Dimension Results and Local Times for Superdiffusions on Fractals[⋆,⋆⋆]

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Abstract

We consider the Dawson-Watanabe superprocess obtained from a spatial motion with sub-Gaussian transition densities on a metric measure space with finite Hausdorff dimension, and examine the dimensions of the range and the set of times when the support intersects a given set, generalising results of Serlet and Tribe. As intermediate results, we prove existence of local times for the superprocess if the spectral dimension of the spatial motion satisfies $d_s < 4$, and prove that $(2 - d_s/2) \wedge 1$ is the critical Hölder-continuity exponent in the time variable. Furthermore, we prove a bound on moments of the integrated superprocess, and give uniform upper bounds on the mass the superprocess assigns to small balls, generalising a result of Perkins.

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1. Introduction and Setup

SuperBrownian motion in $\mathbb{R}^d$ has been an object of extensive study in recent decades as a canonical measure-valued process and for its fundamental role in a wide variety of models. The path properties, such as the Hausdorff measure of the support, have been explored in [25, 32], while other properties of the support and local time can be found in [7, 34, 24]. More recent results have considered the boundary of the support and range and associated local times [22, 20]. In this paper we will be concerned with the path properties of superprocesses in a more general setting.

Our aim is to extend known results concerning the Hausdorff dimension of two kinds of sets related to the support of superBrownian motion, namely

$$\mathcal{T}(A) := \{t \geq 0 \colon A \cap \mathrm{S}(X_t) \neq \varnothing\},$$

for $A \subset \mathbb{R}^d$, studied by Serlet [34], and

$$\mathcal{R}(I) := \bigcup_{t \in I} \mathrm{S}(X_t)$$

for $I \subset (0, \infty)$, studied by Serlet [34] and Tribe [36]. Here, (X_t) denotes a superBrownian motion in $\mathbb{R}^d$, and $\mathrm{S}(\cdot)$ is the closed support of a Borel measure. The proofs in [34] use LeGall's Brownian snake to apply path properties of linear Brownian motion to study the associated superprocess, while [36] uses particle systems and non-standard analysis. We present an approach using the historical process – which allows for a modern formulation of some ideas used by Tribe [36] – and superprocess local times to establish similar results in a more general setting. An advantage of this approach is that the connection between local times and the dimension of certain sets can be exploited in both directions: Hölder continuity of local times imply a lower bound on the dimension – an idea that has previously

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been used by Krone [24] to study superdiffusions on $\mathbb{R}^d$ – and a corresponding upper bound shows that the Hölder continuity is sharp.

Our aim is to consider the properties of superprocesses on less regular sets, such as fractals. A fundamental example is the Sierpinski gasket, where work of Barlow and Perkins [5] showed how to construct a Brownian motion on such a set, and obtained sub-Gaussian bounds on its transition density. In subsequent developments, diffusions have been constructed on more general classes of fractals and shown to have sub-Gaussian heat kernels, see for example [3, 23, 4]. Superprocesses on the Sierpinski gasket were considered in [16, 17], where some path properties were established. Here we wish to consider the dimension properties of the support of superprocesses on a general class of metric spaces, which include such fractals.

We now give the setting and assumptions for our results. Let (E, ρ) be a complete and separable metric space with Hausdorff dimension $d_f \in (0, \infty)$, and assume that there exists a non-zero Borel measure ν on E such that

$$\nu(B(x, r)) \leq c_2 r^{d_f}, \tag{1.1}$$

for all $x \in E$ and $r > 0$, and some $c_2 > 0$, where $B(x, r)$ denotes the open ball centred at x of radius r (with respect to the metric ρ). Note that the existence of such a measure is equivalent to $\mathcal{H}^{d_f}(E) > 0$ by a version of Frostman's lemma, where $\mathcal{H}^{d_f}$ denotes the d_f-dimensional Hausdorff (outer) measure on E.

Let $(\mathbb{P}^x)_{x\in E}$ denote the laws of a continuous E-valued Feller Markov process $Y = (Y_t)_{t\geq 0}$ with transition semigroup $(P_t)_{t\geq 0}$, where by Feller we mean that $P_t(C_0(E)) \subset C_0(E)$, where $C_0(E)$ denotes the space of continuous functions $E \to \mathbb{R}$ that vanish at infinity. We assume that Y has symmetric transition densities $(p_t(x, y) \colon t > 0, x, y \in E)$ with respect to (w.r.t.) ν – which can be chosen jointly measurable in (t, x, y) [37] – and that there exist $\beta, \gamma, c_3, c_4 > 0$ and $t_0 \in (0, \infty]$ such that

$$p_t(x, y) \leq c_3 t^{-d_f/\beta} \exp\left(-c_4 \rho(x, y)^{\beta\gamma} t^{-\gamma}\right), \tag{1.2}$$

for all $x, y \in E$ and $t \in (0, t_0)$. We call $d_s := 2d_f/\beta$ the *spectral dimension* of Y.

These assumptions already imply that there exist $r_0, c_1 > 0$ such that

$$\nu(B(x, r)) \geq c_1 r^{d_f}, \tag{1.3}$$

for all $x \in E$ and $r \in (0, r_0)$, see Lemma B.1. In other words, given the upper bound (1.1), a lower bound of the form (1.3) is necessary in order for a diffusion on E with sub-Gaussian transition densities (w.r.t. ν) of the form (1.2) to exist.

Remark 1.1. *(i) If $t_0 < \infty$ and* $\operatorname{diam}(E) = \infty$*, then $r_0 < \operatorname{diam}(E) = \infty$. Otherwise $r_0 = \operatorname{diam}(E)$, in which case E is called* Ahlfors d_f-regular*, or a* fractional metric space *in the terminology introduced by Barlow [3]. All examples we give below are of this type.*

A standard consequence of (1.1) *and* (1.3) *is that there are constants $c, c' > 0$ such that $c\mathcal{H}^{d_f}(B) \leq \nu(B) \leq c'\mathcal{H}^{d_f}(B)$ for all Borel sets $B \subset E$ [19, Exercise 8.11].*

(ii) We could have allowed another constant $\alpha > 0$ to take the place of d_f/β in (1.2)*, but an argument similar to [3, Lemma 3.8] shows that* (1.1) *and* (1.2) *would then already imply $\alpha = d_f/\beta$ if $t_0 = \infty$. If $t_0 < \infty$, one would get $\alpha \geq d_f/\beta$, and equality would follow if matching lower bounds to* (1.1) *and* (1.2) *held respectively for small r and small $\rho(x, y)$.*

(iii) We note that under our assumptions on E, if the measure is volume doubling and there exists a (strongly regular local) Dirichlet form on E which satisfies an elliptic Harnack inequality, then the diffusion process Y associated with the Dirichlet form will have transition densities w.r.t. ν that satisfy an upper (and a near-diagonal lower) bound of the form (1.2) *given that*

$$cr^\beta \leq \mathbb{E}^x [\inf\{t \geq 0 \colon Y_t \notin B(x, r)\}] \leq Cr^\beta$$

for small $r > 0$, for some $c, C > 0$. See [15] for details.

(iv) *Since we do not assume that the transition densities are continuous, they are unique only up to modification on a ν-null set. For any given choice, the Markov property implies for any $t, s > 0$ and $x \in E$, that*

$$\int_E p_t(x,y)p_s(y,z)\nu(\mathrm{d}y) = p_{t+s}(x,z), \tag{1.4}$$

for ν-a.e. $z \in E$. We assume that there exists a particular choice that satisfies (1.4) *simultaneously for all $t, s > 0$ and $x, z \in E$. Note that this is automatically the case if p_t, $t > 0$, can be chosen to be continuous, which is true for all examples we give below.*

Examples 1.2. (i) Brownian motion, or indeed any well-behaved diffusion on $\mathbb{R}^d$ satisfies these assumptions with $\beta = 2$, $d_f = d_s = d$, $\beta = 2$, $\gamma = 1$, $t_0 = \infty$ [3, Examples 3.6].

(ii) If Y is a fractional diffusion in the sense of Barlow [3], then our assumptions are satisfied with $\beta \geq 2$, $\gamma = 1/(\beta - 1)$, and $t_0 = \mathrm{diam}(E)^\beta$. We note that this includes finitely ramified fractals, such as the Sierpinski gasket, or the pcf self-similar sets of Kigami [23], all of which have $d_s < 2$, that is $d_f < \beta$.

(iii) Brownian motion on generalised Sierpinski carpets in $\mathbb{R}^d$, for any $d \geq 2$ [4], are diffusions on infinitely ramified fractals to which our results apply, with spectral dimensions in the range $1 < d_s < d$. This includes examples with $d_s > 2$.

Remark 1.3. *Brownian motions have been constructed on some classes of random fractals. In this setting there are fluctuations in the measure and heat kernel due to the randomness, giving equivalents for* (1.1) *and* (1.2) *that can be controlled, for small r and t, by expressions of the form*

$$\nu(B(x,r)) \leq c_2 r^{d_f} h_1(r), \qquad p_t(x,y) \leq c_3 t^{-d_f/\beta} h_2(t) \exp\left(-c_4 \rho(x,y)^{\beta\gamma}(t h_3(t))^{-\gamma}\right),$$

where, for any $\varepsilon > 0$, we have $h_1(r) = o(r^\varepsilon)$, and $h_2(t), h_3(t) = o(t^{-\varepsilon})$. See for instance [18] for homogeneous random Sierpinski carpets in $\mathbb{R}^d$, where such results hold and the spectral dimension can take values $1 < d_s < d$. Our arguments can be adapted to this setting, and we expect our main results on local times and dimensions to hold under these weaker assumptions.

The paper is structured as follows. In the next section, we state and discuss our main results. In Section 3, we introduce the historical process and prove our small ball bound. In Sections 4 and 5 we prove our main results respectively on local times and dimensions. Appendix A introduces Dynkin's moment formula, which is a central ingredient in the proofs of our local time results, and Appendix B contains proofs of some technical lemmas.

2. Main Results

Denote by $\mathcal{M}_F(E)$ the Polish space of finite Borel measures on E with the topology of weak convergence. Let $\gamma_b > 0$ be fixed but arbitrary, and denote by $((X_t)_{t\geq 0}, (\mathbb{P}^\mu)_{\mu \in \mathcal{M}_F(E)})$ a superprocess in E with spatial motion Y and branching rate γ_b, a (Y, γ_b)-superprocess for short. It exists because E is Polish, and Y is a continuous Feller Markov process, see [33, Section II] for a definition and basic properties. The distinction of the laws $(\mathbb{P}^\mu)$ of X from the laws $(\mathbb{P}^x)$ of Y will always be clear from superscript and context.

