Spectral asymptotics for stable trees

David Croydon* and Ben Hambly†

UNIVERSITY OF WARWICK UNIVERSITY OF OXFORD

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Abstract

We calculate the mean and almost-sure leading order behaviour of the high frequency asymptotics of the eigenvalue counting function associated with the natural Dirichlet form on α-stable trees, which lead in turn to short-time heat kernel asymptotics for these random structures. In particular, the conclusions we obtain demonstrate that the spectral dimension of an α-stable tree is almost-surely equal to $2\alpha/(2\alpha-1)$, matching that of certain related discrete models. We also show that the exponent for the second term in the asymptotic expansion of the eigenvalue counting function is no greater than $1/(2\alpha-1)$. To prove our results, we adapt a self-similar fractal argument previously applied to the continuum random tree, replacing the decomposition of the continuum tree at the branch point of three suitably chosen vertices with a recently developed spinal decomposition for α-stable trees.

1 Introduction

This work contains a study of the spectral properties of the class of random real trees known as α-stable trees, $\alpha \in (1,2]$. Such objects are natural: arising as the scaling limits of conditioned Galton-Watson trees [1], [6]; admitting constructions in terms of Levy processes [7] and fragmentation processes [12]; as well as having connections to continuous state branching process models [7]. In recent years, a number of geometric properties of α-stable trees have been studied, such as the Hausdorff dimension and measure function [8], [9], [12], degree of branch points [8] and decompositions into subtrees [13], [22], [23]. Here, our goal is to enhance this understanding of α-stable trees by establishing various analytical properties for them, including determining their spectral dimension, with the results we obtain extending those known to hold for the continuum random tree [4], which corresponds to the case $\alpha = 2$.

To allow us to state our main results, we will start by introducing some notation (precise definitions are postponed until Section 2). First, fix $\alpha \in (1,2]$ and let $\mathcal{T} = (\mathcal{T}, d_{\mathcal{T}})$ represent the α-stable tree $\mathcal{T}$ equipped with its natural metric $d_{\mathcal{T}}$. For $\mathbf{P}$-a.e. realisation of $\mathcal{T}$, it is possible to define a canonical non-atomic Borel probability measure, μ say,

*Dept of Statistics, University of Warwick, Coventry, CV4 7AL, UK; d.a.croydon@warwick.ac.uk.

†Mathematical Institute, 24-29 St Giles', Oxford, OX1 3LB, UK; hambly@maths.ox.ac.uk.

whose support is equal to $\mathcal{T}$, where $\mathbf{P}$ is the probability measure on the probability space upon which all the random variables of the discussion are defined. As with other measured real trees, by applying results of [18], one can check that it is $\mathbf{P}$-a.s. possible to construct an associated Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{T},\mu)$ as an electrical energy when we consider $(\mathcal{T},d_{\mathcal{T}})$ to be a resistance network. In particular, $(\mathcal{E},\mathcal{F})$ is characterised by the following identity: for every $x,y\in\mathcal{T}$,

$$d_{\mathcal{T}}(x,y)^{-1}=\inf\{\mathcal{E}(f,f):\ f\in\mathcal{F},\ f(x)=0,\ f(y)=1\}. \tag{1}$$

Our focus will be on the asymptotic growth of the eigenvalues of the triple $(\mathcal{E},\mathcal{F},\mu)$, which are defined to be the numbers λ which satisfy

$$\mathcal{E}(f,g)=\lambda\int_{\mathcal{T}} fg d\mu,\qquad \forall g\in\mathcal{F},$$

for some non-trivial eigenfunction $f\in\mathcal{F}$. The corresponding eigenvalue counting function, N, is obtained by setting

$$N(\lambda):=\#\{\text{eigenvalues of } (\mathcal{E},\mathcal{F},\mu)\leq\lambda\}. \tag{2}$$

Our conclusions for this function are presented in the following theorem, which describes the large λ mean and $\mathbf{P}$-a.s. behaviour of N. In the statement of the result, the notation $\mathbf{E}$ represents the expectation under the probability measure $\mathbf{P}$. Note that the first order result for $\alpha=2$ was established previously as [4], Theorem 2, and our proof is an adaptation of the argument followed there. In particular, in [4] the recursive self-similarity of the continuum random tree described in [2] was used to enable renewal and branching process techniques to be applied to deduce the results of interest. In this article, we proceed similarly by drawing recursive self-similarity for α-stable trees from a spinal decomposition proved in [13].

Theorem 1.1. *For each $\alpha\in(1,2]$ and $\varepsilon>0$, there exists a deterministic constant $C\in(0,\infty)$ such that the following statements hold.*
(a) As $\lambda\to\infty$,

$$\mathbf{E}N(\lambda)=C\lambda^{\frac{\alpha}{2\alpha-1}}+O\left(\lambda^{\frac{1}{2\alpha-1}+\varepsilon}\right).$$

(b) $\mathbf{P}$-a.s., as $\lambda\to\infty$,

$$N(\lambda)\sim C\lambda^{\frac{\alpha}{2\alpha-1}}.$$

Moreover, in $\mathbf{P}$-probability, the second order estimate of part (a) also holds.

Remark 1.2. *In the special case when $\alpha=2$, the estimate of the second order term can be improved to $O(1)$ in part (a) of the above theorem. A similar comment also applies to Corollaries 1.3(a) and 1.4 below.*

For a bounded domain $\Omega\subseteq\mathbb{R}^n$, Weyl's Theorem establishes for the Dirichlet or Neumann Laplacian eigenvalue counting function the limit

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{n/2}}=c_n|\Omega|_n,$$

where $|\Omega|_n$ is the n-dimensional Lebesgue measure of Ω and c_n is a dimension dependent constant. As a result, in the literature on fractal sets, the limit, when it exists,

$$d_S = 2 \lim_{\lambda \to \infty} \frac{\ln N(\lambda)}{\ln \lambda}$$

is frequently referred to as the spectral dimension of a (Laplacian on a) set. In our setting, the previous theorem allows us to immediately read off that an α-stable tree has $d_S = 2\alpha/(2\alpha - 1)$, $\mathbf{P}$-a.s., where the Laplacian considered here is that associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ in the standard way. As the Hausdorff dimension with respect to $d_{\mathcal{T}}$ of an α-stable tree $\mathcal{T}$ is $d_H = \alpha/(\alpha - 1)$ (see [8], [12]), it follows that $d_S = 2d_H/(d_H + 1)$. This relationship between the analytically defined d_S and geometrically defined d_H has been established for p.c.f. self-similar sets and some other finitely ramified random fractals when the Hausdorff dimension is measured with respect to an intrinsic resistance metric (which is identical to $d_{\mathcal{T}}$ in the α-stable tree case), see [14], [19] for example. Furthermore, it is worth remarking that $2\alpha/(2\alpha - 1)$ is also the spectral dimension of the random walk on a Galton-Watson tree whose offspring distribution lies in the domain of attraction of a stable law with index α, conditioned to survive [5]. This final observation could well have been expected given the convergence result proved in [3] that links the random walks on a related family of Galton-Watson trees conditioned to be large and the Markov process X corresponding to $(\mathcal{E}, \mathcal{F}, \mu)$, which can be interpreted as the Brownian motion on the α-stable tree.

Of course we have shown much more than just the existence of the spectral dimension, as we have demonstrated the mean and $\mathbf{P}$-a.s. existence of the Weyl limit (which does not exist for exactly self-similar fractals with a high degree of symmetry [19]). In fact, for a compact manifold with smooth boundary (under a certain geometric condition), it was proved in [15] that the asymptotic expansion of the eigenvalue counting function of the Neumann Laplacian is given by

$$N(\lambda) = c_n |\Omega| \lambda^{n/2} + \frac{1}{4} c_{n-1} |\partial\Omega|_{n-1} \lambda^{(n-1)/2} + o(\lambda^{(n-1)/2}).$$

Analogously, the result we establish here provides an estimate on the size of the second order term for α-stable trees. If our expansion had the same structure as the classical result, in the case $\alpha = 2$, for example, we would expect to see a constant second order term, as the natural boundary is finite. However, despite seeing this in mean, we do not have (or expect) an almost sure or in probability second term of this type. Indeed, although our results do not confirm that the second order exponent is equal to $1/(2\alpha - 1)$, we anticipate that the randomness in the structure leads to fluctuations of this higher order.

As in [4], it is straightforward to transfer our conclusions regarding the leading order spectral asymptotics of α-stable trees to a result about the heat kernel $(p_t(x,y))_{x,y \in \mathcal{T}}$ for the Laplacian associated with $(\mathcal{E}, \mathcal{F}, \mu)$. In particular, a simple application of an Abelian theorem yields the following asymptotics for the trace of the heat semigroup.

Corollary 1.3. *If $\alpha \in (1,2]$, $\varepsilon > 0$, C is the constant of Theorem 1.1 and Γ is the standard gamma function, then the following statements hold.*
(a) As $t \to 0$,

$$\mathbf{E} \int_{\mathcal{T}} p_t(x,x) \mu(dx) = C\Gamma\left(\tfrac{3\alpha-1}{2\alpha-1}\right) t^{-\frac{\alpha}{2\alpha-1}} + O\left(t^{-\frac{1}{2\alpha-1}+\varepsilon}\right).$$

(b) $\mathbf{P}$*-a.s., as* $t \to 0$*,*

$$\int_{\mathcal{T}} p_t(x,x)\mu(dx) \sim C\Gamma\left(\tfrac{3\alpha-1}{2\alpha-1}\right) t^{-\frac{\alpha}{2\alpha-1}}.$$

Finally, α-stable trees are known to satisfy the same root invariance property as the continuum random tree. More specifically, if we select a μ-random vertex $\sigma \in \mathcal{T}$, then the tree $\mathcal{T}$ rooted at σ has the same distribution as the tree $\mathcal{T}$ rooted at its original root, ρ say (see [8], Proposition 4.8). This allows us to transfer part (a) of the previous result to a limit for the annealed on-diagonal heat kernel at ρ (cf. [4], Corollary 4).

Corollary 1.4. *If* $\alpha \in (1,2]$*,* $\varepsilon > 0$*,* C *is the constant of Theorem 1.1 and* Γ *is the standard gamma function, then, as* $t \to \infty$*,*

$$\mathbf{E}p_t(\rho,\rho) = C\Gamma\left(\tfrac{3\alpha-1}{2\alpha-1}\right) t^{-\frac{\alpha}{2\alpha-1}} + O\left(t^{-\frac{1}{2\alpha-1}+\varepsilon}\right).$$

The rest of the article is organised as follows. In Section 2 we describe some simple properties of Dirichlet forms on compact real trees, and also introduce a spinal decomposition for α-stable trees that will be applied recursively. In Section 3 we prove the mean spectral result stated in this section, via a direct renewal theorem proof. By making the changes to [4] that were briefly described above, we then proceed to establishing the almost-sure first order eigenvalue asymptotics in Section 4 using a branching process argument. Finally, in Section 5, we further investigate the second order behaviour of the function $N(\lambda)$ as $\lambda \to \infty$.

2 Dirichlet forms and recursive spinal decomposition

Before describing the particular properties of α-stable trees that will be of interest to us, we present a brief introduction to Dirichlet forms on more general tree-like metric spaces. To this end, for the time being we suppose that $\mathcal{T} = (\mathcal{T}, d_{\mathcal{T}})$ is a deterministic compact real tree (see [21], Definition 1.1) and μ is a non-atomic finite Borel measure on $\mathcal{T}$ of full support. These assumptions easily allow us to check the conditions of [18], Theorem 5.4, to deduce that there exists a unique local regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathcal{T},\mu)$ associated with the metric $d_{\mathcal{T}}$ through the identity at (1). Given the triple $(\mathcal{E},\mathcal{F},\mu)$, we define the corresponding eigenvalue counting function N as at (2). Now, one of the defining features of a Dirichlet form is that, equipped with the norm $\|\cdot\|_{\mathcal{E},\mu}$ defined by

$$\|f\|_{\mathcal{E},\mu} := \left(\mathcal{E}(f,f) + \int_{\mathcal{T}} f^2 d\mu\right)^{1/2}, \qquad \forall f \in \mathcal{F}, \tag{3}$$

the collection of functions $\mathcal{F}$ is a Hilbert space, and moreover, the characterisation of $(\mathcal{E},\mathcal{F})$ at (1) implies that the natural embedding from $(\mathcal{F}, \|\cdot\|_{\mathcal{E},\mu})$ into $L^2(\mathcal{T},\mu)$ is compact (see [17], Lemma 8.6, for example). By standard theory for self-adjoint operators, it follows that $N(\lambda)$ is zero for $\lambda < 0$ and finite for $\lambda \geq 0$ (see [19], Theorem B.1.13, for example). Furthermore, by applying results of [19], Section 2.3, one can deduce that $1 \in \mathcal{F}$, and $\mathcal{E}(f,f) = 0$ if and only if f is constant on $\mathcal{T}$. Thus $N(0) = 1$. When we incorporate this fact into our argument in the next section it will be convenient to have notation for the shifted eigenvalue counting function $\tilde{N} : \mathbb{R} \to \mathbb{R}_+$ defined by setting $\tilde{N}(\lambda) = N(\lambda) - 1$, which clearly satisfies $\tilde{N}(\lambda) = \#\{\text{eigenvalues of } (\mathcal{E}, \mathcal{F}, \mu) \in (0, \lambda]\}$ for $\lambda > 0$.

Later, it will also be useful to consider the Dirichlet eigenvalues of $(\mathcal{E}, \mathcal{F}, \mu)$ when the boundary of $\mathcal{T}$ is assumed to consist of two distinguished vertices $\rho, \sigma \in \mathcal{T}$, $\rho \neq \sigma$. To define these eigenvalues precisely, we first introduce the form $(\mathcal{E}^D, \mathcal{F}^D)$ by setting $\mathcal{E}^D := \mathcal{E}|_{\mathcal{F}^D \times \mathcal{F}^D}$, where $\mathcal{F}^D := \{f \in \mathcal{F} : f(\rho) = 0 = f(\sigma)\}$. Since $\mu(\{\rho, \sigma\}) = 0$, [10], Theorem 4.4.3, implies that $(\mathcal{E}^D, \mathcal{F}^D)$ is a regular Dirichlet form on $L^2(\mathcal{T}, \mu)$. Furthermore, as it is the restriction of $(\mathcal{E}, \mathcal{F})$, we can apply [20], Corollary 4.7, to deduce that

$$N^D(\lambda) \leq N(\lambda) \leq N^D(\lambda) + 2, \tag{4}$$

where N^D is the eigenvalue counting function for $(\mathcal{E}^D, \mathcal{F}^D, \mu)$, and also, since $\mathcal{E}(f, f) = 0$ if and only if f is a constant on $\mathcal{T}$, $N^D(0) = 0$. The eigenvalues of the triple $(\mathcal{E}^D, \mathcal{F}^D, \mu)$ will also be called the Dirichlet eigenvalues of $(\mathcal{E}, \mathcal{F}, \mu)$ and N^D the Dirichlet eigenvalue counting function of $(\mathcal{E}, \mathcal{F}, \mu)$.

