ρ (red $= 0$, increasing in increments of 0.1 to pink $= 0.9$), with all other parameters equal to the standard parameters. For higher tranches, spread increases with correlation, and for lower tranches this effect is reversed. For high tranches, the spread is very small for low correlations.

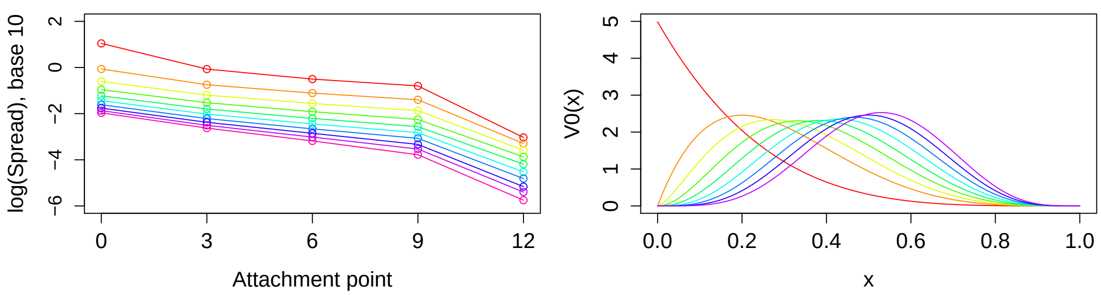

FIGURE 9.5. (Standard parameters) On the left, each line represents the output for a specific choice of β_1 (red $= 0$, increasing in increments of 0.5 to pink $= 4.5$) in the initial condition $V_0(x) \propto x^{\beta_1}(1-x)^{\beta_2}$, with all other parameters equal to the standard parameters (and $\beta_2 = 4$). The corresponding graph of V_0 is presented on the right. Increasing β_1 decreases the spread of every tranche.

the probability of losses in the lower tranches. Hence, increasing ρ decreases the spread for the junior tranches.

In Figure 9.5, the parameter β_1 in the initial density is varied. Decreasing β_1 shifts more of the initial mass towards the default boundary, hence increasing β_1 decreases the spreads.

Pricing example: Distressed market conditions. In this subsection we repeat the above pricing example, except we now fix $\beta_1 = 1$ and $\beta_2 = 10$. As seen from Figure 9.6, this moves most of the initial mass close to the default boundary at zero. With all other parameters set to the standard parameters, this choice captures distressed or sub-prime market conditions, which is reflected in the corresponding default rate curve in Figure 9.6.

Increasing the value of μ shows the same behaviour as in the previous example, except now the spreads are generally higher and more tightly packed (Figure 9.7, left). The explanation for the tight packing is that altering the drift on the range given in the figure has only a small influence on the price compared to that of the distressed initial condition. Varying the value of ρ, however, shows less predictable behaviour (Figure 9.7, right). For all except the most senior tranche, increasing ρ decreases the tranche spread. Again, this is because increasing the correlation increases the probability of entities surviving simultaneously. The crossover effect from Figure 9.4 is no longer observed.

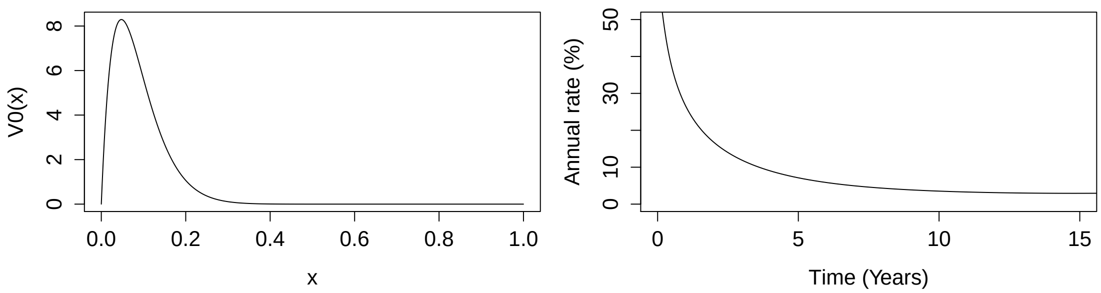

FIGURE 9.6. (Distressed parameters) On the left is the initial density for the choice of distressed parameters $\beta_1 = 1$ and $\beta_2 = 10$. The corresponding default rate curve is on the right. The default rate is extremely high initially and almost all the mass has been lost from the system after 15 years.

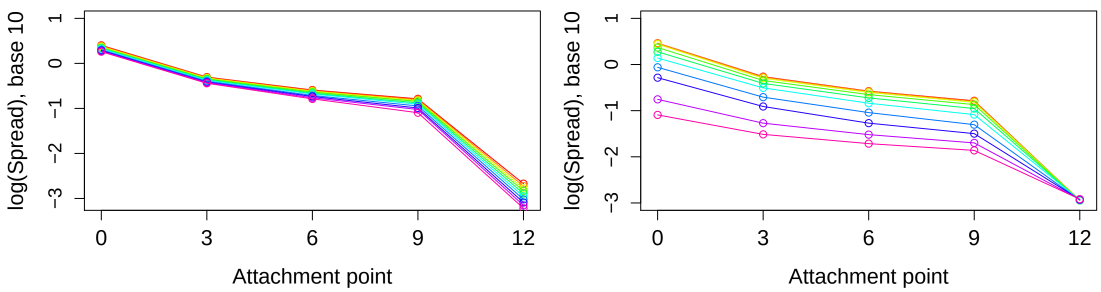

FIGURE 9.7. (Distressed parameters) On the left each line represents the output for a specific choice of μ (red $= 0$, increasing in increments of 0.005 to pink $= 0.045$), with all other parameters equal to the distressed parameters. On the right each line represents the output for a specific choice of ρ (red $= 0$, increasing in increments of 0.1 to pink $= 0.9$), with all other parameters equal to the distressed parameters.

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