We first present our local time result. Several notions of superprocess local times have been studied, the most common of which can be heuristically defined in our setting as

$$L(t,x) = \int_0^t X_s(\delta_x)\,\mathrm{d}s, \tag{2.1}$$

for $x \in E$ and $t \geq 0$, where δ_x denotes the Dirac delta function at x (w.r.t. ν), and where we write $\mu(f)$ for the integral of a function f against a measure μ. More formally, L is defined as a density of the occupation time process $\int_0^t X_s\,\mathrm{d}s$ w.r.t. ν, if it exists. This existence is known to be true in the superBrownian [35] and α-stable [1] cases on $\mathbb{R}^d$ if $d \leq 3$ and $d < 2\alpha$, respectively, and further results have been achieved in the context of multitype superprocesses on $\mathbb{R}^d$ [28] and superprocesses with immigration and dependent spatial motions on $\mathbb{R}$ [26]. A lot of recent work has been on local times associated with the boundary of the support of superBrownian motion [22, 20]. Dynkin [10] introduced an alternative notion of superprocess local times based on representations of functionals of a superprocess by multiple stochastic integrals, which have been shown to coincide with L for superBrownian motion [27]. We show that, in our setting, local times exist in the sense of (2.1), and we establish sharp estimates on their Hölder continuity.

Theorem 2.1 (Local Times)**.** *Suppose that $d_s < 4$. Then, for every $\delta > 0$ and every $\mu \in \mathcal{M}_F(E)$, there exists a family of non-negative random variables $(L_\delta(t,x))_{t\geq\delta,x\in E}$ such that the following hold.*

(i) For all $T > 0$,

$$\sup_{\delta\leq t\leq T} \mathbb{E}^\mu\left[\left|L_\delta(t,x) - \int_\delta^t X_s(p_h(x,\cdot))\,\mathrm{d}s\right|^2\right] \overset{h\to 0}{\longrightarrow} 0.$$

(ii) $\mathbb{P}^\mu$-almost surely (a.s.), for all measurable functions $f\colon E \to [0,\infty]$ and every $t \geq \delta$,

$$\int_\delta^t X_s(f)\,\mathrm{d}s = \int_E f(x)L_\delta(t,x)\nu(\mathrm{d}x).$$

(iii) For $x \in E$, $L_\delta(\cdot,x)$ is increasing and β'-Hölder continuous for all $\beta' < (2 - d_s/2) \wedge 1$.

(iv) For $x \in E$, $L_\delta(\cdot,x)$ is $\mathbb{P}^\mu$-a.s. constant on any interval on which it is β'-Hölder continuous for a $\beta' > (2-d_s/2)\wedge 1$.

Remark 2.2. *(i) Similar to the condition given in [35] for the existence of superBrownian local times, our proofs can be modified to give the assertion of Theorem 2.1 with $\delta = 0$ under the condition that*

$$\sup_{x\in E} \int_0^t \int_E p_s(x,y)\mu(\mathrm{d}y)\,\mathrm{d}s < \infty,$$

for all $t > 0$.

(ii) We consider superprocess local times in this setting mostly as a means to prove Theorem 2.7 below, which is why we did not examine continuity in the space variable.

The central ingredient in the proof of Theorem 2.1 (iii) is the following moment bound for the integrated superprocess. Denote by $\|\cdot\|_{\infty,\nu}$ the ν-essential supremum, and write $(\mu q_t)(x) := \int_0^t \int_E p_s(x,y)\mu(\mathrm{d}y)\,\mathrm{d}s$ for $t > 0$ and $x \in E$.

Theorem 2.3. *If $d_s < 4$, then for any $n \in \mathbb{N}$ there is a $c_5(n) > 0$ such that, for any $\mu \in \mathcal{M}_F(E)$ and $f \in B_+(E)$, and $t \in (0,t_0)$,*

$$\mathbb{E}^\mu\left[\left(\int_0^t X_s(f)\,\mathrm{d}s\right)^n\right] \leq c_5(n)\left[\nu(f)\cdot\left(\|(\mu q_t)\|_{\infty,\nu} \vee t^{2-d_s/2}\right)\right]^n. \tag{2.2}$$

In particular, if μ has a bounded density g w.r.t. ν, then

$$\mathbb{E}^\mu\left[\left(\int_0^t X_s(f)\,\mathrm{d}s\right)^n\right] \leq c_5(n)\left[\nu(f)\,\|g\|_{\infty,\nu}\left(t \vee t^{2-d_s/2}\right)\right]^n, \quad t \in (0,t_0). \tag{2.3}$$

Remark 2.4. *(i) The exponent $2 - d_s/2$ is sharp in the following sense. If $d_s < 2$, then the maximum on the right-hand side (RHS) in (2.2) will be attained by $\|(\mu q_t)\|_{\infty,\nu}$ for small times. If $d_s \geq 2$, then there is no exponent larger than $2 - d_s/2$ for which (2.2) remains true simultaneously for all $n \in \mathbb{N}$ and $\mu \in \mathcal{M}_F(E)$. We prove this along with Theorem 2.3 in Section 4.*

(ii) Theorem 2.3 holds in much more generality: Beyond the assumptions necessary to guarantee existence of the superprocess (Y is a continuous Feller Markov process taking values in a Polish space E), it requires only the existence of transition densities (p_t) for Y w.r.t. some Borel measure ν that satisfy $\|p_t\|_{\infty,\nu} \leq ct^{-\alpha}$ for all $t \in (0,t_0)$ and some $\alpha \in (0,2)$, $t_0 \in (0,\infty]$, and $c > 0$. In that case, α takes the place of $d_s/2$ in (2.2).

(iii) *In a recent paper, Gonzalez, Horton, and Kyprianou [14] analysed the asymptotics of the same moments. In our notation, their result implies*

$$\mathbb{E}^\mu\left[\left(\int_0^t X_s(f)\,\mathrm{d}s\right)^n\right] \sim c(n,f)t^{2n-1}, \quad t\to\infty,$$

for a constant $c(n,f)>0$, where $a(t)\sim b(t)$ denotes $a(t)/b(t)\to 1$ as $t\to\infty$. Note that this would contradict (2.3) *if $t_0=\infty$ and $n>1\vee 2/d_s$. However, their assumption (H1) implies that Y started at any $x\in E$ converges to a stationary distribution, which in our setting necessitates $t_0<\infty$.*

Write $\mathcal{T}(A) := \{t\geq 0\colon A\cap \mathrm{S}(X_t)\neq\varnothing\}$ for $A\subset E$. Denote Hausdorff dimension by $\dim_H$, and write $a_+ := a\vee 0$ for $a\in\mathbb{R}$. Recall that a set $A\subset E$ is called *analytic* if it is the continuous image of a Polish space, and that this is automatically the case if A is Borel measurable.

Theorem 2.5. *Let $\mu\in\mathcal{M}_F(E)$ and $A\subset E$, and put*

$$D_A := \left(2-\frac{d_s}{2}+\frac{\dim_H A}{\beta}\right)_+ \wedge 1.$$

(i) *$\mathbb{P}^\mu$-almost surely,*

$$\dim_H \mathcal{T}(A) \leq D_A.$$

(ii) *If A is analytic, and $\int_\delta^\infty \mathrm{d}t\int \mu(\mathrm{d}x)p_t(x,y)>0$ for all $y\in A$, for some $\delta>0$, then for every $\varepsilon>0$, with positive $\mathbb{P}^\mu$-probability,*

$$\dim_H \mathcal{T}(A)\geq D_A-\varepsilon.$$

(iii) *If, in addition to the assumptions in (ii), $\mathcal{H}^{\dim_H A}(A)>0$, then with positive $\mathbb{P}^\mu$-probability,*

$$\dim_H\mathcal{T}(A) = D_A.$$

Remark 2.6. (i) *The additional assumption in Theorem 2.5 (ii) essentially asks that A be reachable from* $\mathrm{S}(\mu)$*. It is automatically satisfied if $p_t>0$ everywhere for all $t>0$, which is the case for all of the processes listed in Examples 1.2. The requirement for the integral to be positive* away from zero uniformly on *A is technical, and can be dropped if, say, p_t, $t>0$, are continuous (that is, we can then take $\delta=0$).*

(ii) *If specialised to superBrownian motion on $\mathbb{R}^d$, this is the same as Serlet's result [34, Theorem 1], except that A need not be measurable in Theorem 2.5 (i), that it need only be analytic in Theorem 2.5 (ii), and with the addition of Theorem 2.5 (iii). Note that the example given by Serlet for why one cannot in general expect the assertion of (iii) to hold has the property that $\mathcal{H}^{\dim_H A}(A)=0$.*

(iii) *One cannot in general expect equality in Theorem 2.5 (iii) to hold $\mathbb{P}^\mu$-a.s. conditional on X hitting A. Consider, for example, the case where A is the disjoint union of two sets B and C with $D_B<D_C$. Then $\mathbb{P}^\mu$-a.s. if X hits B but not C, by Theorem 2.5 (i)*

$$\dim_H\mathcal{T}(A)=\dim_H\mathcal{T}(B)\leq D_B<D_C=D_A.$$

In the case where A is a singleton, we can give an improved version of Theorem 2.5. For $x_0\in E$, we write $\mathcal{T}(x_0):=\mathcal{T}(\{x_0\})$, and say that X *spends time at* x_0 if there are $t>\delta>0$ with $L_\delta(t,x_0)>0$.

Theorem 2.7. *Let $\mu\in\mathcal{M}_F(E)$, $x_0\in E$. Then, $\mathbb{P}^\mu$-a.s., if X spends time at x_0,*

$$\dim_H\mathcal{T}(x_0) = 1\wedge\left(2-\frac{d_s}{2}\right)_+.$$

The reason why we have to assume that X spends time at x_0 (instead of the more natural assumption $\mathcal{T}(x_0)\neq\varnothing$) is that our proof requires the local time at x_0, which is supported on $\mathcal{T}(x_0)$, to be non-constant. We were not able to prove that this is always the case when $\mathcal{T}(x_0)\neq\varnothing$.

The next result is on the range of X. Write $\mathcal{R}(I):=\bigcup_{t\in I}\mathrm{S}(X_t)$ for $I\subset(0,\infty)$.

Theorem 2.8. *If $\mu \in \mathcal{M}_F(E)$, then $\mathbb{P}^\mu$-a.s. for every $I \subset (0,\infty)$,*

$$\dim_H \mathcal{R}(I) \leq \beta(1 + \dim_H I) \wedge d_f \tag{2.4}$$

and $\dim_H \mathrm{S}(X_t) \geq \beta \wedge d_f$ for all $t > 0$ with $X_t \neq 0$. In particular, $\mathbb{P}^\mu$-a.s. $\dim_H \mathrm{S}(X_t) = \beta \wedge d_f$ for all $t > 0$ with $X_t \neq 0$.