To conclude this general discussion of Dirichlet forms on compact real trees, we prove a lemma that provides a lower bound for the first non-zero eigenvalue of $(\mathcal{E}, \mathcal{F}, \mu)$ and first eigenvalue of $(\mathcal{E}^D, \mathcal{F}^D, \mu)$, which will be repeatedly applied in the subsequent section. In the statement of the result, $\text{diam}_{d_\mathcal{T}}(\mathcal{T}) := \sup_{x,y \in \mathcal{T}} d_\mathcal{T}(x, y)$ is the diameter of the real tree $(\mathcal{T}, d_\mathcal{T})$.

Lemma 2.1. *In the above setting,* $N^D(\lambda) = \tilde{N}(\lambda) = 0$ *whenever*

$$0 \leq \lambda < \frac{1}{\text{diam}_{d_\mathcal{T}}(\mathcal{T})\mu(\mathcal{T})}.$$

Proof. As in the proof of [4], Lemma 20, observe that if $f \in \mathcal{F}^D$ is an eigenfunction of $(\mathcal{E}^D, \mathcal{F}^D, \mu)$ with eigenvalue $\lambda > 0$, then (1) implies that, for $x \in \mathcal{T}$,

$$f(x)^2 = (f(x) - f(\rho))^2 \leq \mathcal{E}(f, f) d_\mathcal{T}(\rho, x) \leq \lambda \text{diam}_{d_\mathcal{T}}(\mathcal{T}) \int_\mathcal{T} f^2 d\mu.$$

Integrating out x with respect to μ yields the result in the Dirichlet case.

Similarly, if $f \in \mathcal{F}$ is an eigenfunction of $(\mathcal{E}, \mathcal{F}, \mu)$ with eigenvalue $\lambda > 0$, then, for $x, y \in \mathcal{T}$,

$$(f(x) - f(y))^2 \leq \lambda \text{diam}_{d_\mathcal{T}}(\mathcal{T}) \int_\mathcal{T} f^2 d\mu.$$

Since by the definition of an eigenfunction $\int_\mathcal{T} f d\mu = \lambda^{-1}\mathcal{E}(f, 1) = 0$, integrating out both x and y with respect to μ yields $\lambda \geq 2/(\text{diam}_{d_\mathcal{T}}(\mathcal{T})\mu(\mathcal{T}))$, which is actually a slightly stronger result for the Neumann case than stated. □

We now turn to α-stable trees. To fix notation, as in the introduction we will henceforth assume that $\mathcal{T} = (\mathcal{T}, d_\mathcal{T})$ is an α-stable tree, $\alpha \in (1, 2]$, μ is the canonical Borel probability measure on $\mathcal{T}$ and all the random variables we consider are defined on a probability space with probability measure $\mathbf{P}$. Since α-stable trees have been reasonably widely studied, we do not feel it essential to provide an explicit construction of such objects, examples of which can be found in [7] and [12]. Instead, we simply observe that the results of [7] imply that $(\mathcal{T}, \mu)$ satisfies all the properties for measured compact real trees that were assumed at the start of this section, and therefore the above discussion applies to the Dirichlet forms $(\mathcal{E}, \mathcal{F})$, $(\mathcal{E}^D, \mathcal{F}^D)$, and eigenvalue counting functions N, $\tilde{N}$, N^D, associated with the α-stable tree $\mathcal{T}$, $\mathbf{P}$-a.s.

Fundamental to our proof of Theorem 1.1 in the case $\alpha \in (1,2)$ is the fine spinal decomposition of $\mathcal{T}$ that was developed in [13], and which we now describe. First, suppose that there is a distinguished vertex $\rho \in \mathcal{T}$, which we call the root, and choose a second vertex $\sigma \in \mathcal{T}$ randomly according to μ. Note that, since μ is non-atomic, $\rho \neq \sigma$, $\mathbf{P}$-a.s. Secondly, let $(\mathcal{T}_i^o)_{i \in \mathbb{N}}$ be the connected components of $\mathcal{T} \backslash [[\rho, \sigma]]$, where $[[\rho, \sigma]]$ is the minimal arc connecting ρ to σ in $\mathcal{T}$. We assume that $(\mathcal{T}_i^o)_{i \in \mathbb{N}}$ have been ordered so that the masses $\Delta_i := \mu(\mathcal{T}_i^o)$, which $\mathbf{P}$-a.s. take values in $(0,1)$ and sum to 1, are non-increasing in i. $\mathbf{P}$-a.s. for each i, the closure of $\mathcal{T}_i^o$ in $\mathcal{T}$ contains precisely one point more than $\mathcal{T}_i^o$, ρ_i say, and we can therefore write it as $\mathcal{T}_i = \mathcal{T}_i^o \cup \{\rho_i\}$. We define a metric $d_{\mathcal{T}_i}$ and probability measure μ_i on $\mathcal{T}_i$ by setting

$$d_{\mathcal{T}_i} := \Delta_i^{\frac{1-\alpha}{\alpha}} d_{\mathcal{T}}|_{\mathcal{T}_i \times \mathcal{T}_i}, \qquad \mu_i(\cdot) := \frac{\mu(\cdot \cap \mathcal{T}_i)}{\Delta_i}. \tag{5}$$

Furthermore, let σ_i be μ_i-random vertices of $\mathcal{T}_i$, chosen independently for each i. The usefulness of this decomposition of $\mathcal{T}$ into the subsets $(\mathcal{T}_i)_{i \in \mathbb{N}}$ is contained in the subsequent proposition, which is a simple modification of parts of [13], Corollary 10, and is stated without proof.

Proposition 2.2. *For every $\alpha \in (1,2)$, $\{((\mathcal{T}_i, d_{\mathcal{T}_i}), \mu_i, \rho_i, \sigma_i)\}_{i \in \mathbb{N}}$ is an independent collection of copies of $((\mathcal{T}, d_{\mathcal{T}}), \mu, \rho, \sigma)$, and moreover, the entire family is independent of $(\Delta_i)_{i \in \mathbb{N}}$, which has a Poisson-Dirichlet $(\alpha^{-1}, 1 - \alpha^{-1})$ distribution.*

Similarly to the argument of [4], we will apply this result recursively, and will label the objects generated by this procedure using the address space of sequences that we now introduce. For $n \geq 0$, let

$$\Sigma_n := \mathbb{N}^n, \qquad \Sigma_* := \bigcup_{m \geq 0} \Sigma_m,$$

where $\Sigma_0 := \{\emptyset\}$. For $i \in \Sigma_m, j \in \Sigma_n$, write $ij = i_1 \dots i_m j_1 \dots j_n$, and for $k \in \Sigma_*$, denote by $|k|$ the unique integer n such that $k \in \Sigma_n$. Later, we will also write for $i \in \Sigma_m$, $i|_n = i_1 \dots i_n$ for any $n \leq m$.

Continuing with our inductive procedure, given $((\mathcal{T}_i, d_{\mathcal{T}_i}), \mu_i, \rho_i, \sigma_i)$ for some $i \in \Sigma_*$, we define $\left\{((\mathcal{T}_{ij}, d_{\mathcal{T}_{ij}}), \mu_{ij}, \rho_{ij}, \sigma_{ij})\right\}_{j \in \mathbb{N}}$ and $(\Delta_{ij})_{j \in \mathbb{N}}$ from $((\mathcal{T}_i, d_{\mathcal{T}_i}), \mu_i, \rho_i, \sigma_i)$ using exactly the same method as that by which $\mathcal{T}$ was decomposed above. Thus, if the σ-algebra generated by the random variables $(\Delta_i)_{1 \leq |i| \leq n}$ is denoted by $\mathcal{F}_n$ for each $n \in \mathbb{N}$, by iteratively applying Proposition 2.2 it is easy to deduce the following result.

Corollary 2.3. *Let $\alpha \in (1,2)$. For each $n \in \mathbb{N}$, $\{((\mathcal{T}_i, d_{\mathcal{T}_i}), \mu_i, \rho_i, \sigma_i)\}_{i \in \Sigma_n}$ is an independent collection of copies of $((\mathcal{T}, d_{\mathcal{T}}), \mu, \rho, \sigma)$, independent of $\mathcal{F}_n$.*

For $i \in \Sigma_* \backslash \{\emptyset\}$, we will write $(\mathcal{E}_i, \mathcal{F}_i)$, $(\mathcal{E}_i^D, \mathcal{F}_i^D)$, N_i, $\tilde{N}_i$, N_i^D to represent the Dirichlet forms and eigenvalue counting functions corresponding to $((\mathcal{T}_i, d_{\mathcal{T}_i}), \mu_i, \rho_i, \sigma_i)$. and set

$$D_i := \Delta_{i|_1} \Delta_{i|_2} \dots \Delta_{i|_{|i|}},$$

which is actually the mass of $\mathcal{T}_i$ with respect to the original measure μ. By convention, we set $D_\emptyset := 1$, and when other objects are indexed by $\emptyset$, we are referring to the relevant quantities defined from the original α-stable tree.

Finally, when discussing the Brownian case ($\alpha = 2$), we will instead consider the decomposition of the real tree $\mathcal{T}$ described in [4]. The first step of this decomposition involves choosing two μ-random vertices σ_1 and σ_2, and then splitting the tree $\mathcal{T}$ at the unique branch point of these and the root ρ. Taking the closure of the resulting three connected components and rescaling as at (5) yields three independent copies of the original tree. Moreover, these three real trees are independent of the mass factors, $(\Delta_i)_{i=1}^3$ say, which have a Dirichlet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ distribution. By repeating this step, one can define a whole collection of trees and related objects indexed by the space of finite sequences of $\{1, 2, 3\}$. Since the procedure for doing this is extremely similar to that of the $\alpha \in (1, 2)$ case outlined above, we leave the reader to refer to [4] for further details.

3 Mean spectral asymptotics

To prove the mean spectral asymptotics for α-stable trees given in Theorem 1.1(a), we will appeal to a renewal theorem argument. In doing this, we depend on a series of inequalities that allow the Neumann and Dirichlet eigenvalue counting functions of $(\mathcal{E}, \mathcal{F}, \mu)$ to be usefully compared with those associated with Dirichlet forms on subsets of $\mathcal{T}$. In particular, the collection of subsets that we consider will be those arising from the fine spinal decomposition of $\mathcal{T}$ described in Section 2, namely $(\mathcal{T}_i)_{i\in\mathbb{N}}$, and the first main result of this section is the following, where until the final paragraph of this section we suppose $\alpha \in (1, 2)$ and define $\gamma := \alpha/(2\alpha - 1)$.

Proposition 3.1. **P**-*a.s., we have, for every* $\lambda \geq 0$,

$$\sum_{i\in\mathbb{N}} N_i^D(\lambda\Delta_i^{1/\gamma}) \leq N^D(\lambda) \leq N(\lambda) \leq 1 + \sum_{i\in\mathbb{N}} \tilde{N}_i(\lambda\Delta_i^{1/\gamma}),$$

with the upper bound being finite.

To derive this result, we will proceed via a sequence of lemmas. The first of these provides an alternative description of $(\mathcal{E}, \mathcal{F})$ that will be useful in proving the lower bound for $N^D(\lambda)$, which appears as Lemma 3.3. We write $(\mathcal{E}_{[[\rho,\sigma]]}, \mathcal{F}_{[[\rho,\sigma]]})$ to represent the local regular Dirichlet form on the compact real tree $([[\rho, \sigma]], d_\mathcal{T}|_{[[\rho,\sigma]]\times[[\rho,\sigma]]})$ equipped with the one-dimensional Hausdorff measure that is constructed using [18], Theorem 5.4 and which therefore satisfies the variational equality analogous to (1). Note that in what follows we apply the convention that if a form E is defined for functions on a set A and f is a function defined on $B \supseteq A$, then we write $E(f, f)$ to mean $E(f|_A, f|_A)$.

Lemma 3.2. **P**-*a.s., we can write*

$$\mathcal{E}(f, f) = \mathcal{E}_{[[\rho,\sigma]]}(f, f) + \sum_{i\in\mathbb{N}} \Delta_i^{\frac{1-\alpha}{\alpha}} \mathcal{E}_i(f, f), \qquad \forall f \in \mathcal{F}, \tag{6}$$

$$\mathcal{F} = \left\{ f \in L^2(\mathcal{T}, \mu) : f|_{[[\rho,\sigma]]} \in \mathcal{F}_{[[\rho,\sigma]]}, \text{ and also, for every } i \in \mathbb{N},\ f|_{\mathcal{T}_i} \in \mathcal{F}_i \right\}. \tag{7}$$

Proof. Let $(\mathcal{E}', \mathcal{F}')$ be defined by setting $\mathcal{E}'(f,f)$ to be equal to the expression on the right-hand side of (6) for any $f \in \mathcal{F}'$, where $\mathcal{F}'$ is defined to be equal to the right-hand side of (7). By results of [19], Section 2.3, to show that $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}', \mathcal{F}')$ are equal and establish the lemma, it will be enough to check that (1) still holds when we replace $(\mathcal{E}, \mathcal{F})$ by $(\mathcal{E}', \mathcal{F}')$.

Suppose $x \in \mathcal{T}_i^o$, $y \in \mathcal{T}_j^o$, for some $i \neq j$, then the infimum of interest can be rewritten as

$$\begin{aligned} &\inf\{\mathcal{E}'(f,f) : \; f \in \mathcal{F}', \; f(x) = 0, \; f(y) = 1\} \\ = &\inf_{a,b \in \mathbb{R}} \inf\{\mathcal{E}'(f,f) : \; f \in \mathcal{F}', f(x) = 0, f(\rho_i) = a, f(\rho_j) = b, f(y) = 1\}. \end{aligned}$$

Now, observe that if f is in the collection of functions over which this double-infimum is taken, then so is g, where g is equal to f on $[[\rho, \sigma]] \cup \mathcal{T}_i \cup \mathcal{T}_j$ and equal to $f(\rho_k)$ on $\mathcal{T}_k$ for $k \neq i, j$. Moreover, g satisfies

$$\mathcal{E}'(g,g) = \Delta_i^{\frac{1-\alpha}{\alpha}} \mathcal{E}_i(f,f) + \mathcal{E}_{[[\rho,\sigma]]}(f,f) + \Delta_j^{\frac{1-\alpha}{\alpha}} \mathcal{E}_j(f,f) \leq \mathcal{E}'(f,f),$$

and so we can neglect functions that are not constant on each $\mathcal{T}_k$, $k \neq i, j$. In particular, we need to compute

$$\inf_{a,b \in \mathbb{R}} \inf\{\Delta_i^{\frac{1-\alpha}{\alpha}} \mathcal{E}_i(f,f) + \mathcal{E}_{[[\rho,\sigma]]}(f,f) + \Delta_j^{\frac{1-\alpha}{\alpha}} \mathcal{E}_j(f,f)\},$$

where the second infimum is taken over functions in $\mathcal{F}'$ that satisfy $f(x) = 0, f(\rho_i) = a, f(\rho_j) = b, f(y) = 1$ and are constant on each $\mathcal{T}_k$, $k \neq i, j$. Since the forms $\mathcal{E}_i$, $\mathcal{E}_{[[\rho,\sigma]]}$ and $\mathcal{E}_j$ are zero on constant functions, we can apply their characterisation in terms of distance to obtain that this is equal to

$$\inf_{a,b \in \mathbb{R}} \left\{ \frac{a^2}{d_{\mathcal{T}}(x, \rho_i)} + \frac{(b-a)^2}{d_{\mathcal{T}}(\rho_i, \rho_j)} + \frac{(1-b)^2}{d_{\mathcal{T}}(\rho_j, y)} \right\},$$

and, from this, a simple quadratic optimisation using the additivity of the metric $d_{\mathcal{T}}$ along paths yields the desired result in this case. The argument is similar for other choices of $x, y \in \mathcal{T}$. □

The method of proof of the next lemma is an adaptation of [20], Proposition 6.3.

Lemma 3.3. **P**-*a.s., we have, for every* $\lambda \geq 0$,

$$N^D(\lambda) \geq \sum_{i \in \mathbb{N}} N_i^D(\lambda \Delta_i^{1/\gamma}).$$

Proof. First, define a quadratic form $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ by setting $\mathcal{E}^{(0)} := \mathcal{E}|_{\mathcal{F}^{(0)} \times \mathcal{F}^{(0)}}$, where

$$\mathcal{F}^{(0)} := \{f \in \mathcal{F} : f(x) = 0, \forall x \in [[\rho, \sigma]] \cup (\cup_{i \in \mathbb{N}} \{\sigma_i\})\}.$$

Since $\mu([[\rho, \sigma]] \cup (\cup_{i \in \mathbb{N}} \{\sigma_i\})) = 0$, it is possible to check that $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ is a regular Dirichlet form on $L^2(\mathcal{T}, \mu)$ by applying [10], Theorem 4.4.3. Moreover, since we have that $\mathcal{F}^{(0)} \subseteq \mathcal{F}^D$ and $\mathcal{E}^{(0)} = \mathcal{E}^D|_{\mathcal{F}^{(0)} \times \mathcal{F}^{(0)}}$, we can again apply [20], Theorem 4.5, to deduce that $N^{(0)}(\lambda) \leq N^D(\lambda)$ for every $\lambda \geq 0$, where $N^{(0)}$ is the eigenvalue counting function for $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)}, \mu)$. Consequently, to complete the proof of the lemma, it will suffice to show that $\mathbf{P}$-a.s. we have, for every $\lambda \geq 0$,

$$N^{(0)}(\lambda) \geq \sum_{i \in \mathbb{N}} N_i^D(\lambda \Delta_i^{1/\gamma}). \tag{8}$$

To demonstrate that this is indeed the case, first fix $i \in \mathbb{N}$ and suppose f is an eigenfunction of $(\mathcal{E}_i^D, \mathcal{F}_i^D, \mu_i)$ with eigenvalue $\lambda \Delta_i^{1/\gamma}$. If we set

$$g(x) := \begin{cases} f(x), & \text{for } x \in \mathcal{T}_i, \\ 0 & \text{otherwise,} \end{cases}$$

then we can apply Lemma 3.2 to deduce that, for $h \in \mathcal{F}^{(0)}$,

$$\mathcal{E}^{(0)}(g,h) = \Delta_i^{\frac{1-\alpha}{\alpha}} \mathcal{E}_i^D(f,h) = \lambda \Delta_i \int_{\mathcal{T}_i} f h d\mu_i = \lambda \int_{\mathcal{T}} g h d\mu.$$

Thus g is an eigenfunction of $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)}, \mu)$ with eigenvalue λ, and (8) follows. □

We now prove the upper bound for $N(\lambda)$. In establishing the corresponding estimates in [4], [14] and [20], extensions of the Dirichlet form of interest for which the eigenvalue counting function could easily be controlled were constructed, and we will follow a similar approach here. However, since the collection of sets $(\mathcal{T}_i)_{i \in \mathbb{N}}$ is infinite, compactness issues prevent us from directly imitating this procedure to define a single suitable Dirichlet form extension of $(\mathcal{E}, \mathcal{F})$. Instead we will consider a sequence of Dirichlet form extensions, each built as a sum of Dirichlet forms on the sets in a finite decomposition of $\mathcal{T}$.

Lemma 3.4. $\mathbf{P}$*-a.s., we have, for every* $\lambda \geq 0$,

$$\tilde{N}(\lambda) \leq \sum_{i \in \mathbb{N}} \tilde{N}_i(\lambda \Delta_i^{1/\gamma}),$$

with the upper bound being finite.

Proof. We start by describing our sequence of Dirichlet form extensions of $(\mathcal{E}, \mathcal{F})$. Fix $k \in \mathbb{N}$, and set $\mathcal{S}_k := \mathcal{T} \backslash \cup_{i=1}^k \mathcal{T}_i^o$, which is a compact real tree when equipped with the restriction of $d_{\mathcal{T}}$ to $\mathcal{S}_k$. Again appealing to [18], Theorem 5.4, let $(\mathcal{E}_{\mathcal{S}_k}, \mathcal{F}_{\mathcal{S}_k})$ be the associated local regular Dirichlet form on $L^2(\mathcal{S}_k, \mu(\cdot \cap \mathcal{S}_k))$. Now, define a pair $(\mathcal{E}^{(k)}, \mathcal{F}^{(k)})$ by setting $\mathcal{F}^{(k)}$ equal to

$$\left\{ f \in L^2(\mathcal{T}, \mu) : \begin{array}{l} \text{for every } i \in \{1, \dots, k\},\ f = f_i \text{ on } \mathcal{T}_i^o \\ \text{for some } f_i \in \mathcal{F}_i, \text{ and also } f|_{\mathcal{S}_k} \in \mathcal{F}_{\mathcal{S}_k} \end{array} \right\},$$

and

$$\mathcal{E}^{(k)}(f,g) := \mathcal{E}_{\mathcal{S}_k}(f,g) + \sum_{i=1}^k \Delta_i^{\frac{1-\alpha}{\alpha}} \mathcal{E}_i(f_i, g_i), \qquad \forall f, g \in \mathcal{F}^{(k)}.$$

Since $\mathcal{F}_i$ is dense in $L^2(\mathcal{T}_i, \mu(\cdot \cap \mathcal{T}_i))$ and $\mathcal{F}_{\mathcal{S}_k}$ is dense in $L^2(\mathcal{S}_k, \mu(\cdot \cap \mathcal{S}_k))$, we clearly have that $\mathcal{F}^{(k)}$ is dense in $L^2(\mathcal{T}, \mu)$. Furthermore, applying the corresponding properties for the

Dirichlet forms in the sum, it is easy to check that $(\mathcal{E}^{(k)},\mathcal{F}^{(k)})$ is a non-negative symmetric bilinear form satisfying the Markov property, by which we mean that if $f\in\mathcal{F}^{(k)}$ and $\overline{f}:=(0\vee f)\wedge 1$, then $\overline{f}\in\mathcal{F}^{(k)}$ and $\mathcal{E}^{(k)}(\overline{f},\overline{f})\leq\mathcal{E}^{(k)}(f,f)$. Hence to prove that $(\mathcal{E}^{(k)},\mathcal{F}^{(k)})$ is a Dirichlet form on $L^2(\mathcal{T},\mu)$ it remains to demonstrate that $(\mathcal{F}^{(k)},\|\cdot\|_{\mathcal{E}^{(k)},\mu})$ is a Hilbert space, where $\|\cdot\|_{\mathcal{E}^{(k)},\mu}$ is the defined as at (3). Given that the number of terms in the above sum is finite, this is elementary, and so $(\mathcal{E}^{(k)},\mathcal{F}^{(k)})$ is indeed a Dirichlet form on $L^2(\mathcal{T},\mu)$. Moreover, by a simple adaptation of the proof of [20], Proposition 6.2(3), it can also be shown that the identity map from $(\mathcal{F}^{(k)},\|\cdot\|_{\mathcal{E}^{(k)},\mu})$ to $L^2(\mathcal{T},\mu)$ is compact, and so the eigenvalue counting function for $(\mathcal{E}^{(k)},\mathcal{F}^{(k)},\mu)$, $N^{(k)}$ say, is finite everywhere on the real line.

In order to demonstrate that $(\mathcal{E}^{(k)},\mathcal{F}^{(k)})$ is an extension of $(\mathcal{E},\mathcal{F})$, we first observe that, by following an identical line of reasoning to that applied in the proof of Lemma 3.2, it is possible to prove that the Dirichlet form $(\mathcal{E},\mathcal{F})$ satisfies

$$\mathcal{E}(f,g):=\mathcal{E}_{\mathcal{S}_k}(f,g)+\sum_{i=1}^{k}\Delta_i^{\frac{1-\alpha}{\alpha}}\mathcal{E}_i(f,g),\qquad\forall f,g\in\mathcal{F},$$

$$\mathcal{F}=\left\{f\in L^2(\mathcal{T},\mu):\text{ for every }i\in\{1,\dots,k\},\ f|_{\mathcal{T}_i}\in\mathcal{F}_i,\text{ and also }f|_{\mathcal{S}_k}\in\mathcal{F}_{\mathcal{S}_k}\ \right\}.$$

From this characterisation of $(\mathcal{E},\mathcal{F})$, it is immediate that $\mathcal{F}\subseteq\mathcal{F}^{(k)}$ and $\mathcal{E}=\mathcal{E}^{(k)}|_{\mathcal{F}\times\mathcal{F}}$, as desired. Consequently a further application of [20], Theorem 4.5, yields that $\tilde{N}(\lambda)\leq\tilde{N}^{(k)}(\lambda):=N^{(k)}(\lambda)-1$, and we complete our proof by establishing suitable upper bounds for $\tilde{N}^{(k)}$.

Let $f\not\equiv 0$ be an eigenfunction of $(\mathcal{E}^{(k)},\mathcal{F}^{(k)})$ with eigenvalue $\lambda>0$. If $i\in\{1,\dots,k\}$ and $g\in\mathcal{F}_i$, then define a function $h\in\mathcal{F}^{(k)}$ by setting

$$h(x):=\begin{cases}g(x), & \text{if } x\in\mathcal{T}_i^o,\\ 0, & \text{otherwise.}\end{cases}$$

By the definition of $\mathcal{E}^{(k)}$ and this construction, we have that

$$\mathcal{E}_i(f,g)=\Delta_i^{\frac{\alpha-1}{\alpha}}\mathcal{E}^{(k)}(f,h)=\lambda\Delta_i^{\frac{\alpha-1}{\alpha}}\int_{\mathcal{T}}fhd\mu=\lambda\Delta_i^{1/\gamma}\int_{\mathcal{T}_i}fgd\mu_i.$$

Thus if f is not identically zero on $\mathcal{T}_i$, then it must be the case that $\lambda\Delta_i^{1/\gamma}$ is an eigenvalue of $(\mathcal{E}_i,\mathcal{F}_i,\mu_i)$. Similarly, if $g\in\mathcal{F}_{\mathcal{S}_k}$, h is defined by

$$h(x):=\begin{cases}g(x), & \text{if } x\in\mathcal{S}_k,\\ 0, & \text{otherwise,}\end{cases}$$

and f is not identically zero on $\mathcal{S}_k$, then λ is an eigenvalue of $(\mathcal{E}_{\mathcal{S}_k},\mathcal{F}_{\mathcal{S}_k},\mu(\cdot\cap\mathcal{S}_k))$. Combining these facts, it follows that, for $\lambda\geq 0$,

$$\tilde{N}^{(k)}(\lambda)\leq\tilde{N}_{\mathcal{S}_k}(\lambda)+\sum_{i=1}^{k}\tilde{N}_i(\lambda\Delta_i^{1/\gamma}),\tag{9}$$

where $\tilde{N}_{\mathcal{S}_k}$ is the (strictly positive) eigenvalue counting function for $(\mathcal{E}_{\mathcal{S}_k},\mathcal{F}_{\mathcal{S}_k},\mu(\cdot\cap\mathcal{S}_k))$.

Now note that, by Lemma 2.1, the first term in (9) is zero whenever λ is strictly less than $1/\mathrm{diam}_{d_\mathcal{T}}(\mathcal{S}_k)\mu(\mathcal{S}_k)$. Thus we can conclude that, for each $k \in \mathbb{N}$,

$$\tilde{N}(\lambda) \leq \sum_{i=1}^{k} \tilde{N}_i(\lambda\Delta_i^{1/\gamma}), \qquad \forall \lambda < \frac{1}{\mathrm{diam}_{d_\mathcal{T}}(\mathcal{T})(1-\Delta_1-\cdots-\Delta_k)}.$$

Since $\mathrm{diam}_{d_\mathcal{T}}(\mathcal{T}) < \infty$ and $\Delta_1+\cdots+\Delta_k \to 1$ as $k \to \infty$, $\mathbf{P}$-a.s., the upper bound of the lemma follows.