Remark 2.9. *A lower bound corresponding to* (2.4) *was proved for superBrownian motion by Serlet [34, Theorem 3] and Tribe [36, Theorem 2.13], and we believe that a proof along the lines of Tribe's works in our setting as well. However, an additional precise result about superBrownian motion would need to be reproved in this setting, see (2.39) in [36], proved by Perkins in [31, Proof of Theorem 4.5]. This is an interesting direction for future work, but we felt it would go beyond the scope of the present paper.*

Our last result is a small ball bound that generalises a result of Perkins on superBrownian motion [33, Theorem III.3.4].

Theorem 2.10. *There exists $c_6 > 0$ (depending only on the constants in our assumptions on ν and (p_t)) such that, for any $\mu \in \mathcal{M}_F(E)$, $\mathbb{P}^\mu$-a.s., for every $\delta > 0$ there exists an $r_1 = r_1(\delta, \omega) \in (0,1)$ with*

$$\sup_{\substack{t \geq \delta \\ x \in E}} X_t(B(x,r)) \leq \gamma_b c_6 \xi(r) \tag{2.5}$$

for all $0 < r < r_1$, where

$$\xi(r) = \xi_{d_f,\beta}(r) = \begin{cases} r^\beta \left(\log \frac{1}{r}\right)^{1+\mathbb{1}_{\{d_s=2\}}} & , d_s \geq 2, \\ r^{d_f} \left(\log \frac{1}{r}\right)^{2-d_s/2} & , d_s < 2. \end{cases} \tag{2.6}$$

This bound is sharp in the sense that, if Theorem 2.10 were true with $\xi(r) = O(r^a)$ for some $a > d_f \wedge \beta$, then $\dim_H \mathrm{S}(X_t) \geq a > \beta \wedge d_f$ [19, page 61], which contradicts (2.4).

Remark 2.11. *It is known that superBrownian motion in one dimension has a density w.r.t. Lebesgue measure, and, in $d \geq 2$, is at fixed times a deterministic multiple of the Hausdorff measure with dimension function h_d satisfying*

$$h_d(r) = \begin{cases} r^2 (\log 1/r)(\log\log\log 1/r) & , d = 2, \\ r^2 \log\log 1/r & , d \geq 3, \end{cases}$$

for small $r > 0$. In light of Theorems 2.8 and 2.10, this suggests the conjecture that in our setting, perhaps under some additional assumptions, X has a density w.r.t. ν if $d_s < 2$, and is a multiple of the Hausdorff measure with some dimension function having leading order r^β if $d_s \geq 2$.

3. Historical Process and Small Ball Bound

We begin this section by introducing the historical process associated to our superdiffusion. It was first introduced by Dawson and Perkins [8], and has since been an indispensable tool in the study of path properties of superprocesses. The idea, heuristically, is that if we think of points in the support of X_t as individuals alive at time t, then a point in the support of the associated historical process is the *entire trajectory* of an individual alive at time t. This allows us to apply path properties of the underlying spatial motion directly to the superprocess – leading e.g. to a modulus of continuity, see Theorem 3.3 – and to make precise notions of ancestry, see Theorem 3.1 below.

We follow the notation of [33]. Denote by Y the canonical process on the space $C := C(E)$ of continuous paths $[0,\infty) \to E$ with the topology of uniform convergence on compacts, so that $\mathbb{P}^x$ is a member of the space $\mathcal{M}_1(C)$ of Borel probability measures on C, equipped with the topology of weak convergence. The historical process at time t will be a member of $\mathcal{M}_F^t(C) := \{\mu \in \mathcal{M}_F(C) \colon \mu(C^t) = \mu(C)\}$ for $t \geq 0$, where $y^t := y(t \wedge \cdot)$ for $y \in C$ denotes a path stopped at time t, and $C^t := \{y \in C \colon y^t = y\}$ is the set of paths that are constant on $[t,\infty)$. In words, $\mathcal{M}_F^t(C)$ is the set of probability measures on C whose support is comprised of paths that evolve only until time t. The historical process will be a random element of $\Omega_H := \Omega_H[0,\infty)$, where

$$\Omega_H[s,\infty) := \left\{G \in C([s,\infty), \mathcal{M}_F(C)) \colon \forall t \geq s \colon G_t \in \mathcal{M}_F^t(C)\right\}, \quad s \geq 0,$$

which are closed subsets of $C([s,\infty),\mathcal{M}_F(C))$, $s \geq 0$, and thus Polish. We now let $H = (H_t)_{t\geq 0} := \mathrm{id}_{\Omega_H}$ be the canonical process on Ω_H. Denote by $\mathcal{F}^H$ the Borel σ-algebra on Ω_H, and write $\mathcal{F}^H_{s,t} := \bigcap_{\varepsilon>0}\sigma(H_r\colon s\leq r\leq t+\varepsilon)$ for $0\leq s\leq t\leq\infty$. Denote by

$$\left(\mathbb{Q}_{s,m}\in\mathcal{M}_1(\Omega_H[s,\infty))\colon s\geq 0, m\in\mathcal{M}^s_F(C)\right)$$

the family of laws that are the unique solutions to the historical martingale problem associated with X [33, Theorem II.8.3]. This solution exists under the assumption that the map $E\to\mathcal{M}_1(C(E))$; $x\mapsto\mathbb{P}^x$ is continuous, which is the case here, as we prove in Lemma B.2 in the appendix. In particular, if $m\in\mathcal{M}^0_F(C)$, then $(\pi_t(H_t))_{t\geq 0}$ w.r.t. $\mathbb{Q}_{0,m}$ is a (Y,γ_b)-superprocess started at $\pi_0(m)$, where $\pi_t\colon C\to E$; $y\mapsto y(t)$, and $f(\mu)$ denotes the pushforward of a measure μ under a measurable map f. Note that m can be chosen so that $\pi_0(m)=\mu$, for any given $\mu\in\mathcal{M}_F(E)$.

The following fundamental result heuristically states that for fixed $0<s<t$, all individuals alive at time t descend from a *finite* number of individuals that were alive at time s, drawn as a Poisson point process with intensity proportional to $\frac{1}{t-s}X_s$. It follows from [33, Theorem III.1.1]. Denote for $x\in E$ by $\delta_x\in\mathcal{M}_F(E)$ the point mass at x.

Theorem 3.1 (Historical Cluster Representation)**.** *For $t>s$ fixed, there exist measurable maps $M\colon\Omega_H\to\mathbb{N}_0$, $e_i\colon\Omega_H\to(0,\infty)$, and $y_i\colon\Omega_H\to C$, $i\in\mathbb{N}$, such that, for any $0\leq u\leq s$ and $m\in\mathcal{M}^u_F(C)$,*

$$r_s(H_t)(\cdot) := H_t(y\colon y^s\in\cdot) = \sum_{i=1}^{M} e_i\delta_{y_i}(\cdot), \quad \mathbb{Q}_{u,m}\text{-a.s.},$$

and a conditional distribution of $((e_i,y_i))_{i=1}^M$, given $\mathcal{F}^H_{u,s}$, is that of a Poisson point process on $(0,\infty)\times C$ with intensity measure $\frac{2}{\gamma_b(t-s)}\left(Exp\left(\frac{2}{\gamma_b(t-s)}\right)\otimes H_s\right)$.

The next result is a modulus of continuity for the trajectories in the historical process. It follows from the following tail bound for the spatial motion.

Lemma 3.2. *There are $c_7, c_8>0$ such that, for $x\in E$, $r>0$, and $t\in(0,t_0)$,*

$$\mathbb{P}^x(\rho(x,Y_t)>r)\leq c_7\exp\left(-c_8 r^{\beta\gamma}t^{-\gamma}\right). \tag{3.1}$$

Proof. Follows from (1.1) and (1.2), see [3, Lemma 3.9]. □

Write, for $r>0$ and $\delta, c>0$,

$$\begin{aligned} h(r) &:= h_{\beta,\gamma}(r) := r^{1/\beta}\log\left(\frac{1}{r}\right)^{1/(\beta\gamma)}, \\ K^{\delta,c} &:= \{y\in C\colon \forall s,t\geq 0, |t-s|<\delta\colon \rho(y(s),y(t))<ch(|t-s|)\}. \end{aligned} \tag{3.2}$$

Theorem 3.3 (Historical Modulus of Continuity)**.** *For any $c>0$ sufficiently large, and any $m\in\mathcal{M}^0_F(C)$, there is $c_9>0$ such that, $\mathbb{Q}_{0,m}$-a.s., there exists $\delta=\delta(c,\omega)>0$ with*

$$\mathrm{S}(H_t)\subset K^{\delta,c}, \quad t\geq 0.$$

Proof. This follows by replacing the Gaussian tail estimate in the proof of Theorem III.1.3 in [33] by (3.1). □

The following assertions are consequences of Theorem 3.3, and can be obtained through the same line of argument presented in [33] following Theorem III.1.3. Note that the arguments regarding compactness in $\mathbb{R}^d$ employed in [33] work here as well because every bounded subset of E is relatively compact (see Lemma 3.9). Write $\rho(\cdot,A) := \inf\{\rho(\cdot,z)\colon z\in A\}$, and

$$A^\varepsilon := \{x\in E\colon \rho(x,A)<\varepsilon\}, \tag{3.3}$$

for $x\in E$, $A\subset E$, $\varepsilon>0$.

Corollary 3.4. *Let $m\in\mathcal{M}^0_F(C)$ and $\mu\in\mathcal{M}_F(E)$.*

(i) $\mathbb{Q}_{0,m}$*-a.s.,* $\mathrm{S}(H_t)$ *is compact and* $\mathrm{S}(\pi_t(H_t)) = \pi_t(\mathrm{S}(H_t))$ *for all* $t > 0$*. In particular,* $\mathrm{S}(X_t)$ *is compact for all* $t > 0$*,* $\mathbb{P}^\mu$*-a.s.*

(ii) *For* $c > 0$ *sufficiently large (independent of* μ*), there* $\mathbb{P}^\mu$*-a.s. exists* $\delta = \delta(c, \omega) > 0$ *such that, if* $0 < t - s < \delta$*, then* $\mathrm{S}(X_t) \subset \mathrm{S}(X_s)^{ch(t-s)}$*.*

(iii) $\mathcal{R}([\delta, \infty)) = \bigcup_{t \geq \delta} \mathrm{S}(X_t)$ *is compact for all* $\delta > 0$*,* $\mathbb{P}^\mu$*-a.s.*

Remark 3.5. *The only assumption on Y necessary to establish Theorem 3.3 and Corollary 3.4 (beyond being a continuous Feller Markov process) is the assertion of Lemma 3.2.*

We now prove the small ball bound Theorem 2.10, closely following the proof of Theorem III.3.4 in [33]. The generalisation to this setting is not entirely trivial – amongst other things due to the fact that the bound in Lemma 3.8 below only holds up to a possibly finite time t_0 – so we give the proof in full. It is technical and need not to be read in detail to understand the subsequent sections.