It still remains to show the $\mathbf{P}$-a.s. finiteness of $\sum_{i\in\mathbb{N}} \tilde{N}_i(\lambda\Delta_i^{1/\gamma})$. To show this is the case, we again apply Lemma 2.1 to obtain that the ith term is zero whenever

$$\lambda < \Delta_i^{-1/\gamma}\left(\mathrm{diam}_{d_{\mathcal{T}_i}}(\mathcal{T}_i)\mu_i(\mathcal{T}_i)\right)^{-1} = \left(\Delta_i \mathrm{diam}_{d_\mathcal{T}}(\mathcal{T}_i)\right)^{-1}.$$

The result is readily obtained from this on noting that $(\Delta_i\mathrm{diam}_{d_\mathcal{T}}(\mathcal{T}_i))^{-1}$ is bounded below by $(\Delta_i\mathrm{diam}_{d_\mathcal{T}}(\mathcal{T}))^{-1} \to \infty$ as $i \to \infty$ and so only a finite number of terms (each of which is finite) in the sum are non-zero, $\mathbf{P}$-a.s. □

Given the eigenvalue counting function comparison result of Proposition 3.1, which follows from (4), Lemma 3.3 and Lemma 3.4, we now turn to our renewal theorem argument to derive mean spectral asymptotics for α-stable trees. Similarly to [4], define the functions $(\eta_i)_{i\in\Sigma_*}$ by, for $t \in \mathbb{R}$,

$$\eta_i(t) := N_i^D(e^t) - \sum_{j\in\mathbb{N}} N_{ij}^D(e^t\Delta_{ij}^{1/\gamma}),$$

and let $\eta := \eta_\emptyset$. By Proposition 3.1, $\eta_i(t)$ is non-negative and finite for every $t \in \mathbb{R}$, $\mathbf{P}$-a.s., and the dominated convergence theorem implies that η_i has cadlag paths, $\mathbf{P}$-a.s. It will also be useful to note that Corollary 2.3 implies η_i is independent of $(\Delta_j)_{|j|\leq|i|}$. If we set $X_i(t) := N_i^D(e^t)$, and $X := X_\emptyset$, then it is immediate that the following evolution equation holds:

$$X(t) = \eta(t) + \sum_{i\in\mathbb{N}} X_i(t+\gamma^{-1}\ln\Delta_i). \tag{10}$$

We introduce associated discounted mean processes

$$m(t) := e^{-\gamma t}\mathbf{E}X(t), \qquad u(t) := e^{-\gamma t}\mathbf{E}\eta(t),$$

define a measure ν by $\nu([0,t]) = \sum_{i\in\mathbb{N}}\mathbf{P}(\Delta_i \geq e^{-\gamma t})$, and let ν_γ be the measure that satisfies $\nu_\gamma(dt) = e^{-\gamma t}\nu(dt)$. The properties we require of m, u and ν_γ are collected in the following lemma. In the proof of this result, which is an adaptation of [4], Lemma 20, it will be convenient to define, for $x \geq 0$,

$$\psi(x) := \sum_{i\in\mathbb{N}}\mathbf{E}(\Delta_i^x). \tag{11}$$

By [25], equation (6), this quantity is infinite for $x \leq \alpha^{-1}$, and otherwise satisfies

$$\psi(x) = \frac{\alpha-1}{\alpha x-1}. \tag{12}$$

Moreover, we set

$$\beta := \frac{\alpha-1}{2\alpha-1} \equiv \gamma - \frac{1}{2\alpha-1}. \tag{13}$$

Lemma 3.5. *(a) The function m is bounded.*
(b) The function u is in $L^1(\mathbb{R})$ and, for any $\varepsilon > 0$, $u(t) = O(e^{-(\beta-\varepsilon)t})$ as $t \to \infty$.
(c) The measure ν_γ is a Borel probability measure on $[0,\infty)$, and the integral $\int_0^\infty t\nu_\gamma(dt)$ is finite.

Proof. First observe that by iterating (10) we obtain for each $k \in \mathbb{N}$ that

$$X(t) = \sum_{|i|<k} \eta_i(t + \gamma^{-1} \ln D_i) + \sum_{i \in \Sigma_k} X_i(t + \gamma^{-1} \ln D_i).$$

Thus establishing the $\mathbf{P}$-a.s. limit

$$\lim_{k\to\infty} \sum_{i\in\Sigma_k} X_i(t + \gamma^{-1} \ln D_i) = 0, \qquad \forall t \in \mathbb{R}, \tag{14}$$

will also confirm that we can $\mathbf{P}$-a.s. write

$$X(t) = \sum_{i\in\Sigma_*} \eta_i(t + \gamma^{-1} \ln D_i), \qquad \forall t \in \mathbb{R}. \tag{15}$$

To prove that (14) does indeed hold, we first note that, since $X_i(t + \gamma^{-1} \ln D_i) = N_i^D(e^t D_i^{1/\gamma}) = 0$ for $e^t D_i^{1/\gamma} < \mathrm{diam}_{d_{\mathcal{T}_i}}(\mathcal{T}_i)^{-1}$, the sum appearing in (14) is zero if

$$\sup_{i\in\Sigma_k} D_i^{1/\gamma} \mathrm{diam}_{d_{\mathcal{T}_i}}(\mathcal{T}_i) < e^{-t}.$$

Hence, to prove (14), it will be enough to show that this supremum converges $\mathbf{P}$-a.s. to zero as $k \to \infty$. To establish that this is the case, we will apply the following bound: for $\varepsilon, \theta > 0$,

$$\begin{aligned}
\sum_{k=0}^\infty \mathbf{P}\left(\sup_{i\in\Sigma_k} D_i^{1/\gamma} \mathrm{diam}_{d_{\mathcal{T}_i}}(\mathcal{T}_i) \geq \varepsilon\right) &\leq \sum_{k=0}^\infty \mathbf{P}\left(\sum_{i\in\Sigma_k} D_i^{\theta/\gamma} \mathrm{diam}_{d_{\mathcal{T}_i}}(\mathcal{T}_i)^\theta \geq \varepsilon^\theta\right) \\
&\leq \varepsilon^{-\theta} \mathbf{E}\left(\mathrm{diam}_{d_\mathcal{T}}(\mathcal{T})^\theta\right) \sum_{k=0}^\infty \sum_{i\in\Sigma_k} \mathbf{E}\left(D_i^{\theta/\gamma}\right) \\
&= \varepsilon^{-\theta} \mathbf{E}\left(\mathrm{diam}_{d_\mathcal{T}}(\mathcal{T})^\theta\right) \sum_{k=0}^\infty \psi(\theta\gamma^{-1})^k,
\end{aligned} \tag{16}$$

where we have made use of the recursive decomposition result of Corollary 2.3. Exploiting the fragmentation process description of α-stable trees proved in [22], it is possible to apply [11], Proposition 14, to check that the expectation $\mathbf{E}(\mathrm{diam}_{d_\mathcal{T}}(\mathcal{T})^\theta)$ is finite for any $\theta > 0$. Furthermore, by (12), we have that $\psi(\theta\gamma^{-1}) < 1$ for $\theta > \gamma$. Thus, by choosing $\theta > \gamma$, we obtain that the expression at (16) is finite, and therefore the Borel-Cantelli lemma can be applied to complete the proof that (14) and (15) hold.

From the characterisation of X at (15) and the definition of η_i we see that

$$m(t) = e^{-\gamma t} \sum_{i\in\Sigma_*} \mathbf{E}\left(N_i^D(e^t D_i^{1/\gamma}) - \sum_{j\in\mathbb{N}} N_{ij}^D(e^t D_{ij}^{1/\gamma})\right).$$

Since

$$\begin{aligned}\eta_i(t+\gamma^{-1}\ln D_i) &= N_i^D(e^t D_i^{1/\gamma}) - \sum_{j\in\mathbb{N}} N_{ij}^D(e^t D_{ij}^{1/\gamma}) \\ &\leq \mathbf{1}_{\{D_i^{1/\gamma}\mathrm{diam}_{d_{\mathcal{T}_i}}(\mathcal{T}_i)\geq e^{-t}\}} + \sum_{j\in\mathbb{N}}\left(\tilde{N}_{ij}(e^t D_{ij}^{1/\gamma}) - N_{ij}^D(e^t D_{ij}^{1/\gamma})\right), \\ &\leq \mathbf{1}_{\{D_i^{1/\gamma}\mathrm{diam}_{d_{\mathcal{T}_i}}(\mathcal{T}_i)\geq e^{-t}\}} + \sum_{j\in\mathbb{N}} \mathbf{1}_{\{D_{ij}^{1/\gamma}\mathrm{diam}_{d_{\mathcal{T}_{ij}}}(\mathcal{T}_{ij})\geq e^{-t}\}}, \end{aligned} \tag{17}$$

where we have applied (4), Lemma 2.1 and Proposition 3.1, it follows that

$$\begin{aligned} m(t) &\leq 2e^{-\gamma t}\sum_{i\in\Sigma_*}\mathbf{P}\left(D_i^{1/\gamma}\mathrm{diam}_{d_{\mathcal{T}_i}}(\mathcal{T}_i)\geq e^{-t}\right) \\ &= 2e^{-\gamma t}\mathbf{E}\left(\#\left\{i\in\Sigma_*: -\gamma^{-1}\ln D_i \leq t+\ln \mathrm{diam}_{d_{\tilde{\mathcal{T}}}}(\tilde{\mathcal{T}})\right\}\right), \end{aligned}$$

where $(\tilde{\mathcal{T}}, d_{\tilde{\mathcal{T}}})$ is an independent copy of $(\mathcal{T}, d_{\mathcal{T}})$. Similarly to the corresponding argument in [4], by considering the Crump-Mode-Jagers branching process with particles $i\in\Sigma_*$, where $i\in\Sigma_*$ has offspring ij at time $-\ln\Delta_{ij}$ after its birth, $j\in\mathbb{N}$, it is possible to show that $\mathbf{E}(\#\{i\in\Sigma_*: -\ln D_i\leq t\})\leq Ce^t$ for every $t\in\mathbb{R}$, where C is a finite constant. Hence $m(t)\leq 2C\mathbf{E}\left(\mathrm{diam}_{d_{\mathcal{T}}}(\mathcal{T})^\gamma\right)$ for every $t\in\mathbb{R}$. As already noted, the moments of the diameter of an α-stable tree are finite and so this bound establishes that m is bounded.

For part (b), first observe that

$$u(t)=e^{-\gamma t}\mathbf{E}\eta(t)\leq e^{-\gamma t}\sum_{i\in\Sigma_*}\mathbf{E}\eta_i(t+\gamma^{-1}\ln D_i)=m(t),$$

and so u is bounded. Thus, since η is $\mathbf{P}$-a.s. cadlag, then u is also measurable. Furthermore, multiplying (17) by $e^{-\gamma t}$ and taking expectations yields, for any $\theta>0$,

$$\begin{aligned} u(t) &\leq e^{-\gamma t}\left(\mathbf{P}\left(\mathrm{diam}_{d_{\mathcal{T}}}(\mathcal{T})\geq e^{-t}\right)+\sum_{i\in\mathbb{N}}\mathbf{P}\left(\Delta_i^{1/\gamma}\mathrm{diam}_{d_{\mathcal{T}_i}}(\mathcal{T}_i)\geq e^{-t}\right)\right) \\ &\leq e^{(\theta-\gamma)t}\mathbf{E}\left(\mathrm{diam}_{d_{\mathcal{T}}}(\mathcal{T})^\theta\right)\left(1+\sum_{i\in\mathbb{N}}\mathbf{E}\left(\Delta_i^{\theta/\gamma}\right)\right) \\ &= C_\theta e^{(\theta-\gamma)t}, \end{aligned}$$

where the second inequality is a simple application of Chebyshev's inequality and $C_\theta := \mathbf{E}(\mathrm{diam}_{d_{\mathcal{T}}}(\mathcal{T})^\theta)(1+\psi(\theta\gamma^{-1}))$. As all the positive moments of $\mathrm{diam}_{d_{\mathcal{T}}}(\mathcal{T})$ are finite and $\psi(\theta\gamma^{-1})$ is finite for $\theta>\gamma\alpha^{-1}$, C_θ is a finite constant for any $\theta>(2\alpha-1)^{-1}$. In particular, choosing $\theta=(2\alpha-1)^{-1}+\varepsilon$, we obtain $u(t)=O(e^{-(\beta-\varepsilon)t})$ as $t\to\infty$, which is the second claim of part (b). We further note that by setting $\theta=1+\gamma$, the above bound implies $u(t)=O(e^t)$ as $t\to-\infty$, which, in combination with our earlier observations, establishes that $u\in L^1(\mathbb{R})$ as desired.

Finally, to demonstrate that ν_γ is a Borel probability measure on $[0,\infty)$ is elementary given that $\psi(1)=\sum_{i\in\mathbb{N}}\Delta_i=1$, $\mathbf{P}$-a.s. Moreover, by definition the integrability condition can be rewritten $\sum_{i\in\mathbb{N}}\mathbf{E}(\Delta_i|\ln\Delta_i|)<\infty$, and this can be confirmed by a second application of equation (6) of [25]. □

Applying this lemma, it would be possible to apply the renewal theorem of [16] exactly as in [4] to deduce the convergence of $m(t)$ as $t \to \infty$. However, in order to establish an estimate for the second order term, we present a direct proof of the renewal theorem in our setting. The β in the statement of the result is defined as at (13), and $m(\infty)$ is the constant defined by

$$m(\infty) := \frac{\int_{-\infty}^{\infty} u(t)dt}{\int_0^{\infty} t\nu_\gamma(dt)}. \tag{18}$$

That $m(\infty)$ is finite and non-zero is an easy consequence of Lemma 3.5.

Proposition 3.6. *For any $\varepsilon > 0$, the function m satisfies*

$$|m(t) - m(\infty)| = O(e^{-(\beta-\varepsilon)t}),$$

as $t \to \infty$.

Proof. From (15) and Fubini's theorem, we obtain

$$\begin{aligned} m(t) &= e^{-\gamma t}\mathbf{E}X(t) \\ &= \sum_{i\in\Sigma_*} e^{-\gamma t}\mathbf{E}\eta_i(t+\gamma^{-1}\ln D_i) \\ &= \sum_{i\in\Sigma_*}\int_0^\infty e^{-\gamma(t-s)}\mathbf{E}\eta_i(t-s)e^{-\gamma s}\mathbf{P}(-\gamma^{-1}\ln D_i \in ds) \\ &= \int_0^\infty u(t-s)\sum_{i\in\Sigma_*} e^{-\gamma s}\mathbf{P}(-\gamma^{-1}\ln D_i \in ds), \end{aligned}$$

where to deduce the third equality, we apply the independence of η_i and D_i that follows from Corollary 2.3. We will analyse the measure in this integral. Let $\lambda > 0$, then

$$\begin{aligned} \int_0^\infty e^{-\lambda s}\sum_{i\in\Sigma_*} e^{-\gamma s}\mathbf{P}(-\gamma^{-1}\ln D_i \in ds) &= \sum_{i\in\Sigma_*}\mathbf{E}D_i^{1+\lambda/\gamma} \\ &= \sum_{n=0}^{\infty}\psi(1+\lambda\gamma^{-1})^n \\ &= \frac{1}{1-\psi(1+\lambda\gamma^{-1})}. \end{aligned}$$

Furthermore, observe that $M := (\int_0^\infty s\nu_\gamma(ds))^{-1}$ satisfies

$$M^{-1} = -\gamma^{-1}\psi'(1) = \frac{2\alpha-1}{\alpha-1}.$$

It follows that

$$\int_0^\infty e^{-\lambda s}\left[Mds - \sum_{i\in\Sigma_*} e^{-\gamma s}\mathbf{P}(-\gamma^{-1}\ln D_i \in ds)\right] = \frac{M}{\lambda} - \frac{1}{1-\psi(1+\lambda\gamma^{-1})} = -1,$$

and inverting this Laplace transform yields

$$Mds - \sum_{i\in\Sigma_*} e^{-\gamma s}\mathbf{P}(-\gamma^{-1}\ln D_i \in ds) = -\delta_0(s)ds,$$

where $\delta_0(s)$ is the Dirac delta function. Therefore

$$\begin{aligned} & m(\infty)-m(t) \\ &= M\int_0^\infty u(t+s)ds+\int_0^\infty u(t-s)\left[Mds-\sum_{i\in\Sigma_*}e^{-\gamma s}\mathbf{P}(-\gamma^{-1}\ln D_i\in ds)\right] \\ &= M\int_0^\infty u(t+s)ds-u(t), \end{aligned}$$

and the result follows from Lemma 3.5. □

Rewriting the above result in terms of N^D and using (4) to compare N^D with N yields Theorem 1.1(a) for $\alpha\in(1,2)$. Before we conclude this section, though, let us briefly discuss the case $\alpha=2$, so as to explain how the corresponding parts of the theorem and Remark 1.2 can be verified. Letting m, u and ν_γ be defined as in [4] (which closely matches the notation of this article), then by repeating an almost identical argument to the previous proof, with the Poisson-Dirichlet random variables $(\Delta_i)_{i\in\mathbb{N}}$ of this article being replaced by the Dirichlet $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ triple of that, it is possible to show that

$$m(\infty)-m(t)=\int_0^\infty u(t+s)ds-u(t).$$

(To do this, it is necessary to apply the observations that, in the $\alpha=2$ setting, the constant M is equal to 1, and the function corresponding to $\psi(x)$ can be computed to be $3(2x+1)^{-1}$.) Since u was shown in [4] to satisfy $u(t)\leq Ce^{-2t/3}$ for $t\geq0$, the right-hand side is bounded by a constant when multiplied by $e^{2t/3}$, and it follows that Theorem 1.1(a) holds for $\alpha=2$ with the second order term reduced to $O(1)$.

4 Almost-sure spectral asymptotics

Our task for this section is to establish the $\mathbf{P}$-a.s. convergence of $e^{-\gamma t}X(t)$ as $t\to\infty$, where $X(t)$ is defined as in the previous section and $\alpha\in(1,2)$ is fixed throughout. For this, we follow the branching process argument of [4], which extends [14], making changes where necessary to deal with the infinite number of offspring. This approach relies on a second moment bound for $X(t)$, which we prove via a sequence of lemmas. For brevity, we will henceforth write $\delta_i:=\mathrm{diam}_{d_{\mathcal{T}_i}}(\mathcal{T}_i)$. It will also be convenient to let $m(i,j)=\sup\{n:i|_n=j|_n\}$ be the generation of the most recent common ancestor of the addresses $i,j\in\Sigma_*$ and for $j=ik$ to write $D^i_j=\prod_{l=|i|+1}^{|j|}\Delta_{j|_l}$.

We first state an elementary extension of Markov's inequality.

Lemma 4.1. *Let X,Y be positive random variables. Then for all $x,y>0$,*

$$\mathbf{P}(X>x,Y>y)\leq\frac{1}{xy}\mathbf{E}XY. \tag{19}$$

Lemma 4.2. *Let $i\in\Sigma_k$, $j\in\Sigma_l$ with $k\leq l$, and $\theta>0$. Writing $m:=m(i,j)$ and $\delta:=\delta_\emptyset$, we have that*

$$\begin{aligned} &\mathbf{P}(D_i^{1/\gamma}\delta_i\geq e^{-t},D_j^{1/\gamma}\delta_j\geq e^{-t}) \\ &\leq\; e^{2\theta t}(\mathbf{E}\delta^{4\theta})^{1/2}(\mathbf{E}(\Delta_{i|_{m+1}}^{4\theta/\gamma})\mathbf{E}(\Delta_{j|_{m+1}}^{4\theta/\gamma}))^{1/4}\mathbf{E}(D_{i|_m}^{2\theta/\gamma})\mathbf{E}\left((D_i^{i|_{m+1}})^{\theta/\gamma}\right)\mathbf{E}\left((D_j^{j|_{m+1}})^{\theta/\gamma}\right) \end{aligned}$$

whenever $m < k$*, and if* $\varepsilon > 0$*, then*

$$\mathbf{P}(D_i^{1/\gamma}\delta_i \geq e^{-t}, D_j^{1/\gamma}\delta_j \geq e^{-t}) \leq e^{2\theta t}\mathbf{E}(\delta^{2(1+\varepsilon^{-1})\theta})^{1/(1+\varepsilon^{-1})}\mathbf{E}(D_i^{2\theta/\gamma})\mathbf{E}((D_j^i)^{(1+\varepsilon)\theta/\gamma})^{1/(1+\varepsilon)}$$

whenever $m = k$*.*

Proof. We start by assuming $m < k$ or, if $k = l$, then $m < k-1$. By definition, we have that

$$\begin{aligned}
&\mathbf{P}(D_i^{1/\gamma}\delta_i \geq e^{-t}, D_j^{1/\gamma}\delta_j \geq e^{-t}) \\
&= \mathbf{P}(D_{i|_{m+1}}^{1/\gamma}(D_i^{i|_{m+1}})^{1/\gamma}\delta_i \geq e^{-t}, D_{j|_{m+1}}^{1/\gamma}(D_j^{j|_{m+1}})^{1/\gamma}\delta_j \geq e^{-t}) \\
&= \mathbf{E}(\mathbf{P}((D_i^{i|_{m+1}})^{1/\gamma} \geq x_i, (D_j^{j|_{m+1}})^{1/\gamma} \geq x_j | x_i, x_j))
\end{aligned} \tag{20}$$

where $x_i^{-1} = e^t D_{i|_{m+1}}^{1/\gamma}\delta_i, x_j^{-1} = e^t D_{j|_{m+1}}^{1/\gamma}\delta_j$. Now, as $D_i^{i|_{m+1}}$ and $D_j^{j|_{m+1}}$ are independent, we have

$$\begin{aligned}
&\mathbf{P}(D_i^{1/\gamma}\delta_i \geq e^{-t}, D_j^{1/\gamma}\delta_j \geq e^{-t}) \\
&\leq \mathbf{E}(\mathbf{P}((D_i^{i|_{m+1}})^{1/\gamma} \geq x_i | x_i, x_j)\mathbf{P}((D_j^{j|_{m+1}})^{1/\gamma} \geq x_j | x_i, x_j)) \\
&\leq \mathbf{E}(x_i^{-\theta}x_j^{-\theta}\mathbf{E}((D_i^{i|_{m+1}})^{\theta/\gamma} | x_i, x_j)\mathbf{E}((D_j^{j|_{m+1}})^{\theta/\gamma} | x_i, x_j)) \\
&\leq e^{2\theta t}\mathbf{E}(\delta_i^\theta\delta_j^\theta\Delta_{i|_{m+1}}^{\theta/\gamma}\Delta_{j|_{m+1}}^{\theta/\gamma})\mathbf{E}(D_{i|_m}^{2\theta/\gamma})\mathbf{E}((D_i^{i|_{m+1}})^{\theta/\gamma})\mathbf{E}((D_j^{j|_{m+1}})^{\theta/\gamma})
\end{aligned}$$

A repeated application of Cauchy-Schwarz to $\mathbf{E}(\delta_i^\theta\delta_j^\theta\Delta_{i|_{m+1}}^{\theta/\gamma}\Delta_{j|_{m+1}}^{\theta/\gamma})$ then gives the result.

For the case where $k = l$ and $m = k-1$, that is i, j have the same parent we cannot use independence in the same way and instead use (19) in (20) to get

$$\begin{aligned}
\mathbf{P}(D_i^{1/\gamma}\delta_i \geq e^{-t}, D_j^{1/\gamma}\delta_j \geq e^{-t}) &\leq \mathbf{E}(x_i^{-\theta}x_j^{-\theta}\mathbf{E}(\Delta_{i|_k}^{\theta/\gamma}\Delta_{j|_k}^{\theta/\gamma} | x_i, x_j)) \\
&\leq e^{2\theta t}\mathbf{E}(\delta_i^\theta\delta_j^\theta\Delta_{i|_k}^{\theta/\gamma}\Delta_{j|_k}^{\theta/\gamma})\mathbf{E}(D_{i|_m}^{2\theta/\gamma})
\end{aligned}$$

and Cauchy-Schwarz again gives the result.

For the case where $m = k$ we have by (19) that

$$\begin{aligned}
&\mathbf{P}(D_i^{1/\gamma}\delta_i \geq e^{-t}, D_i^{1/\gamma}(D_j^i)^{1/\gamma}\delta_j \geq e^{-t}) \\
&\leq e^{2\theta t}\mathbf{E}\left(D_i^{2\theta/\gamma}\delta_i^\theta(D_j^i)^{\theta/\gamma}\delta_j^\theta\right) \\
&= e^{2\theta t}\mathbf{E}(D_i^{2\theta/\gamma})\mathbf{E}\left(\delta_i^\theta(D_j^i)^{\theta/\gamma}\delta_j^\theta\right)
\end{aligned}$$

Applying Hölder twice to $\mathbf{E}\left(\delta_i^\theta(D_j^i)^{\theta/\gamma}\delta_j^\theta\right)$, we have the result in this case as well. □

For the following result, we define $\psi_r := \psi(r\theta\gamma^{-1})$ for $r = 1, 2$, where the function $(\psi(x))_{x\geq 0}$ was introduced at (11). We also set

$$\psi_{1,\varepsilon} := \sum_{i\in\mathbb{N}} \mathbf{E}(\Delta_i^{(1+\varepsilon)\theta/\gamma})^{1/(1+\varepsilon)}.$$

If $\theta > \gamma/\alpha$, we observe that $\psi_1 \leq \psi_{1,\varepsilon} < \infty$, where the lower inequality is simply Jensen's and the upper inequality is a consequence of [25], equation (50).

Lemma 4.3. *For $k \le l$, $\theta > \gamma/\alpha$ and $\varepsilon > 0$, we have that*

$$\sum_{i\in\Sigma_k}\sum_{j\in\Sigma_l}\mathbf{E}(\eta_i(t+\gamma^{-1}\ln D_i)\eta_j(t+\gamma^{-1}\ln D_j)) \le Ce^{2\theta t}(k+1)\psi_{1,\varepsilon}^{k+l}\left(\frac{\psi_2}{\psi_{1,\varepsilon}^2}\vee 1\right)^k$$

for some finite constant C.

Proof. Let $i\in\Sigma_k, j\in\Sigma_l$ for some $k\le l$, then (17) implies that

$$\begin{aligned}
&\mathbf{E}(\eta_i(t+\gamma^{-1}\ln D_i)\eta_j(t+\gamma^{-1}\ln D_j))\\
&\quad\le \mathbf{E}(\mathbf{1}_{A_i,A_j}+\mathbf{1}_{A_i}\sum_{n\in\mathbb{N}}\mathbf{1}_{A_{jn}}+\mathbf{1}_{A_j}\sum_{n\in\mathbb{N}}\mathbf{1}_{A_{in}}+\sum_{n,n'\in\mathbb{N}}\mathbf{1}_{A_{in},A_{jn'}})\\
&\quad= \mathbf{P}(A_i,A_j)+\sum_{n\in\mathbb{N}}(\mathbf{P}(A_i,A_{jn})+\mathbf{P}(A_j,A_{in}))+\sum_{n,n'\in\mathbb{N}}\mathbf{P}(A_{in},A_{jn'}),
\end{aligned}\tag{21}$$

where $A_i := \{D_i^{1/\gamma}\delta_i \ge e^{-t}\}$. We now apply Lemma 4.2 to deduce that

$$\begin{aligned}
&\sum_{i\in\Sigma_k}\sum_{j\in\Sigma_l:j|_k\ne i}\mathbf{E}(\eta_i(t+\gamma^{-1}\ln D_i)\eta_j(t+\gamma^{-1}\ln D_j))\\
&\quad\le e^{2\theta t}(\mathbf{E}\delta^{4\theta})^{1/2}\sum_{i\in\Sigma_k}\sum_{j\in\Sigma_l:j|_k\ne i}(\mathbf{E}(\Delta_{i|_{m+1}}^{4\theta/\gamma})\mathbf{E}(\Delta_{j|_{m+1}}^{4\theta/\gamma}))^{1/4}\mathbf{E}(D_{i|_m}^{2\theta/\gamma})(I_1+I_2+I_3),
\end{aligned}$$

where $m := m(i,j)$ is strictly less than k for the i and j in the above sum, $\delta := \delta_\emptyset$, and

$$\begin{aligned}
I_1 &:= \mathbf{E}\left((D_i^{i|_{m+1}})^{\theta/\gamma}\right)\mathbf{E}\left((D_j^{j|_{m+1}})^{\theta/\gamma}\right)\\
I_2 &:= \sum_{n\in\mathbb{N}}\left(\mathbf{E}\left((D_i^{i|_{m+1}})^{\theta/\gamma}\right)\mathbf{E}\left((D_{jn}^{j|_{m+1}})^{\theta/\gamma}\right)+\mathbf{E}\left((D_{in}^{i|_{m+1}})^{\theta/\gamma}\right)\mathbf{E}\left((D_j^{j|_{m+1}})^{\theta/\gamma}\right)\right)\\
I_3 &:= \sum_{n,n'\in\mathbb{N}}\mathbf{E}\left((D_{in}^{i|_{m+1}})^{\theta/\gamma}\right)\mathbf{E}\left((D_{jn'}^{j|_{m+1}})^{\theta/\gamma}\right).
\end{aligned}$$