For $f \geq 0$ measurable and bounded, and $t \geq 0$, define $G(f, t) := \int_0^t \|P_s f\|_\infty \, \mathrm{d}s$, where $\|\cdot\|_\infty$ is the supremum norm. Denote by $A \colon \mathcal{D}(A) \subset C_0(E) \to C_0(E)$ the generator of Y.

Lemma 3.6. *If* $f \geq 0$ *is measurable and bounded, and* $t \geq 0$ *such that* $G(f, t) < 2/\gamma_b$*, and* $P_s f \in \mathcal{D}(A)$ *for all* $s > 0$*, then, for any* $\mu \in \mathcal{M}_F(E)$*,*

$$\mathbb{E}^\mu\left[\mathrm{e}^{X_t(f)}\right] \leq \exp\left(\mu(P_t f)\left(1 - \frac{\gamma_b}{2} G(f, t)\right)^{-1}\right).$$

Proof. Under the assumption that $P_s f \in \mathcal{D}(A)$ for all $s > 0$, the proof of Lemma III.3.6 in [33] works in this setting without modification. □

Corollary 3.7. *If* (Z_t) *is a Feller diffusion with branching rate* γ_b*, started at* $z > 0$*, then, for* $\lambda < 2/\gamma_b$ *and* $t > 0$*,*

$$\mathbb{E}\left[\mathrm{e}^{\lambda Z_t}\right] \leq \exp\left(\frac{2z\lambda}{2 - \gamma_b \lambda t}\right).$$

Proof. Apply Lemma 3.6 with $f \equiv \lambda$ and recall that $(X_t(1))$, under $\mathbb{P}^\mu, \mu \in \mathcal{M}_F(E)$, is a Feller diffusion with branching rate γ_b started at $\mu(1)$. □

The following lemmas are easy consequences of (1.1) to (1.3).

Lemma 3.8. *There is* $c_{10} > 0$ *such that, if* $d_f \geq \beta$*,*

$$G(\mathbb{1}_{B(x_0,r)}, t) \leq c_{10} r^\beta \left[1 + \mathbb{1}_{\{d_f = \beta\}} \log\left(\frac{t}{r^\beta}\right)\right], \quad r \in (0, \mathrm{diam}(E)), t \in (0, t_0).$$

If $d_f < \beta$*, then* $G(\mathbb{1}_{B(x_0,r)}, t) \leq c_{10} r^{d_f} t^{1 - d_f/\beta}$*.*

Lemma 3.9. *There is a* $c_{11} > 0$ *such that, for every* $x_0 \in E$*,* $r > 0$*, and* $\varepsilon < r \wedge r_0$*,* $B(x_0, r)$ *can be covered by at most* $c_{11}(\varepsilon/r)^{-d_f}$ *balls of radius* ε*.*

We close the section with the proof of Theorem 2.10. Recall that $d_s = 2d_f/\beta$. Although the statement of Theorem 2.10 is in terms of d_s, for clarity we will use d_f and β in the proof.

Proof of Theorem 2.10. Fix $\mu \in \mathcal{M}_F(E)$. Let $m \in \mathcal{M}_F^0(E)$ with $\pi_0(m) = \mu$, so that the distribution of (X_t) under $\mathbb{P}^\mu$ is the same as that of $(\pi_t(H_t))$ under $\mathbb{Q}_{0,m}$. We work with the latter, and slightly abuse notation by writing $X_t := \pi_t(H_t)$ for the duration of the proof.

The left-hand side (LHS) in the claim is increasing as $\delta \downarrow 0$, so we may prove it for fixed $\delta \in (0, t_0 \wedge 1)$ with $\delta 2^n \in \mathbb{N}$ for some $n \in \mathbb{N}$. By Corollary 3.4 (iii), it suffices to prove the claim with $\sup_{t \geq \delta, x \in E}$ on the LHS of (2.5) replaced by $\sup_{t \geq \delta, x \in A}$, where $A = B(x_0, N)$ for some $x_0 \in E$ and fixed but arbitrary $N \in \mathbb{N}$. By Lemma 3.9, for $n \in \mathbb{N}$ with $2^{-n/\beta} < r_0$ there are $(x_i)_{i \in I_n}$ such that $(B(x_i, 2^{-n/\beta}))_{i \in I_n}$ cover A, and $|I_n| \leq c_{11} N^{d_f} 2^{nd_f/\beta}$.

Let $c > 0$ be such that $\mathbb{Q}_{0,m}$-a.s. existence of $\delta' = \delta'(c,\omega) > 0$ according to Theorem 3.3 is ensured. If $x \in A$ and $t > 0$, then $x \in B(x_i, 2^{-n/\beta})$ for some $i \in I_n$ and $j2^{-n} < t \le (j+1)2^{-n}$ for some $j \in \mathbb{N}_0$. Hence, if $2^{-n} < \delta'$, then for H_t-almost all $y \in C(E)$, $y(t) \in B(x, h(2^{-n}))$ implies

$$\begin{aligned} \left|y(j2^{-n}) - x_i\right| &\le \left|y(j2^{-n}) - y(t)\right| + |y(t) - x| + |x - x_i| \\ &\le ch(2^{-n}) + h(2^{-n}) + 2^{-n/\beta} \\ &\le (2+c)h(2^{-n}). \end{aligned}$$

Thus, putting $B_{in} := B(x_i, (2+c)h(2^{-n}))$,

$$\begin{aligned} X_t(B(x, h(2^{-n}))) &= H_t(y\colon y(t) \in B(x, h(2^{-n}))) \\ &\le H_t(y\colon y(j2^{-n}) \in B_{in})) \\ &= M_{j2^{-n},B_{in}}(t - j2^{-n}), \end{aligned}$$

where, for $\tau \ge 0$, and a Borel set $B \subset E$,

$$M_{\tau,B}(t) := H_{\tau+t}(y\colon y(\tau) \in B), \quad t > 0.$$

Hence, for $j \in \mathbb{N}$ and if $2^{-n} < \delta'$,

$$\sup_{\substack{j2^{-n} < t \le (j+1)2^{-n} \\ x \in A}} X_t(B(x, h(2^{-n}))) \le \sup_{B \in \mathcal{B}_n} \sup_{t \le 2^{-n}} M_{j2^{-n},B}(t),$$

where

$$\mathcal{B}_n := \{B_{in}\colon i \in I_n\} = \{B(x_i, (2+c)h(2^{-n}))\colon i \in I_n\}.$$

$M_{\tau,B}$ is a Feller diffusion under $\mathbb{Q}_{0,m}$ with (random) starting point $H_\tau(y\colon y(\tau) \in B) = X_\tau(B)$ (which follows from [33, Equation (III.1.3)], say, and the strong Markov property of H [33, Section II.8]), in particular a non-negative martingale. Hence for any $\lambda_n, \varepsilon_n > 0$ and for sufficiently large n (so that $\delta 2^n \in \mathbb{N}$, $2^{-n} < \delta'$, and $2^{-n/\beta} < r_0$), say $n \ge n_0$, and for $b \in (0,1)$ which we choose later,

$$\begin{aligned} \mathbb{P}^\mu\Bigg(\sup_{\substack{\delta < t < \lfloor n^b \rfloor \\ x \in A}} X_t(B(x, h(2^{-n}))) \ge \varepsilon_n\Bigg) &\le \mathbb{Q}_{0,m}\Bigg(\sup_{\delta 2^n < j \le n^b 2^n} \sup_{B \in \mathcal{B}_n} \sup_{t \le 2^{-n}} M_{j2^{-n},B}(t) \ge \varepsilon_n\Bigg) \\ &\le n^b 2^n\, |\mathcal{B}_n| \sup_{\delta 2^n < j \le n^b 2^n} \sup_{B \in \mathcal{B}_n} \mathbb{Q}_{0,m}\Bigg(\sup_{t \le 2^{-n}} M_{j2^{-n},B}(t) \ge \varepsilon_n\Bigg) \\ &\le n^b 2^n \cdot c_{11} N^{d_f} 2^{n d_f/\beta} \cdot \mathrm{e}^{-\lambda_n \varepsilon_n} \sup_{\delta 2^n < j \le n^b 2^n} \sup_{B \in \mathcal{B}_n} \mathbb{Q}_{0,m}\left[\mathrm{e}^{\lambda_n M_{j2^{-n},B}(2^{-n})}\right], \end{aligned} \tag{3.4}$$

where in the last step we applied Doob's inequality to the submartingale $\left(\exp\left(\lambda_n M_{j2^{-n},B}(t)\right)\right)_{t \ge 0}$. For a constant $c_{12} > 0$ which we choose sufficiently large later, put

$$\lambda_n := \frac{1}{2\gamma_b c_{12} n^a} \begin{cases} 2^n & , d_f \ge \beta, \\ 2^{n d_f/\beta} & , d_f < \beta, \end{cases} \tag{3.5}$$

$$\text{where} \quad a := \begin{cases} 1/\gamma + \mathbb{1}_{\{d_f = \beta\}} & , d_f \ge \beta, \\ 1 - d_f/\beta + d_f/(\gamma\beta) & , d_f < \beta. \end{cases}$$

We claim that, if $c_{12} > 0$ is large enough, then there is an $n_1 \in \mathbb{N}$ (which we may choose so that $n_1 \ge n_0$) with

$$c_{13} := \sup_{n \ge n_1} \sup_{\delta 2^n < j \le n^b 2^n} \sup_{B \in \mathcal{B}_n} \mathbb{Q}_{0,m}\left[\mathrm{e}^{\lambda_n M_{j2^{-n},B}(2^{-n})}\right] < \infty. \tag{3.6}$$