Noting as in the proof of Lemma 3.5 that $\mathbf{E}\delta^{4\theta}$ is finite, it will suffice to bound the sums over the terms involving I_1, I_2 and I_3. Firstly, we have that

$$\begin{aligned}
&\sum_{i\in\Sigma_k}\sum_{j\in\Sigma_l:j|_k\ne i}(\mathbf{E}(\Delta_{i|_{m+1}}^{4\theta/\gamma})\mathbf{E}(\Delta_{j|_{m+1}}^{4\theta/\gamma}))^{1/4}\mathbf{E}(D_{i|_m}^{2\theta/\gamma})I_1\\
&\quad\le \sum_{m'=0}^{k-1}\sum_{i'\in\Sigma_{m'}}\mathbf{E}(D_{i'}^{2\theta/\gamma})\\
&\qquad\times\sum_{i\in\Sigma_k:i|_{m'}=i'}\sum_{j\in\Sigma_l:j|_{m'}=i'}(\mathbf{E}(\Delta_{i|_{m'+1}}^{4\theta/\gamma})\mathbf{E}(\Delta_{j|_{m'+1}}^{4\theta/\gamma}))^{1/4}\mathbf{E}\left((D_i^{i|_{m'+1}})^{\theta/\gamma}\right)\mathbf{E}\left((D_j^{j|_{m'+1}})^{\theta/\gamma}\right)\\
&\quad\le C\sum_{m'=0}^{k-1}\psi_2^{m'}\psi_1^{k+l-2m'-2}\\
&\quad\le Ck\psi_1^{k+l}\left(\frac{\psi_2}{\psi_1^2}\vee 1\right)^k,
\end{aligned}$$

where C is a finite constant and we have applied [25], equation (50) to deal with the $(m+1)$st generation terms. Similar calculations show that the analogous sums involving I_2 and I_3 can be bounded by the same expression after suitable modification of the constant.

We now consider the sum of $\mathbf{E}(\eta_i(t+\gamma^{-1}\ln D_i)\eta_j(t+\gamma^{-1}\ln D_j))$ over $i \in \Sigma_k$ and $j \in \Sigma_l$ in the case when i is an ancestor of j. Again applying Lemma 4.2, we deduce that

$$\begin{aligned}
&\sum_{i\in\Sigma_k}\sum_{j\in\Sigma_l:j|_k=i}\left(\mathbf{P}(A_i,A_j)+\mathbf{P}(A_{j|_{k+1}},A_j)+\sum_{n\in\mathbb{N}}\left(\mathbf{P}(A_i,A_{jn})+\mathbf{P}(A_{jn|_{k+1}},A_{jn})\right)\right)\\
&\quad\leq\ e^{2\theta t}\mathbf{E}(\delta^{2(1+\varepsilon^{-1})\theta})^{1/(1+\varepsilon^{-1})}\sum_{i\in\Sigma_k}\mathbf{E}(D_i^{2\theta/\gamma})\sum_{j\in\Sigma_l:j|_k=i}\Bigg[\mathbf{E}((D_j^i)^{(1+\varepsilon)\theta/\gamma})^{1/(1+\varepsilon)}\\
&\qquad+\mathbf{E}(\Delta_{j|_{k+1}}^{2\theta/\gamma})\mathbf{E}((D_j^{j|_{k+1}})^{(1+\varepsilon)\theta/\gamma})^{1/(1+\varepsilon)}+\sum_{n\in\mathbb{N}}\mathbf{E}((D_{jn}^i)^{(1+\varepsilon)\theta/\gamma})^{1/(1+\varepsilon)}\\
&\qquad+\mathbf{E}(\Delta_{jn|_{k+1}}^{2\theta/\gamma})\sum_{n\in\mathbb{N}}\mathbf{E}((D_{jn}^{jn|_{k+1}})^{(1+\varepsilon)\theta/\gamma})^{1/(1+\varepsilon)}\Bigg]\\
&\quad\leq\ Ce^{2\theta t}\psi_2^k\psi_{1,\varepsilon}^{l-k}. \qquad (22)
\end{aligned}$$

Note that if $l=k$, then the first term involving $j|_{k+1}$ should be deleted from the above argument. Another appeal to Lemma 4.2 yields that we also have

$$\sum_{i\in\Sigma_k}\sum_{j\in\Sigma_l:j|_k=i}\sum_{n\in\mathbb{N}:in\neq j|_{k+1}}\left(\mathbf{P}(A_{in},A_j)+\sum_{n'\in\mathbb{N}}\mathbf{P}(A_{in},A_{jn'})\right)\leq Ce^{2\theta t}\psi_2^k\psi_1^{l-k}. \qquad (23)$$

Summing (22) and (23), the bound at (21) implies

$$\sum_{i\in\Sigma_k}\sum_{j\in\Sigma_l:j|_k=i}\mathbf{E}(\eta_i(t+\gamma^{-1}\ln D_i)\eta_j(t+\gamma^{-1}\ln D_j))\leq Ce^{2\theta t}\psi_2^k\psi_{1,\varepsilon}^{l-k}. \qquad (24)$$

On combining our estimates, we obtain the lemma. □

We can now proceed with our second moment bound for $X(t)$.

Lemma 4.4. *For $\theta>\gamma$, there is a finite constant C such that*

$$\mathbf{E}(X(t)^2)\leq Ce^{2\theta t},\qquad \forall t\in\mathbb{R}.$$

Proof. This is a simple application of the preceding lemma. Firstly, applying (15), we have that

$$\begin{aligned}
\mathbf{E}X(t)^2 &= \mathbf{E}\left(\sum_{i,j\in\Sigma_*}\eta_i(t+\gamma^{-1}\ln D_i)\eta_j(t+\gamma^{-1}\ln D_j)\right),\\
&\leq 2\sum_{k=0}^{\infty}\sum_{l=k}^{\infty}\sum_{i\in\Sigma_k}\sum_{j\in\Sigma_l}\mathbf{E}\left(\eta_i(t+\gamma^{-1}\ln D_i)\eta_j(t+\gamma^{-1}\ln D_j)\right).
\end{aligned}$$

From Lemma 4.3, it follows that

$$\mathbf{E}\left(X(t)^2\right) \leq Ce^{2\theta t}\sum_{k=0}^{\infty}\sum_{l=k}^{\infty}(k+1)\psi_{1,\varepsilon}^{k+l}\left(\frac{\psi_2}{\psi_{1,\varepsilon}^2}\vee 1\right)^k,$$

for some finite constant C, which may depend on $\varepsilon > 0$. Noting that, in the range of θ considered, $\psi_r < 1$ for $r = 1, 2$ and $\psi_{1,\varepsilon} \to \psi_1$ as $\varepsilon \to 0$ (by the dominated convergence theorem), it is clear that the double sum is finite for suitably small ε. Thus the proof is complete. □

For the purposes of proving almost-sure convergence, we introduce the following notation to represent a cut-set of Σ_*: for $t > 0$,

$$\Lambda_t := \{i \in \Sigma_* : -\gamma^{-1}\ln D_i \geq t > -\gamma^{-1}\ln D_{i|_{|i|-1}}\}.$$

We will also have cause to refer to the subset of Λ_t defined by, for $t, c > 0$,

$$\Lambda_{t,c} := \{i \in \Sigma_* : -\gamma^{-1}\ln D_i \geq t + c,\ t > -\gamma^{-1}\ln D_{i|_{|i|-1}}\}.$$

We note that the sets $\Lambda_t, \Lambda_{t,c}$ are countably infinite, but that $\Lambda_t \backslash \Lambda_{t,c}$ is a finite set $\mathbf{P}$-a.s. The following is the main result of this section.

Proposition 4.5. $\mathbf{P}$*-a.s we have that*

$$e^{-\gamma t}X(t) \to m(\infty), \quad \text{as } t \to \infty,$$

where $m(\infty)$ is the constant defined at (18).

Proof. We follow the earlier proofs of such results which originate with [24]. First, we truncate the characteristics η_i (this term is meant in the generalised sense of [24], Section 7) by defining, for fixed $c > 0$, $\eta_i^c(t) := \eta_i(t)\mathbf{1}_{\{t \leq n_0 c\}}$, where n_0 is an integer that will be chosen later in the proof. From these truncated characteristics construct the processes X_i^c as

$$X_i^c(t) := \sum_{j \in \Sigma_*}\eta_{ij}^c(t + \gamma^{-1}\ln(D_{ij}/D_i)),$$

and set $X^c := X_\emptyset^c$. The corresponding discounted mean process is $m^c(t) := e^{-\gamma t}\mathbf{E}X^c(t)$, and this may be checked to converge to $m^c(\infty) \in (0, \infty)$ as $t \to \infty$ using the renewal theorem of [16]. From a branching process decomposition of X^c, we can deduce the following bound for $n_1 \geq n_0$, $n \in \mathbb{N}$,

$$|e^{-\gamma c(n+n_1)}X^c(c(n+n_1)) - m^c(\infty)| \leq S_1(n, n_1) + S_2(n, n_1) + S_3(n, n_1),$$

where

$$S_1(n, n_1) := \left|\sum_{i \in \Lambda_{cn}\backslash\Lambda_{cn,cn_1}}\left(e^{-\gamma c(n+n_1)}X_i^c(c(n+n_1) + \gamma^{-1}\ln D_i) - D_i m^c(c(n+n_1) + \gamma^{-1}\ln D_i)\right)\right|,$$

$$S_2(n,n_1) := \left| \sum_{i\in\Lambda_{cn}\backslash\Lambda_{cn,cn_1}} D_i m^c(c(n+n_1)+\gamma^{-1}\ln D_i) - m^c(\infty) \right|,$$

$$S_3(n,n_1) := e^{-\gamma c(n+n_1)} \sum_{i\in\Lambda_{cn,cn_1}} X_i^c(c(n+n_1)+\gamma^{-1}\ln D_i).$$

For the first two terms we can apply exactly the same argument as in [14] to deduce that, $\mathbf{P}$-a.s.,

$$\lim_{n_1\to\infty} \limsup_{n\to\infty} S_j(n,n_1) = 0, \qquad \text{for } j=1,2.$$

More explicitly, the result for $S_1(n,n_1)$ is an immediate application of the strong law of large numbers proved as [24], Proposition 4.1. For $S_2(n,n_1)$ on the other hand, we first note that this term is bounded above by

$$\sum_{i\in\Lambda_{cn}\backslash\Lambda_{cn,cn_1/2}} D_i \left| m^c(c(n+n_1)+\gamma^{-1}\ln D_i) - m^c(\infty) \right| + 3\sup_{t\in\mathbb{R}} m(t) \sum_{i\in\Lambda_{cn,cn_1/2}} D_i.$$

The first summand is bounded above by $\sup_{t\geq cn_1/2}|m^c(t)-m^c(\infty)|$, and so clearly decays to 0 as n and then n_1 diverges. Checking that the second summand does the same depends on recalling from Lemma 3.5 that m is bounded and applying the general branching process result of [24], Theorem 5.4, which yields an estimate of the form $\lim_{n\to\infty}\sum_{i\in\Lambda_{cn,cn_1/2}} D_i \leq c_1 e^{-c_2 n_1}$, where c_1 and c_2 are constants not depending on n_1.

We will now show that $S_3(n,n_1)$ decays in a similar fashion. We need to modify the approach of [4] slightly to deal with the infinite number of offspring. Firstly we introduce a set of characteristics, ϕ_i^{c,n_1}, defined by

$$\phi_i^{c,n_1}(t) := \sum_{j\in\mathbb{N}} X_{ij}(0)\mathbf{1}_{\{t+cn_1+\ln\delta_{ij}\geq-\gamma^{-1}\ln\Delta_{ij}>t+cn_1,\, t>0\}},$$

where the bound involving $\delta_{ij} = \mathrm{diam}_{d_{\mathcal{T}_{ij}}}(\mathcal{T}_{ij})$ is included to ensure that only a finite number of terms contribute to the sum. For $t>0$, set

$$Y^{c,n_1}(t) := \sum_{i\in\Sigma_*} \phi_i^{c,n_1}(t+\gamma^{-1}\ln D_i).$$

Note that from the definition of the cut-set Λ_{cn,cn_1} we can deduce that

$$Y^{c,n_1}(cn) = \sum_{i\in\Lambda_{cn,cn_1}} X_i(0)\mathbf{1}_{\{c(n+n_1)+\gamma^{-1}\ln D_i\geq-\ln\delta_i\}} \geq e^{\gamma c(n+n_1)} S_3(n,n_1),$$

where for the second inequality we apply the monotonicity of the X_is and the fact that $X_i(t)=0$ for $t<\ln\delta_i^{-1}$. Now, Y^{c,n_1} is a branching process with random characteristic ϕ_i^{c,n_1}, and we will proceed by checking that the conditions of the extension of [24], Theorem 5.4, that is stated as [14], Theorem 3.2, are satisfied by it. There are two conditions, one on the characteristic, the other on the reproduction process.