Assuming this for the moment, we now choose

$$\varepsilon_n := c_{14}n/\lambda_n = 2c_{14}c_{12}\gamma_b n^{1+a}\begin{cases}2^{-n} & , d_f \geq \beta,\\ 2^{-nd_f/\beta} & , d_f < \beta,\end{cases} \tag{3.7}$$

for some $c_{14} > 0$, so that, by (3.4) and (3.6), for $n \geq n_1$,

$$\begin{aligned}\mathbb{P}^\mu\Bigg(\sup_{\substack{\delta<t<\lfloor n^b\rfloor\\ x\in A}} X_t(B(x,h(2^{-n}))) \geq \varepsilon_n\Bigg) &\leq c_{11}c_{13}N^{d_f}n^b2^{n(1+d_f/\beta)}\mathrm{e}^{-\lambda_n\varepsilon_n}\\ &\leq c_{11}c_{13}N^{d_f}n^b2^{n(1+d_f/\beta)}\mathrm{e}^{-c_{14}n},\end{aligned}$$

which is summable if we choose $c_{14} > 0$ large enough. In that case, by Borel-Cantelli, there $\mathbb{Q}_{0,m}$-a.s. exists $n_2 = n_2(\omega) \in \mathbb{N}$ such that, for all $n \geq n_2$,

$$\sup_{\substack{\delta<t<n^b\\ x\in A}} X_t(B(x,h(2^{-n}))) < \varepsilon_n \leq \gamma_b c_6 \xi(h(2^{-n-1})),$$

for some $c_6 > 0$ that does not depend on ω, δ, or m. To confirm that the second inequality holds, recall the definitions of ε_n, h, and ξ respectively from (3.7), (3.2), and (2.6), and that $d_s = 2d_f/\beta$. By making n_2 larger, we may assume that $X_t \equiv 0$ for all $t \geq n_2^b$, and that h is increasing on $(0, 2^{-n_2}]$. Then, for any $0 < r < r_1 := h(2^{-n_2})$, if $n \geq n_2$ is such that $h(2^{-n-1}) \leq r < h(2^{-n})$,

$$\sup_{\substack{\delta<t\\ x\in A}} X_t(B(x,r)) \leq \sup_{\substack{\delta<t<n^b\\ x\in A}} X_t(B(x,h(2^{-n}))) \leq \gamma_b c_6\xi(h(2^{-n-1})) \leq \gamma_b c_6 \xi(r).$$

It remains to prove the claim, that is, that the constant c_{12} in the definition of λ_n, see (3.5), can be chosen large enough so that there is $n_1 \in \mathbb{N}$ satisfying (3.6). Fix $n \in \mathbb{N}$, $B \in \mathscr{B}_n$, and $j \in \mathbb{N}$ with $\delta 2^n < j \leq n^b 2^n$, and put $\tau := j2^{-n} \in (\delta, n^b]$. Recall Corollary 3.7 and that $(M_{\tau,B}(t))_{t\geq 0}$, under $\mathbb{Q}_{0,m}$, is a Feller diffusion started at $X_\tau(B)$ with branching rate γ_b, so the expectation inside the suprema in (3.6) is

$$\begin{aligned}\mathbb{Q}_{0,m}\left[\mathbb{Q}_{0,m}\left[\mathrm{e}^{\lambda_n M_{\tau,B}(2^{-n})}\,\middle|\,\mathcal{F}_\tau^H\right]\right] &\leq \mathbb{Q}_{0,m}\left[\exp\left(2X_\tau(B)\lambda_n\left(2-\gamma_b\lambda_n 2^{-n}\right)^{-1}\right)\right]\\ &\leq \mathbb{E}^\mu\left[\exp\left(2\lambda_n X_\tau(B)\right)\right],\end{aligned} \tag{3.8}$$

for n so large that $\gamma_b\lambda_n 2^{-n} < 1$. We would now like to apply Lemma 3.6 with $f = 2\lambda_n \mathbb{1}_B$, but we do not know that $P_s\mathbb{1}_B \in \mathcal{D}(A)$. However, $\mathcal{D}(A)_+$ is dense in $C_0(E)_+$ w.r.t. $\|\cdot\|_\infty$ [12, Theorems 4.9 and 4.23], so there is an $f = f_{n,B} \in \mathcal{D}(A)_+$ with $\mathbb{1}_B \leq f \leq 2\mathbb{1}_{2B} + \eta_n$, where $2B$ denotes the ball B with doubled radius, and $\eta_n := 1/(4\gamma_b n\lambda_n)$. Take $c > 4$, so that the radius of $2B$ is at most $2(2+c)h(2^{-n}) < 3ch(2^{-n})$. Then, by Lemma 3.8, there exists $c_{10} > 0$ such that, if $d_f \geq \beta$,

$$\begin{aligned}G(\mathbb{1}_{2B},\delta) &\leq G(\mathbb{1}_{2B}, t_0\wedge n)\\ &\leq c_{10}(3ch(2^{-n}))^\beta\left[1+\mathbb{1}_{\{d_f=\beta\}}\log\left(\frac{t_0\wedge n}{(3ch(2^{-n}))^\beta}\right)\right]\\ &\leq c_{10}(3c)^\beta 2^{-n}(n\log 2)^{1/\gamma}\left[1+\mathbb{1}_{\{d_f=\beta\}}\log\left(\frac{n2^n}{(3c)^\beta(n\log 2)^{1/\gamma}}\right)\right]\\ &\leq c_{10}' 2^{-n}n^a,\end{aligned}$$

for some $c_{10}' > 0$. If $d_f < \beta$, then

$$\begin{aligned}G(\mathbb{1}_{2B},\delta) &\leq G(\mathbb{1}_{2B}, t_0\wedge n)\\ &\leq c_{10}(3ch(2^{-n}))^{d_f}(t_0\wedge n)^{1-d_f/\beta}\\ &\leq c_{10}(3c)^{d_f}2^{-nd_f/\beta}(n\log 2)^{d_f/(\beta\gamma)}n^{1-d_f/\beta}\\ &\leq c_{10}'' 2^{-nd_f/\beta}n^a,\end{aligned}$$

for some $c''_{10} > 0$. Thus if $c_{12} \geq 4(c'_{10} \vee c''_{10})$ (recall we are proving that (3.6) holds for some $n_1 \in \mathbb{N}$ given that c_{12} is sufficiently large), then

$$G(2\lambda_n f, \delta) \leq 2\lambda_n\Big(2G(\mathbb{1}_{2B}, \delta) + \eta_n\delta\Big) \leq 2\lambda_n\left(\frac{1}{4\gamma_b\lambda_n} + \frac{1}{4\gamma_b n\lambda_n}\right) \leq 1/\gamma_b < 2/\gamma_b,$$

and Lemma 3.6 implies

$$\begin{aligned}\mathbb{E}^\mu\left[\exp\left(2\lambda_n X_\tau(B)\right)\right] &\leq \mathbb{E}^\mu\left[\exp\left(2\lambda_n X_\tau(f)\right)\right]\\ &= \mathbb{E}^\mu\left[\mathbb{E}^{X_{\tau-\delta}}\left[\exp\left(2\lambda_n X_\delta(f)\right)\right]\right]\\ &\leq \mathbb{E}^\mu\left[\exp\left(2\lambda_n X_{\tau-\delta}(P_\delta f)(1-\frac{\gamma_b}{2}G(2\lambda_n f,\delta))^{-1}\right)\right]\\ &\leq \mathbb{E}^\mu\left[\exp\left(4\lambda_n \|P_\delta f\|_\infty X_{\tau-\delta}(1)\right)\right].\end{aligned} \tag{3.9}$$

Note that we could not have applied Lemma 3.6 directly to $\mathbb{E}^\mu\left[\exp\left(2\lambda_n X_\tau(f)\right)\right]$ because the bound on $G(2\lambda_n f, \tau)$ obtained from Lemma 3.8 is only valid for $\tau < t_0$, and τ is up to n^b. Now note that $\|P_\delta f\|_\infty \leq 2\|P_\delta \mathbb{1}_{2B}\|_\infty + \eta_n$, and

$$\|P_\delta \mathbb{1}_{2B}\|_\infty = \sup_{x\in E}\int_{2B} p_\delta(x,y)\nu(\mathrm{d}y) \leq \|p_\delta\|_\infty\, \nu(2B) \leq c_2(3ch(2^{-n}))^{d_f}\,\|p_\delta\|_\infty\,,$$

and, putting $c_{15} := \frac{(\log 2)^{d_f/(\beta\gamma)}}{2\gamma_b c_{12}}$,

$$\lambda_n h(2^{-n})^{d_f} = c_{15}2^{-nd_f/\beta}n^{d_f/(\beta\gamma)-a}\begin{cases}2^n & ,d_f \geq \beta,\\ 2^{nd_f/\beta} & ,d_f < \beta,\end{cases} = \begin{cases}O(2^{n(1-d_f/\beta)}) & ,d_f > \beta,\\ O(n^{-1}) & ,d_f = \beta,\\ O(n^{-1+d_f/\beta}) & ,d_f < \beta.\end{cases}$$

Furthermore, recall that $\lambda_n\eta_n = 1/(4\gamma_b n)$, so $\lambda_n\|P_\delta f\|_\infty \leq c_{16}n^{-b'}$ for all $n \in \mathbb{N}$ for some $b' > 0$ and $c_{16} = c_{16}(\delta) > 0$. Now choose $b = b'/2$, and recall that $\tau \in (\delta, n^b]$. Then, and by Corollary 3.7,

$$\begin{aligned}\mathbb{E}^\mu\left[\exp\left(4\lambda_n\|P_\delta f\|_\infty X_{\tau-\delta}(1)\right)\right] &\leq \exp\left(\frac{8\mu(1)\lambda_n\|P_\delta f\|_\infty}{2-\gamma_b(\tau-\delta)4\lambda_n\|P_\delta f\|_\infty}\right)\\ &\leq \exp\left(\frac{4c_{16}\mu(1)n^{-b'}}{1-2c_{16}\gamma_b n^{-b'/2)}}\right),\end{aligned}$$

given that $2c_{16}\gamma_b n^{-b'/2} < 1$. This bound is independent of j and B, and no more than, say, exp(1), for all $n \geq n_1$ for some $n_1 = n_1(\delta) \in \mathbb{N}$. Recalling (3.8) and (3.9), this finishes the proof. □

4. Local Times

Fix $\delta > 0$ and $\mu \in \mathcal{M}_F(E)$ throughout this section.

4.1. Existence of Local Times

For the proof of existence of the family $(L_\delta(t,x))$, we follow the approach used by Sugitani [35] in the superBrownian case. For $h > 0$, $t \geq \delta$, $x \in E$, put

$$\begin{aligned}L^h(t,x) &:= L^h_\delta(t,x) := \int_\delta^t X_s(p_h(x,\cdot))\,\mathrm{d}s,\\ \widetilde{L}^h(t,x) &:= \widetilde{L}^h_{\delta,\mu}(t,x) := L^h(t,x) - \mathbb{E}^\mu\left[L^h(t,x)\right].\end{aligned}$$

For $0 \leq s \leq t$, and $x \in E$, write $p^x_s := p_s(x,\cdot)$, $q^x_s := q_s(x,\cdot)$, where $q_{s,t}(x,\cdot) := \int_s^t p^x_r(\cdot)\,\mathrm{d}r$, and $q_s := q_{0,s}$. Note that $P_s p^x_r = p^x_{s+r}$ and thus $P_s q^x_{a,b} = q^x_{a+s,b+s}$ for $r, s > 0$, $0 \leq a < b$, $x \in E$, by (1.4). Write $(\mu f)(x) := \mu(f(x,\cdot))$ for $f\colon E\times E \to [0,\infty]$ measurable.

Proposition 4.1. *For* $0 < a \le b$, $\left\|(\mu q_{a,b})\right\|_\infty < \infty$, *and*

$$\sup_{t\ge\delta, x\in E} \left|\mathbb{E}^\mu\left[L^h(t,x)\right] - (\mu q_{\delta,t})(x)\right| \longrightarrow 0, \quad h \to 0.$$

Proof. Recall that $\|p_s\|_\infty \le \left\|p_{s\wedge t_0}\right\|_\infty \le c_3(s\wedge t_0)^{-d_f/\beta}$ for all $s > 0$ by (1.4) and (1.2). Hence, $\left\|(\mu q_{a,b})\right\|_\infty \le \mu(1)\left\|q_{a,b}\right\|_\infty \le \mu(1)(b-a)\sup_{t\ge a}\|p_t\|_\infty < \infty$ for all $0 < a \le b$.

Fix $t \ge \delta$ and $x \in E$, and recall that $\mathbb{E}^\mu\left[X_t(f)\right] = \mu(P_t f)$ for $t \ge 0$ and $f \in B_+(E) \cup B_b(E)$ (see (A.1)), where $B(E)$ denotes Borel functions on E and subscripts $+$ and b denote non-negative and bounded functions respectively. Then,

$$\mathbb{E}^\mu\left[L^h(t,x)\right] = \int_\delta^t \mathbb{E}^\mu\left[X_s(p_h^x)\right]\mathrm{d}s = \int_\delta^t \mu(P_s p_h^x)\,\mathrm{d}s = \int_\delta^t \mu(p_{s+h}^x)\,\mathrm{d}s = \mu(q_{\delta+h,t+h}^x) = (\mu q_{\delta+h,t+h})(x),$$

where we used Tonelli's theorem. Thus,

$$\begin{aligned}\left|\mathbb{E}^\mu\left[L^h(t,x)\right] - (\mu q_{\delta,t})(x)\right| &= \left|(\mu q_{\delta+h,t+h})(x) - (\mu q_{\delta,t})(x)\right| \\ &= \left|(\mu q_{t,t+h})(x) - (\mu q_{\delta,\delta+h})(x)\right| \\ &\le \left\|(\mu q_{t,t+h})\right\|_\infty + \left\|(\mu q_{\delta,\delta+h})\right\|_\infty.\end{aligned}$$

For any $s \ge \delta$, $x \in E$, and $h > 0$,

$$(\mu q_{s,s+h})(x) = \int_s^{s+h}\int_E p_r(x,y)\mu(\mathrm{d}y)\,\mathrm{d}r \le \mu(1)\int_s^{s+h}\|p_r\|_\infty\,\mathrm{d}r \le c_{17}h,$$

where $c_{17} = \mu(1)\sup_{s\ge\delta}\|p_s\|_\infty < \infty$, so

$$\left|\mathbb{E}^\mu\left[L^h(t,x)\right] - (\mu q_{\delta,t})(x)\right| \le 2c_{17}h$$

is a vanishing bound that is valid for all $t \ge \delta$ and $x \in E$. □

Proposition 4.2. *If* $d_s < 4$, *then there exists a family* $(\widetilde{L}(t,x))_{t\ge\delta,x\in E}$ *of random variables such that*

$$\lim_{h\to 0}\sup_{\substack{x\in E\\ \delta\le t\le T}} \mathbb{E}^\mu\left[\left|\widetilde{L}^h(t,x) - \widetilde{L}(t,x)\right|^2\right] = 0, \quad T > 0.$$

Proof. Fix $T > \delta$, $t \in [\delta, T]$, and $x \in E$. We start by showing that $(\widetilde{L}^h(t,x))_{h>0}$ is L^2-Cauchy w.r.t. $\mathbb{P}^\mu$. Let $h, h' > 0$, say $h' \le h$. Put

$$Z := L^h(t,x) - L^{h'}(t,x) = \int_\delta^t X_s(\underbrace{p_h^x - p_{h'}^x}_{=:f})\,\mathrm{d}s = \int_\delta^t X_s(f)\,\mathrm{d}s,$$

where we temporarily suppress h and h' to lighten notation, so $\widetilde{L}^h(t,x) - \widetilde{L}^{h'}(t,x) = Z - \mathbb{E}^\mu[Z]$, and by the second moment formula for superprocesses (A.2),

$$\begin{aligned}\mathbb{E}^\mu\left[\left|\widetilde{L}^h(t,x) - \widetilde{L}^{h'}(t,x)\right|^2\right] &= \mathbb{E}^\mu\left[Z^2\right] - \mathbb{E}^\mu[Z]^2 \\ &= \int_\delta^t\int_\delta^t \Big(\mathbb{E}^\mu\left[X_s(f)X_{s'}(f)\right] - \mathbb{E}^\mu\left[X_s(f)\right]\mathbb{E}^\mu\left[X_{s'}(f)\right]\Big)\mathrm{d}s\,\mathrm{d}s' \\ &= \gamma_b\int_\delta^t\int_\delta^t\int_0^{s\wedge s'} \mu\Big(P_r\left((P_{s-r}f)(P_{s'-r}f)\right)\Big)\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}s' \\ &= \gamma_b\int_0^t\int_{r\vee\delta}^t\int_{r\vee\delta}^t \mu\Big(P_r\left((P_{s-r}f)(P_{s'-r}f)\right)\Big)\mathrm{d}s\,\mathrm{d}s'\,\mathrm{d}r \\ &= \gamma_b\int_0^t \mu\left(P_r\left(\left(\int_{0\vee(\delta-r)}^{t-r} P_s f\,\mathrm{d}s\right)^2\right)\right)\mathrm{d}r,\end{aligned} \tag{4.1}$$

where the changes in integration order are warranted by Tonelli's theorem. For $r \geq 0$, since $h' \leq h$, putting $a :=$ $0 \vee (\delta - r)$ and $b := t - r$ for the moment,

$$
\begin{aligned}
\int_{0\vee(\delta-r)}^{t-r} P_s f \,\mathrm{d}s &= \int_a^b P_s\left(p_h^x - p_{h'}^x\right)\mathrm{d}s \\
&= \int_a^b (p_{h+s}^x - p_{h'+s}^x)\,\mathrm{d}s \\
&= \int_{a+h}^{b+h} p_s^x \,\mathrm{d}s - \int_{a+h'}^{b+h'} p_s^x \,\mathrm{d}s \\
&= \int_{b+h'}^{b+h} p_s^x \,\mathrm{d}s - \int_{a+h'}^{a+h} p_s^x \,\mathrm{d}s \\
&= P_b q_{h',h}^x - P_a q_{h',h}^x.
\end{aligned}
$$

Using this in (4.1), and since $0 \leq q_{h',h}^x \leq q_h^x$ for $h' \leq h$, we have

$$
\begin{aligned}
\mathbb{E}^\mu\left[\left|\widetilde{L}^h(t,x) - \widetilde{L}^{h'}(t,x)\right|^2\right] \leq \gamma_b \underbrace{\int_0^t \mu\left(P_{t-r}\big((P_r q_h^x)^2\big)\right)\mathrm{d}r}_{=:I_1} &+ 2\gamma_b \underbrace{\int_0^t \mu\left(P_r\left(P_{0\vee(\delta-r)} q_h^x P_{t-r} q_h^x\right)\right)\mathrm{d}r}_{=:I_2} \\
&+ \gamma_b \underbrace{\int_0^t \mu\left(P_r\big((P_{0\vee(\delta-r)} q_h^x)^2\big)\right)\mathrm{d}r}_{=:I_3},
\end{aligned}
$$

where we substituted $r \to t - r$ in I_1. We proceed by showing that I_1, I_2, I_3 converge to zero as $h \downarrow 0$, uniformly in $t \leq T$ and $x \in E$. The following fact will be useful.

$$
\forall T > 0\colon \int_0^T \left\|q_{r,r+h}\right\|_\infty \mathrm{d}r \longrightarrow 0, \quad h \to 0. \tag{4.2}
$$

Indeed, for $0 \leq r \leq T$,

$$
\left\|q_{r,r+h}\right\|_\infty = \left\|\int_r^{r+h} p_s \,\mathrm{d}s\right\|_\infty \leq \int_r^{r+h} \|p_s\|_\infty \,\mathrm{d}s \leq c_3 \int_r^{r+h} (t_0 \wedge s)^{-d_f/\beta} \,\mathrm{d}s,
$$

and, since $d_f/\beta = d_s/2 < 2$,

$$
\int_0^T \int_r^{r+h} (s \wedge t_0)^{-d_f/\beta} \,\mathrm{d}s\,\mathrm{d}r \longrightarrow 0 \text{ as } h \to 0.
$$

(I_1) $(P_r q_h^x)^2 = (q_{r,r+h}^x)^2 \leq q_{r,r+h}^x \left\|q_{r,r+h}^x\right\|_\infty \leq q_{r,r+h}^x \left\|q_{r,r+h}\right\|_\infty$, so, assuming $h < 1$,

$$
\begin{aligned}
I_1 \leq \int_0^t \left\|q_{r,r+h}\right\|_\infty \mu\left((P_{t-r} q_{r,r+h}^x)\right)\mathrm{d}r = \int_0^t \left\|q_{r,r+h}\right\|_\infty \mu\left(q_{t,t+h}^x\right)\mathrm{d}r &= (\mu q_{t,t+h})(x) \int_0^t \left\|q_{r,r+h}\right\|_\infty \mathrm{d}r \\
&\leq \left\|(\mu q_{\delta,T+1})\right\|_\infty \int_0^T \left\|q_{r,r+h}\right\|_\infty \mathrm{d}r.
\end{aligned}
$$

This bound is independent of $t \leq T$ and x, and goes to zero as $h \to 0$ by Proposition 4.1 and (4.2).