For the reproduction process, it is enough to show that there is a non-increasing, bounded positive integrable function g such that $\int_0^\infty g(t)^{-1}\nu_\gamma(dt)<\infty$. If we take $g(t)=1\wedge t^{-2}$, then by equation (6) of [25], we see that

$$\int_0^\infty (1\vee t^2)e^{-\gamma t}\nu(dt) \leq \mathbf{E}\sum_{i\in\mathbb{N}}\Delta_i\left(1+(\gamma^{-1}\ln\Delta_i)^2\right) < \infty.$$

For the characteristic, we need to prove the existence of a non-increasing, bounded positive integrable function h such that $\mathbf{E}\sup_{t\geq 0} e^{-\gamma t}\phi_\emptyset^{c,n_1}(t)/h(t) < \infty$. Taking $h(t) := e^{-\beta t/2}$, where β is the constant defined at (13), we find that

$$\begin{aligned}\sup_{t\in\mathbb{R}} \frac{e^{-\gamma t}\phi_\emptyset^{c,n_1}(t)}{h(t)} &\leq e^{\left(\frac{\beta}{2}-\gamma\right)t}\sum_{i\in\mathbb{N}} X_i(0)\left(e^{t+cn_1}\delta_i\Delta_i^{1/\gamma}\right)^{\frac{1+\alpha}{2(2\alpha-1)}} \\ &= e^{cn_1(1+\alpha)/2(2\alpha-1)}\sum_{i\in\mathbb{N}} X_i(0)\delta_i^{\frac{1+\alpha}{2(2\alpha-1)}}\Delta_i^{\frac{1+\alpha}{2\alpha}}. \end{aligned} \qquad (25)$$

Thus it will suffice to prove that the final expression here has a finite first moment. Since $(X_i(0))_{i\in\mathbb{N}}$ and $(\delta_i)_{i\in\mathbb{N}}$ are independent of $(\Delta_i)_{i\in\mathbb{N}}$, we deduce that

$$\mathbf{E}\left(\sum_{i\in\mathbb{N}} X_i(0)\delta_i^{\frac{1+\alpha}{2(2\alpha-1)}}\Delta_i^{\frac{1+\alpha}{2\alpha}}\right) \leq \left(\mathbf{E}(X(0)^2)\mathbf{E}\left(\delta_\emptyset^{\frac{1+\alpha}{2\alpha-1}}\right)\right)^{1/2}\psi((1+\alpha)/2\alpha), \qquad (26)$$

where we have applied Cauchy-Schwarz to separate the expectations involving $\delta_\emptyset$ and $X(0)$. Now observe that, by Lemma 4.4, $\mathbf{E}(X(0)^2) < \infty$, the moments of the diameter of a α-stable tree are finite and $\psi((1+\alpha)/2\alpha) < \infty$, which means that the condition on the characteristics is fulfilled.

Consequently, applying [14], Theorem 3.2, we find that $\mathbf{P}$-a.s.,

$$e^{-\gamma t}Y^{c,n_1}(t) \to \frac{\int_0^\infty e^{-\gamma t}\mathbf{E}\phi_\emptyset^{c,n_1}(t)dt}{\int_0^\infty t\nu_\gamma(dt)}, \qquad \text{as } t\to\infty.$$

By (25) and (26), the above limit is bounded by $Ce^{cn_1(1+\alpha)/2(2\alpha-1)}$, where C is a constant not depending on n_1. Hence, $\mathbf{P}$-a.s.,

$$\lim_{n_1\to\infty}\limsup_{n\to\infty} S_3(n,n_1) \leq \lim_{n_1\to\infty} Ce^{cn_1(1+\alpha)/2(2\alpha-1)}e^{-\gamma cn_1} = \lim_{n_1\to\infty} Ce^{-cn_1\beta/2} = 0,$$

and combining the three limit results for S_1, S_2 and S_3, it is easy to deduce that $\mathbf{P}$-a.s.,

$$\lim_{n\to\infty} |e^{-\gamma cn}X^c(cn) - m^c(\infty)| = 0. \qquad (27)$$

We now show that the process X, when suitably scaled, converges along the subsequence $(cn)_{n\geq 0}$. From (27) we have that $\mathbf{P}$-a.s.,

$$\limsup_{n\to\infty} |e^{-\gamma cn}X(cn) - m(\infty)| \leq |m(\infty) - m^c(\infty)| + \limsup_{n\to\infty} e^{-\gamma cn}|X(cn) - X^c(cn)|. \qquad (28)$$

Recall that the process X^c and its discounted mean process m^c depend on the integer n_0. By the dominated convergence theorem, the first of the terms in (28), which is deterministic, converges to zero as $n_0\to\infty$. To show the corresponding result for the second term, we start by introducing a collection of random variables $(U_i)_{i\in\Sigma_*}$ satisfying

$$U_i := \sup_{t\in\mathbb{R}} \frac{e^{-\gamma t}\eta_i(t)}{h(t)},$$

where, similarly to above, $h(t) := e^{-\beta t/2}$. By applying ideas from the proof of Lemma 3.5, it is an elementary exercise to check that $\mathbf{E}U_i < \infty$. Now, if we define characteristics

$\phi_i(t) := U_i \mathbf{1}_{\{t\in[0,c]\}}$, then this finite integrability of U_i readily implies the conditions of [14], Theorem 3.2, which yields that, $\mathbf{P}$-a.s.,

$$e^{-\gamma t}\sum_{i\in\Sigma_*}\phi_i(t+\gamma^{-1}\ln D_i)\to\frac{\int_0^c e^{-\gamma t}\mathbf{E}U_i dt}{\int_0^\infty t\nu_\gamma(dt)},\qquad \text{as } t\to\infty.$$

This we can rewrite as, $\mathbf{P}$-a.s.,

$$e^{-\gamma t}\sum_{i\in A_t\setminus A_{t-c}} U_i\to\frac{\int_0^c e^{-\gamma t}\mathbf{E}U_i dt}{\int_0^\infty t\nu_\gamma(dt)},\qquad \text{as } t\to\infty,$$

where $A_t := \{i\in\Sigma_* : -\gamma^{-1}\ln D_i\le t\}$. Hence, we can proceed similarly to the proof of [24], Lemma 5.8, to obtain that, $\mathbf{P}$-a.s., for $n>n_0$,

$$\begin{aligned}
e^{-\gamma cn}|X(cn)-X^c(cn)| &= e^{-\gamma cn}\sum_{i\in\Sigma_*}\eta_i(cn+\gamma^{-1}\ln D_i)\mathbf{1}_{\{cn+\gamma^{-1}\ln D_i>cn_0\}}\\
&\le \sum_{i\in\Sigma_*} D_iU_ih(cn+\gamma^{-1}\ln D_i)\mathbf{1}_{\{i\in A_{c(n-n_0)}\}}\\
&\le U_\emptyset h(cn)+\sum_{k=1}^{n-n_0}\sum_{i\in A_{ck}\setminus A_{c(k-1)}} D_iU_ih(c(n-k))\\
&\le U_\emptyset h(cn)+\sum_{k=1}^{n-n_0} e^{-c((n-k)\beta/2+(k-1)\gamma)}\sum_{i\in A_{ck}\setminus A_{c(k-1)}} U_i\\
&\le U_\emptyset e^{-\beta cn/2}+C\sum_{k=1}^{n-n_0} e^{-c(n-k)\beta/2}\\
&= U_\emptyset e^{-\beta cn/2}+C\sum_{k=n_0}^{\infty} e^{-ck\beta/2}.
\end{aligned}$$

This yields in particular that, $\mathbf{P}$-a.s.,

$$\limsup_{n\to\infty} e^{-\gamma cn}|X(cn)-X^c(cn)|\le Ce^{-cn_0\beta/2}.$$

Consequently, by choosing n_0 suitably large, the upper bound in (28) can be made arbitrarily small, which has as a result that $e^{-\gamma cn}X(cn)\to m(\infty)$ as $n\to\infty$, $\mathbf{P}$-a.s., for each c. The proposition is readily deduced from this using the monotonicity of X. □

5 The second order term

In this section we proceed to extend the result of the previous section so as to obtain an estimate on the second order term. We continue to assume that $\alpha\in(1,2)$, and recall from (13) the definition of $\beta=(\alpha-1)/(2\alpha-1)$. In particular, in terms of the process $X(t)=N^D(e^t)$, it is our aim to prove the following proposition.

Proposition 5.1. *For each $\varepsilon > 0$, in $\mathbf{P}$-probability, as $t \to \infty$,*

$$|e^{-\gamma t}X(t) - m(\infty)| = O(e^{-(\beta-\varepsilon)t}).$$

Let us start by introducing the notation $Y(t) := e^{-\gamma t}X(t) - m(t)$ for the rescaled and centred version of $X(t)$. Using the decomposition of X given at (10), we have

$$Y(t) = \zeta(t) + \sum_{i\in\mathbb{N}} \Delta_i Y_i(t + \gamma^{-1}\ln \Delta_i),$$

where Y_i is defined in the obvious way and

$$\zeta(t) = e^{-\gamma t}(\eta(t) - \mathbf{E}\eta(t)) + \sum_{i\in\mathbb{N}}(\Delta_i m(t + \gamma^{-1}\ln \Delta_i) - \mathbf{E}(\Delta_i m(t + \gamma^{-1}\ln \Delta_i))).$$

Hence

$$Y(t)^2 = Z(t) + \sum_{i\in\mathbb{N}} \Delta_i^2 Y_i(t + \gamma^{-1}\ln \Delta_i)^2, \tag{29}$$

where

$$Z(t) = \zeta^2(t)+2\zeta(t)\sum_{i\in\mathbb{N}} \Delta_i Y_i(t+\gamma^{-1}\ln \Delta_i)+ \sum_{i,j\in\mathbb{N},i\neq j} \Delta_i\Delta_j Y_i(t+\gamma^{-1}\ln \Delta_i)Y_j(t+\gamma^{-1}\ln \Delta_j).$$

Iterating (29), we have for any $k \in \mathbb{N}$,

$$Y(t)^2 = \sum_{|i|<k} D_i^2 Z_i(t + \gamma^{-1}\ln D_i) + \sum_{i\in\Sigma_k} D_i^2 Y_i(t + \gamma^{-1}\ln D_i)^2,$$

where the definition of Z_i is also the obvious one. The following lemma shows that the value of the remainder term here converges to zero as $k \to \infty$, from which we obtain a useful decomposition of $Y(t)^2$.

Lemma 5.2. *We have, $\mathbf{P}$-a.s., that*

$$\lim_{k\to\infty} \sum_{i\in\Sigma_k} D_i^2 Y_i(t + \gamma^{-1}\ln D_i)^2 = 0,$$

and hence we have the representation, $\mathbf{P}$-a.s.,

$$Y(t)^2 = \sum_{i\in\Sigma_*} D_i^2 Z_i(t + \gamma^{-1}\ln D_i), \qquad \forall t \in \mathbb{R}. \tag{30}$$

Proof. From the second moment estimates of Lemma 4.4 and the boundedness of m (see Lemma 3.5), we have that

$$\mathbf{E}Y(t)^2 \leq 2\mathbf{E}e^{-2\gamma t}X(t)^2 + 2m(t)^2 \leq C\left(e^{2\varepsilon t} \vee 1\right).$$

Thus

$$\begin{aligned}
\mathbf{E}\sum_{i\in\Sigma_k} D_i^2 Y_i(t + \gamma^{-1}\ln D_i)^2 &\leq \sum_{i\in\Sigma_k} C\mathbf{E}\left(D_i^2\left(e^{2\varepsilon(t+\gamma^{-1}\ln D_i)} \vee 1\right)\right). \\
&= \sum_{i\in\Sigma_k} C\left(e^{2\varepsilon t} \vee 1\right)\mathbf{E}(D_i^2) \\
&= C\left(e^{2\varepsilon t} \vee 1\right)\psi(2)^k,
\end{aligned}$$

where ψ was defined at (11). As $\psi(2) < 1$, we therefore have that

$$\sum_{k=0}^{\infty} \mathbf{P}\left(\sum_{i\in\Sigma_k} D_i^2 Y_i(t+\gamma^{-1}\ln D_i)^2 > \delta\right) \leq C\delta^{-1}\left(e^{2\varepsilon t}\vee 1\right)\sum_{k=0}^{\infty}\psi(2)^k < \infty,$$

where we have applied Chebyshev to deduce the first inequality. Hence, Borel-Cantelli implies the representation of $Y(t)^2$ for each fixed t, $\mathbf{P}$-a.s. By countability, it follows that the same result holds for each rational t. Since Y is cadlag, the representation can easily be extended to hold for all $t \in \mathbb{R}$. □

We use this result to derive a second moment estimate for $Y(t)$.

Lemma 5.3. *For each $\varepsilon > 0$, there exists a constant C such that*

$$\mathbf{E}Y(t)^2 \leq Ce^{-(2\beta-\varepsilon)t}.$$

Proof. We need to estimate the terms on the right-hand side of (30). Firstly, from the definition of Z_i, conditioning on Δ_i and using $\mathbf{E}Y_i(t) = 0$ we have

$$\begin{aligned}
&\mathbf{E}D_i^2 Z_i(t+\gamma^{-1}\ln D_i)\\
&= \mathbf{E}D_i^2\zeta_i(t+\gamma^{-1}\ln D_i)^2 + 2\mathbf{E}D_i^2\zeta_i(t+\gamma^{-1}\ln D_i)\sum_{j\in\mathbb{N}}\Delta_{ij}Y_{ij}(t+\gamma^{-1}\ln D_{ij})\\
&\leq 2\mathbf{E}\left(e^{-2\gamma t}D_i^2(\eta_i(t+\gamma^{-1}\ln D_i) - \mathbf{E}(\eta_i(t+\gamma^{-1}\ln D_i)|D_i))^2\right) + 2\mathbf{E}\left(\left(\sum_{j\in\mathbb{N}}\kappa_{ij}\right)^2\right)\\
&\qquad +2\mathbf{E}D_i^2 e^{-\gamma t}\eta_i(t+\gamma^{-1}\ln D_i)\sum_{j\in\mathbb{N}}\Delta_{ij}Y_{ij}(t+\gamma^{-1}\ln D_{ij}),
\end{aligned}$$

where we define $\kappa_{ij} := D_{ij}m(t+\gamma^{-1}\ln D_{ij}) - \mathbf{E}(D_{ij}m(t+\gamma^{-1}\ln D_{ij})|D_i)$. Hence $\mathbf{E}Y(t)^2 \leq 2(I_1+I_2+I_3)$, where

$$\begin{aligned}
I_1 &= \sum_{i\in\Sigma_*}\mathbf{E}\left(e^{-2\gamma t}D_i^2(\eta_i(t+\gamma^{-1}\ln D_i) - \mathbf{E}(\eta_i(t+\gamma^{-1}\ln D_i)|D_i))^2\right),\\
I_2 &= \sum_{i\in\Sigma_*}\mathbf{E}\left(\left(\sum_{j\in\mathbb{N}}\kappa_{ij}\right)^2\right),\\
I_3 &= \sum_{i\in\Sigma_*}\mathbf{E}\left(D_i^2 e^{-\gamma t}\eta_i(t+\gamma^{-1}\ln D_i)\sum_{j\in\mathbb{N}}\Delta_{ij}Y_{ij}(t+\gamma^{-1}\ln D_{ij})\right).
\end{aligned}$$