(I_2) For $0 \leq r \leq T$, and $a := a_r := 0 \vee (\delta - r)$,

$$
(P_a q_h^x)(P_{t-r} q_h^x) = q_{a,a+h}^x q_{t-r,t-r+h}^x \leq q_{a,a+h}^x \left\|q_{t-r,t-r+h}\right\|_\infty,
$$

so, again assuming $h < 1$,

$$\begin{aligned}
I_2 &\leq \int_0^t \left\|q_{t-r,t-r+h}\right\|_\infty \mu\left(P_r q^x_{a,a+h}\right) \mathrm{d}r \\
&= \int_0^t \left\|q_{t-r,t-r+h}\right\|_\infty \mu(q^x_{r\vee\delta,r\vee\delta+h}) \,\mathrm{d}r \\
&\leq \mu(1) \int_0^t \left\|q_{t-r,t-r+h}\right\|_\infty \left\|q_{r\vee\delta,r\vee\delta+h}\right\|_\infty \mathrm{d}r \\
&\leq \mu(1) \int_0^{t/2} \left\|q_{t/2,t+h}\right\|_\infty \underbrace{\left\|q_{r\vee\delta,r\vee\delta+h}\right\|_\infty}_{\leq \left\|q_{r,r+h}\right\|_\infty + \left\|q_{\delta,\delta+h}\right\|_\infty} \mathrm{d}r + \mu(1) \int_{t/2}^t \left\|q_{t-r,t-r+h}\right\|_\infty \left\|q_{t/2,t+h}\right\|_\infty \mathrm{d}r \\
&\leq \mu(1) \left\|q_{\delta/2,T+1}\right\|_\infty \left(2\int_0^T \left\|q_{r,r+h}\right\|_\infty \mathrm{d}r + T\left\|q_{\delta,\delta+h}\right\|_\infty\right),
\end{aligned}$$

which goes to zero as $h \to 0$ by (4.2), and the fact that $\left\|q_{\delta,\delta+h}\right\|_\infty \leq h \sup_{s\geq\delta} \|p_s\|_\infty \longrightarrow 0$ as $h \to 0$.

(I_3) We first split the integral according to

$$I_3 = \underbrace{\int_0^\delta \mu\left(P_r\left((q^x_{\delta-r,\delta-r+h})^2\right)\right)\mathrm{d}r}_{=:I_{3.1}} + \underbrace{\int_\delta^t \mu\left(P_r\left((q^x_h)^2\right)\right)\mathrm{d}r}_{=:I_{3.2}}.$$

Then,

$$I_{3.1} \leq \int_0^\delta \left\|q_{\delta-r,\delta-r+h}\right\|_\infty \mu(\underbrace{P_r q^x_{\delta-r,\delta-r+h}}_{=q^x_{\delta,\delta+h}}) \,\mathrm{d}r \leq \mu(1)\left\|q_{\delta,\delta+h}\right\|_\infty \int_0^\delta \left\|q_{r,r+h}\right\|_\infty \mathrm{d}r,$$

which goes to zero by (4.2). We turn to $I_{3.2}$. Assuming $h < t_0/2$,

$$\begin{aligned}
I_{3.2} &= \int_\delta^t \mathrm{d}r \int_E \mu(\mathrm{d}y) \int_E \nu(\mathrm{d}z)\, p_r(y,z) \int_0^h \mathrm{d}s \int_0^h \mathrm{d}s'\, p_s(x,z) p_{s'}(x,z) \\
&= \int_0^h \mathrm{d}s \int_0^h \mathrm{d}s' \int_E \nu(\mathrm{d}z)\,(\mu q_{\delta,t})(z) p_s(x,z) p_{s'}(x,z) \\
&\leq \left\|(\mu q_{\delta,T})\right\|_\infty \int_0^h \int_0^h \underbrace{\left(\int_E p_s(x,z) p_{s'}(z,x) \nu(\mathrm{d}z)\right)}_{=p_{s+s'}(x,x)} \mathrm{d}s\,\mathrm{d}s' \\
&\leq \left\|(\mu q_{\delta,T})\right\|_\infty c_3 \int_0^h \int_0^h (s+s')^{-d_f/\beta}\,\mathrm{d}s\,\mathrm{d}s',
\end{aligned} \tag{4.3}$$

which is a bound independent of $t \leq T$ and $x \in E$. The integral is no more than $\int_0^{2h} u^{1-d_f/\beta}\,\mathrm{d}u$, which tends to zero as $h \to 0$ since $d_f/\beta = d_s/2 < 2$.

We have shown that, for every $T > \delta$,

$$\sup_{\delta\leq t\leq T, x\in E} \mathbb{E}^\mu\left[\left|\widetilde{L}^h(t,x) - \widetilde{L}^{h'}(t,x)\right|^2\right] \longrightarrow 0$$

as $h \vee h' \to 0$, so there exist random variables $(\widetilde{L}(t,x))_{t\geq\delta, x\in E}$ with

$$\sup_{\delta\leq t\leq T, x\in E} \mathbb{E}^\mu\left[\left|\widetilde{L}^h(t,x) - \widetilde{L}(t,x)\right|^2\right] \xrightarrow{h\to 0} 0.$$

□

Putting $L_\delta(t,x) \coloneqq \widetilde{L}_\delta(t,x) + (\mu q_t)(x)$ for $t \geq 0$ and $x \in E$, we infer from Propositions 4.1 and 4.2 that

$$\sup_{\delta \leq t \leq T, x \in E} \mathbb{E}^\mu \left[\left| L^h_\delta(t,x) - L_\delta(t,x) \right|^2 \right] \longrightarrow 0, \quad T > 0, \tag{4.4}$$

which proves Theorem 2.1 (i).

4.2. Continuity of Local Times

Proposition 4.3. *Suppose that $d_s < 4$. Then, for any $0 < a < (2 - d_s/2) \wedge 1$, and $n \in \mathbb{N}$,*

$$\sup_{x \in E} \sup_{\substack{s,t \geq \delta \\ |t-s|<h}} \mathbb{E}^\mu \left[\frac{|L_\delta(t,x) - L_\delta(s,x)|^n}{|t-s|^{na}} \right] \overset{h \to 0}{\longrightarrow} 0. \tag{4.5}$$

Given this, Theorem 2.1 (iii) is a straight-forward consequence of Kolmogorov's continuity criterion.

Proof of Theorem 2.1 (iii). Let $x \in E$. Proposition 4.3 and Kolmogorov's continuity criterion imply that, for any given $n \in \mathbb{N}$ and $a \in (0, 1 \wedge (2 - d_s/2))$, there is a modification of $L_\delta(x, \cdot)$ which is β'-Hölder continuous for all $0 < \beta' < a - 1/n$. Since all of these modifications coincide $\mathbb{P}^\mu$-a.s. (they are continuous), there is in fact a single modification which is β'-Hölder continuous for all $0 < \beta' < 1 \wedge (2 - d_s/2)$. Since Theorem 2.1 (i) remains true when replacing $L_\delta(\cdot, x)$ by a modification for every $x \in E$, we may assume that $L_\delta(\cdot, x)$ has the desired Hölder continuity for every $x \in E$.

By Theorem 2.1 (i) and a standard diagonalisation argument, we can choose a sequence $h_n \downarrow 0$ such that there is a single set Ω' of $\mathbb{P}^\mu$-probability one on which $L^{h_n}(t,x) \longrightarrow L(t,x)$ for all $t \in [\delta, \infty) \cap \mathbb{Q}$. $L^{h_n}(\cdot, x)$ is increasing by definition, so, on Ω', $L(\cdot, x)$ is increasing on $[\delta, \infty) \cap \mathbb{Q}$, hence by continuity on $[\delta, \infty)$. By putting $L(\cdot, x) \equiv 0$ outside Ω', we obtain a modification that is increasing and of the desired continuity. □

Theorem 2.3 and the following lemma are the central ingredients in the proof of Proposition 4.3.

Lemma 4.4. *Suppose that $d_s < 4$. Then there is $c_{18} > 0$ such that $\mathbb{P}^\mu$-a.s., there exists for each $\delta > 0$ an $h_0 = h_0(\delta, \omega) \in (0,1)$ with*

$$\sup_{s \geq \delta} \|(X_s q_h)\|_\infty \leq c_{18} \zeta(h), \qquad 0 < h < h_0,$$

where

$$\zeta(h) = \zeta_{d_s,\gamma}(h) \coloneqq \begin{cases} h^{2-d_s/2} \log\left(\frac{1}{h}\right)^{1+1/\gamma + \mathbb{1}\{d_s=2\}}, & d_s \geq 2, \\ h \log\left(\frac{1}{h}\right)^{2-d_s/2+d_s/(2\gamma)}, & d_s < 2, \end{cases} \qquad h \in (0,1).$$

With these, we can prove Proposition 4.3.

Proof of Proposition 4.3. Put $D \coloneqq (2 - d_s/2) \wedge 1$, and fix $a \in (0, D)$ and $n \in \mathbb{N}$. Let $x \in E$ and $t > s \geq \delta$ with $|t - s| < t_0$. By Theorem 2.1 (i), there is $h_m \downarrow 0$ such that $\int_\delta^q X_r(p^x_{h_m})\, \mathrm{d}r \longrightarrow L_\delta(q,x)$ for $q = s,t$ on a common set of $\mathbb{P}^\mu$-probability one. Let $c_{19}(n) \coloneqq c_5(n) \vee c_{18} \vee 1 > 0$, where $c_5(n), c_{18} > 0$ are the constants from Theorem 2.3 and Lemma 4.4. Then, by the Markov property and Fatou's lemma,

$$\begin{aligned}
\mathbb{E}^\mu\Big[\big|L_\delta(t,x) - L_\delta(s,x)\big|^n\Big] &= \mathbb{E}^\mu \left[\lim_{m\to\infty} \left(\int_s^t X_r(p^x_{h_m}) \right)^n \right] \\
&= \mathbb{E}^\mu \left[\mathbb{E}^{X_s} \left[\lim_{m\to\infty} \left(\int_0^{t-s} X_r(p^x_{h_m})\, \mathrm{d}r \right)^n \right] \right] \\
&\leq \mathbb{E}^\mu \left[\varliminf_{m\to\infty} \mathbb{E}^{X_s} \left[\left(\int_0^{t-s} X_r(p^x_{h_m})\, \mathrm{d}r \right)^n \right] \right] \\
&\leq c_{19}(n) \mathbb{E}^\mu \Big[\varliminf_{m\to\infty} \left(\nu(p^x_{h_m}) \cdot \|(X_s q_{t-s})\|_{\infty,\nu} \vee (t-s)^{2-d_s/2} \right)^n \Big] \\
&= c_{19}(n) \mathbb{E}^\mu \left[\left(\|(X_s q_{t-s})\|_{\infty,\nu} \vee (t-s)^{2-d_s/2} \right)^n \right].
\end{aligned}$$

Note that in applying the Markov property, we used that $\mathcal{M}_F(E) \to \mathbb{R};\ \eta \mapsto \|(\eta q_t)\|_{\infty,\nu}$ is measurable, which follows by writing $\|\cdot\|_{\infty,\nu} = \lim_{p\to\infty}\left(\int_E |\cdot|^p\,\mathrm{d}\nu\right)^{1/p}$, and the fact that $\eta \mapsto \int_E f(x,\cdot)\eta(\mathrm{d}x)$ is measurable for any measurable $f\colon E\times E \to [0,\infty]$ by a standard Dynkin argument.