For I_1, we apply Lemma 4.2 similarly to the proof of Lemma 4.3 to deduce that, for suitably chosen $\theta > \gamma/\alpha$,

$$\begin{aligned}
I_1 &\leq \sum_{k=0}^{\infty}\sum_{i\in\Sigma_k} e^{-2\gamma t}\mathbf{E}(\eta_i(t+\gamma^{-1}\ln D_i)^2)\\
&\leq Ce^{-2\gamma t}e^{2\theta t}\sum_{k=0}^{\infty}\psi(2\theta\gamma^{-1})^k\\
&= Ce^{-(2(\alpha-1)\gamma/\alpha-\varepsilon)t},
\end{aligned}$$

which is a bound of the appropriate magnitude. For I_2, we use an extension of [25], equation (6), coupled with the estimate on the convergence rate of $m(t)$ to its limit. Specifically, we begin by writing $\kappa_{ij} = D_i A_j(t_i)$ where $t_i := t + \gamma^{-1} \ln D_i$ and

$$A_j(t) := \Delta_j m(t + \gamma^{-1} \ln \Delta_j) - \mathbf{E}\Delta_j m(t + \gamma^{-1} \ln \Delta_j).$$

As $\sum_{j\in\mathbb{N}} \Delta_j = 1$, we can write

$$\sum_{j\in\mathbb{N}} A_j(t) = \sum_{j\in\mathbb{N}} \left(\Delta_j \hat{m}(t + \gamma^{-1} \ln \Delta_j) - \mathbf{E}\Delta_j \hat{m}(t + \gamma^{-1} \ln \Delta_j)\right),$$

where $\hat{m}(t) := m(t) - m(\infty)$. By Proposition 3.6 and the boundedness of m (Lemma 3.5(a)), there is a constant C such that $|\hat{m}(t)| = |m(t) - m(\infty)| \le Ce^{-(\beta-\varepsilon)t}$ for $t \in \mathbb{R}$, and hence

$$\left|\sum_{j\in\mathbb{N}} A_j(t))\right| \le C \left(\sum_{j\in\mathbb{N}} \Delta_j^{1-(\beta-\varepsilon)/\gamma} + \mathbf{E}\sum_{j\in\mathbb{N}} \Delta_j^{1-(\beta-\varepsilon)/\gamma}\right) e^{-(\beta-\varepsilon)t}.$$

Now, to obtain our estimate, we note that

$$\begin{aligned}
&\mathbf{E}\left(\left(\sum_{j\in\mathbb{N}} \kappa_{ij}\right)^2\right) \\
&= \mathbf{E}\left(D_i^2 \mathbf{E}\left(\left(\sum_{j\in\mathbb{N}} A_j(t_i)\right)^2 \middle| D_i\right)\right) \\
&\le C\mathbf{E}\left(D_i^2 \mathbf{E}\left(\left(\sum_{j\in\mathbb{N}} \Delta_j^{1-(\beta-\varepsilon)/\gamma}\right)^2 + \left(\mathbf{E}\sum_{j\in\mathbb{N}} \Delta_j^{1-(\beta-\varepsilon)/\gamma}\right)^2\right) D_i^{-2(\beta-\varepsilon)/\gamma}\right) e^{-2(\beta-\varepsilon)t}.
\end{aligned}$$

Using the lemma in the appendix, and the fact that $1 - \beta\gamma^{-1} = \alpha^{-1}$, we can compute the first term as follows:

$$\begin{aligned}
\mathbf{E}\left(\sum_{j\in\mathbb{N}} \Delta_j^{1-(\beta-\varepsilon)/\gamma}\right)^2 &= \mathbf{E}\left(\sum_{j\in\mathbb{N}} \Delta_j^{2-2(\beta-\varepsilon)/\gamma} + \sum_{j,l\in\mathbb{N}:j\ne l} \Delta_j^{1-(\beta-\varepsilon)/\gamma} \Delta_l^{1-(\beta-\varepsilon)/\gamma}\right) \\
&= \psi(2 - 2(\beta-\varepsilon)\gamma^{-1}) + \frac{\Gamma(2-\alpha^{-1})^2}{(\varepsilon\gamma^{-1})^2 \Gamma(1-\alpha^{-1})\Gamma(1+\alpha^{-1})}.
\end{aligned}$$

Thus we obtain that

$$\mathbf{E}\left(\left(\sum_{j\in\mathbb{N}} \kappa_{ij}\right)^2\right) \le C\mathbf{E}D_i^{2/\alpha+\epsilon} e^{-2(\beta-\varepsilon)t}.$$

As $2/\alpha + \epsilon > 1$ for $\alpha \in (1, 2]$, this can be summed over $i \in \Sigma_*$ to give the bound

$$I_2 \le ce^{-2(\beta-\varepsilon)t}.$$

Finally, for I_3, we first observe that by (15) and the definition of $Y(t)$ we can write

$$Y(t) = e^{-\gamma t} \sum_{i \in \Sigma_*} \left(\eta_i(t + \gamma^{-1} \ln D_i) - \mathbf{E}\eta_i(t + \gamma^{-1} \ln D_i) \right).$$

Hence

$$\begin{aligned} I_3 &\leq e^{-2\gamma t} \sum_{i \in \Sigma_*} \mathbf{E} \left(D_i^2 \eta_i(t + \gamma^{-1} \ln D_i) \sum_{j \in \Sigma_* \setminus \{\emptyset\}} \eta_{ij}(t + \gamma^{-1} \ln D_{ij}) \right) \\ &\leq e^{-2\gamma t} \sum_{k=0}^{\infty} \sum_{l=k+1}^{\infty} \sum_{i \in \Sigma_k} \sum_{j \in \Sigma_l : j|_k = i} \mathbf{E} \left(\eta_i(t + \gamma^{-1} \ln D_i) \eta_j(t + \gamma^{-1} \ln D_j) \right). \end{aligned}$$

To bound the inner two sums, we follow the arguments of Section 4, but taking different powers to those used there. For example, in the case when i is an ancestor of j, we can replace the second statement of Lemma 4.2 by: for $\theta_1, \theta_2, \varepsilon' \geq 0$,

$$\mathbf{P}(A_i \cap A_j) \leq C e^{t(\theta_1 + \theta_2)} \mathbf{E} \left(D_i^{(\theta_1 + \theta_2)/\gamma} \right) \mathbf{E} \left((D_j^i)^{\theta_2 (1+\varepsilon')/\gamma} \right)^{1/(1+\varepsilon')},$$

where A_i is defined as in the proof of Lemma 4.3 and C is a constant that depends only on θ_1, θ_2 and ε'. After proceeding similarly with the other relevant terms and taking $\theta_1 := (2\alpha^{-1} - 1)\gamma$, $\theta_2 := \gamma + \varepsilon$, we are consequently able to show that (cf. (24)): for any $\theta > \gamma/\alpha$,

$$\sum_{i \in \Sigma_k} \sum_{j \in \Sigma_l : j|_k = i} \mathbf{E}(\eta_i(t + \gamma^{-1} \ln D_i) \eta_j(t + \gamma^{-1} \ln D_j)) \leq C e^{2\theta t} \psi_2^k \psi(1 + \varepsilon\gamma^{-1}, \varepsilon')^{l-k},$$

where, as previously, $\psi_2 := \psi(2\theta\gamma^{-1})$, and

$$\psi(1 + \varepsilon\gamma^{-1}, \varepsilon') := \sum_{i \in \mathbb{N}} \mathbf{E} \left(\Delta_i^{(1+\varepsilon\gamma^{-1})(1+\varepsilon')} \right)^{1/(1+\varepsilon')} \to \psi(1 + \varepsilon\gamma^{-1}),$$

as $\varepsilon' \to 0$. Since $\psi(1 + \varepsilon\gamma^{-1}), \psi_2 < 1$, if ε' is chosen small enough, we find from these results that

$$I_3 \leq C e^{-(2\beta - \varepsilon)t} \sum_{k=0}^{\infty} \sum_{l=k+1}^{\infty} \psi_2^k \psi(1 + \varepsilon\gamma^{-1}, \varepsilon')^{l-k} \leq C e^{-(2\beta - \varepsilon)t},$$

as desired. □

Given this bound, it is now straightforward to prove the result of interest.

Proof of Proposition 5.1. By Chebyshev and Lemma 5.3, there exists a C such that for all $t \geq 0$

$$\mathbf{P}(|Y(t)| > x) \leq x^{-2} \mathbf{E}(Y(t)^2) \leq x^{-2} C e^{-(2\beta - \varepsilon)t}.$$

Now choose $x = e^{-t(\beta - \varepsilon)}$ to see that

$$\mathbf{P}(|Y(t)| > e^{-t(\beta - \varepsilon)}) \leq C e^{-\varepsilon t},$$

and hence we have the desired result in probability. □

To completely establish Theorem 1.1, it remains to demonstrate that part (b) holds in the case $\alpha = 2$. However, since the appropriate first order asymptotic behaviour was already obtained in [4] and the second order term requires us to make only very minor changes to the above argument, we omit the proof of this part of the theorem.

Unfortunately, the arguments of this section are not enough to yield an almost-sure result regarding the size of second order term in the asymptotic expansion of the eigenvalue counting function for α-stable trees. By Borel-Cantelli, the results we have proved so far would be good enough to show that for any $c > 0$ it is $\mathbf{P}$-a.s. the case that

$$\limsup_{n\to\infty} |Y(nc)| e^{cn(\beta-\varepsilon)} \leq 1.$$

To extend this to all t and establish that, $\mathbf{P}$-a.s.,

$$\limsup_{t\to\infty} |e^{-\gamma t} X(t) - m(t)| e^{-t(\beta-\varepsilon)} \leq C,$$

it would be enough to have moment estimates of the form

$$\mathbf{E} Y(t)^k \leq C e^{-k(\beta-\varepsilon)t},$$

for all $k \in \mathbb{N}$. Although it appears that suitable extensions of the techniques used here would, after much effort, yield such a result, we will leave such a calculation to an interested reader. Finally, let us remark that, by analogy with the results known to hold for related branching processes, it might also be hoped that a central limit theorem-type result of the following form holds, establishing the second order term for the eigenvalue counting function of α-stable trees.

Conjecture 5.4. *As $\lambda \to \infty$,*

$$\frac{N^D(\lambda) - m(\infty)\lambda^{\alpha/(2\alpha-1)}}{\lambda^{1/(2\alpha-1)}} \to Z_\alpha, \quad \textit{in distribution},$$

where Z_α is an α-stable random variable.

A Appendix

The following result, which is a straightforward extension of [25], equation (6), is applied in the proof of Lemma 5.3.

Lemma A.1. *Suppose $(V_i)_{i\in\mathbb{N}}$ has the Poisson-Dirichlet (α, θ) distribution. For measurable functions f, g we have*

$$\begin{aligned} &\mathbf{E} \sum_{i=1}^{\infty} \sum_{j=1, j\neq i}^{\infty} f(V_i) g(V_j) \\ &= C_{\alpha,\theta} \int_0^1 \int_0^1 f(x) g((1-x)y) x^{-1-\alpha} (1-x)^{\theta+\alpha-1} y^{-1-\alpha} (1-y)^{\theta+2\alpha-1} dx dy, \end{aligned}$$

where

$$C_{\alpha,\theta} = \frac{\Gamma(\theta+1)\Gamma(\theta+\alpha+1)}{\Gamma(1-\alpha)^2 \Gamma(\theta+\alpha)\Gamma(\theta+2\alpha)}.$$

Proof. This is an application of size-biased sampling. Following the set up in [25], define $\tilde{V}_1$ to be a size biased pick from $(V_i)_{i\in\mathbb{N}}$, that is

$$P(\tilde{V}_1 = V_n|\{V_i\}) = V_n, \quad n \in \mathbb{N}.$$

Also, let $\tilde{V}_2$ be the second size biased pick, that is a random variable with distribution

$$P(\tilde{V}_2 = V_n|\tilde{V}_1, \{V_i\}) = \frac{V_n \mathbf{1}_{\{V_n \neq \tilde{V}_1\}}}{1-\tilde{V}_1}, \quad n \in \mathbb{N}.$$

It is then possible to show that we can write

$$\tilde{V}_1 = \tilde{Y}_1, \quad \tilde{V}_2 = (1-\tilde{Y}_1)\tilde{Y}_2,$$

where $\tilde{Y}_i$, $i = 1, 2$, are independent random variables with Beta$(1-\alpha, \theta+i\alpha)$ distribution (see [25], Proposition 2, for example). Applying this result,

$$\begin{aligned}
&\mathbf{E}\sum_{i=1}^{\infty}\sum_{j=1,j\neq i}^{\infty} f(V_i)g(V_j) \\
&\quad = \mathbf{E}\frac{f(\tilde{V}_1)}{\tilde{V}_1}\frac{g(\tilde{V}_2)(1-\tilde{V}_1)}{\tilde{V}_2} \\
&\quad = \mathbf{E}\frac{f(\tilde{Y}_1)}{\tilde{Y}_1}\frac{g((1-\tilde{Y}_1)\tilde{Y}_2)}{\tilde{Y}_2} \\
&\quad = \frac{\Gamma(\theta+1)\Gamma(\theta+\alpha+1)}{\Gamma(1-\alpha)^2\Gamma(\theta+\alpha)\Gamma(\theta+2\alpha)} \\
&\qquad \times \int_0^1\int_0^1 f(x)g((1-x)y)x^{-1-\alpha}y^{-1-\alpha}(1-x)^{\theta+\alpha-1}(1-y)^{\theta+2\alpha-1}dxdy,
\end{aligned}$$

as required. □

To apply this in our setting, we use $f(x) = g(x) = x^{\alpha^{-1}+\varepsilon\gamma^{-1}}$ with Poisson-Dirichlet parameters $(\alpha^{-1}, 1-\alpha^{-1})$.

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