Fix any $a' \in (a, D)$, so that $h^{2-d_s/2} < \zeta(h) < h^{a'}$ for small $h > 0$. The upper bound above is independent of x, so we conclude with Lemma 4.4 that

$$\begin{aligned}
\lim_{h\to 0}\sup_{x\in E}\sup_{\substack{s,t\geq\delta\\ |t-s|<h}} \frac{\mathbb{E}^\mu\left[|L_\delta(t,x)-L_\delta(s,x)|^n\right]}{|t-s|^{na}} &\leq c_{19}(n)\lim_{h\to 0}\sup_{\substack{\delta\leq s<t\\ |t-s|<h}} \frac{1}{|t-s|^{na}}\mathbb{E}^\mu\left[\left(\|(X_s q_{t-s})\|_{\infty,\nu} \vee (t-s)^{2-d_s/2}\right)^n\right] \\
&\leq c_{19}(n)\mathbb{E}^\mu\left[\lim_{h\to 0}\sup_{\substack{\delta\leq s<t\\ |t-s|<h}} \frac{1}{|t-s|^{na}}\left(\|(X_s q_{t-s})\|_{\infty,\nu} \vee (t-s)^{2-d_s/2}\right)^n\right] \\
&\leq c_{19}(n)^{n+1}\lim_{h\to 0}\sup_{\substack{\delta\leq s<t\\ |t-s|<h}} \frac{(\zeta(t-s)\vee(t-s)^{2-d_s/2})^n}{|t-s|^{na}} \\
&\leq c_{19}(n)^{n+1}\lim_{h\to 0} h^{n(a'-a)} = 0,
\end{aligned}$$

where we used monotone convergence in the second step, □

It remains to prove Theorem 2.3 and Lemma 4.4. For the former, we use a moment formula derived from the work of Dynkin [11]: If $f \in B_+(E)\cup B_b(E)$, and $t \geq 0$, then, for $\mu \in \mathcal{M}_F(E)$,

$$\mathbb{E}^\mu\left[\left(\int_0^t X_r(f)\,\mathrm{d}r\right)^n\right] = \sum_{\{D_i\}_{i=1}^m \in \mathbb{G}_n} \prod_{i=1}^m \mu\left(D_i(t;f)\right). \tag{4.6}$$

This statement and the associated notation can be found in Appendix A. The proof of Theorem 2.3 now relies on an induction to bound the RHS of (4.6).

Proof of Theorem 2.3. Fix $f \in B_+(E)$ and $t \in (0,t_0)$. By (4.6), we must find bounds on $\mu(D(t;f))$ for $D \in \mathbb{D}$. For $r \geq 0$ and $D \in \mathbb{D}$, write $D(r,\cdot) := D(r;f)(\cdot)$. Then, if $|D| \geq 2$, say $D = D_1 \vee D_2$, and $x \in E$, by Lemma A.6,

$$\begin{aligned}
D(t,x) &= \gamma_b\int_0^t \mathrm{d}r\,P_r\Big(\underbrace{D_1(t-r,\cdot)D_2(t-r,\cdot)}_{\leq D_1(t,\cdot)D_2(t,\cdot)}\Big)(x) \\
&\leq \gamma_b\int_0^t \mathrm{d}r\int_E \nu(\mathrm{d}y)\,p_r(x,y)D_1(t,y)D_2(t,y) \\
&= \gamma_b\int_E \nu(\mathrm{d}y)\,q_t(x,y)D_1(t,y)D_2(t,y),
\end{aligned} \tag{4.7}$$

so

$$\mu(D(t;f)) = \gamma_b\int_E \nu(\mathrm{d}y)\,(\mu q_t)(y)D_1(t,y)D_2(t,y) \leq \|(\mu q_t)\|_{\infty,\nu}\,D(t), \tag{4.8}$$

where

$$D(t) := \gamma_b\int_E D_1(t,x)D_2(t,x)\nu(\mathrm{d}x). \tag{4.9}$$

We shall find a bound on $D(t)$ inductively. Put $\alpha := d_f/\beta = d_s/2 \in (0,2)$ for this proof. If $D = D_1 \vee D_2 \in \mathbb{D}$ with $|D_1| \geq 2$, $|D_2| \geq 2$ (so that (4.9) defines $D_1(t)$ and $D_2(t)$), we claim that

$$D(t) \leq c_{20}t^{2-\alpha}D_1(t)D_2(t), \tag{4.10}$$

where $c_{20} = \frac{4\gamma_b c_3}{2-\alpha}$. Indeed, we can write $D_1 = D_1' \vee D_1''$, and $D_2 = D_2' \vee D_2''$, so, by (4.7),

$$\begin{aligned} D(t) &= \gamma_b \int_E \nu(\mathrm{d}x)\, D_1(t,x) D_2(t,x) \\ &\le \gamma_b^3 \int_0^t \int_0^t \int_E \int_E \Bigg(\underbrace{\int_E \nu(\mathrm{d}x)\, p_r(y,x) p_{r'}(x,y')}_{=p_{r+r'}(y,y') \le c_3 (r+r')^{-\alpha}} \Bigg) D_1'(t,y) D_1''(t,y) D_2'(t,y') D_2''(t,y')\, \mathrm{d}r\, \mathrm{d}r'\, \nu(\mathrm{d}y)\nu(\mathrm{d}y') \\ &\le \gamma_b c_3 \left(\int_0^t \mathrm{d}r \int_0^t \mathrm{d}r'\, (r+r')^{-\alpha} \right) D_1(t) D_2(t), \end{aligned} \tag{4.11}$$

where

$$\int_0^t \mathrm{d}r \int_0^t \mathrm{d}r'\, (r+r')^{-\alpha} = \int_0^t \mathrm{d}r \int_r^{t+r} \mathrm{d}s\, s^{-\alpha} = \int_0^{2t} \mathrm{d}s \int_{(s-t)_+}^{s \wedge t} \mathrm{d}r\, s^{-\alpha} \le \int_0^{2t} s^{1-\alpha}\, \mathrm{d}s = \frac{(2t)^{2-\alpha}}{2-\alpha} \le \frac{4t^{2-\alpha}}{2-\alpha}.$$

If $D = D_1 \vee D_2 \in \mathbb{D}$, and $|D_1| = 1$, $|D_2| \ge 2$, then $D_1(t,\cdot) = \int_0^t P_r f\, \mathrm{d}r = \int_0^t \mathrm{d}r \int_E \nu(\mathrm{d}y)\, p_r(\cdot,y) f(y)$, and D_2 as in (4.11), and the same argument gives

$$D(t) \le c_{20} t^{2-\alpha} \nu(f) D_2(t), \tag{4.12}$$

and if $|D_1| = |D_2| = 1$, then

$$D(t) \le c_{20} t^{2-\alpha} \nu(f)^2. \tag{4.13}$$

Given $D \in \mathbb{D}$ with $|D| \ge 2$, we can now bound $D(t)$ by recursively applying (4.10), (4.12) and (4.13). We obtain a factor $\nu(f)$ for every leaf, and a factor $c_{20} t^{2-\alpha}$ for every inner node, of which there are $|D|$ and $|D|-1$, respectively, so

$$D(t) \le (c_{20} t^{2-\alpha})^{|D|-1} \nu(f)^{|D|},$$

thus by (4.8),

$$\mu(D(t;f)) \le \|(\mu q_t)\|_{\infty,\nu}\, (c_{20} t^{2-\alpha})^{|D|-1} \nu(f)^{|D|}. \tag{4.14}$$

If $|D| = 1$, then

$$\mu(D(t;f)) = \int_E \mu(\mathrm{d}x) \int_0^t \mathrm{d}r \int_E \nu(\mathrm{d}y)\, p_r(x,y) f(y) = \int_E \nu(\mathrm{d}y)\, (\mu q_t)(y) f(y) \le \|(\mu q_t)\|_{\infty,\nu}\, \nu(f),$$

so (4.14) holds in that case as well. Combining (4.6) and (4.14) gives

$$\begin{aligned} \mathbb{E}^\mu \left[\left(\int_0^t X_s(f)\, \mathrm{d}s \right)^n \right] &= \sum_{\{D_i\}_{i=1}^m \in \mathbb{G}_n} \prod_{i=1}^m \mu(D_i(t;f)) \\ &\le \sum_{\{D_i\}_{i=1}^m \in \mathbb{G}_n} \prod_{i=1}^m \left(\|(\mu q_t)\|_{\infty,\nu}\, (c_{20} t^{2-\alpha})^{|D_i|-1} \nu(f)^{|D_i|} \right) \\ &= \sum_{\{D_i\}_{i=1}^m \in \mathbb{G}_n} \left(\|(\mu q_t)\|_{\infty,\nu}^m\, (c_{20} t^{2-\alpha})^{n-m} \nu(f)^n \right) \\ &\le |\mathbb{G}_n| \left(\|(\mu q_t)\|_{\infty,\nu} \vee (c_{20} t^{2-\alpha}) \right)^n \nu(f)^n. \end{aligned}$$

This proves (2.2) with $c_5(n) := |\mathbb{G}_n|\, (c_{20} \vee 1)^n$.

Finally, if μ has a bounded density g w.r.t. ν, then for any $x \in E$,

$$(\mu q_t)(x) = \int_0^t \int_E p_s(x,y) g(y) \nu(\mathrm{d}y)\, \mathrm{d}s \le \|g\|_{\infty,\nu} \int_0^t \underbrace{\int_E p_s(x,y) \nu(\mathrm{d}y)}_{=1}\, \mathrm{d}s = t\, \|g\|_{\infty,\nu}\,,$$

and the RHS of (2.2) is upper bounded by that of (2.3). □

To prove sharpness of this result in the sense of Remark 2.4 (i), we observe that a stronger bound would imply a stronger Hölder-continuity of the local times, contradicting Theorem 2.1 (iv) (which will be proved later).

Proof of Remark 2.4 (i). It is easy to show that $\|(\mu q_t)\|_{\infty,\nu} \geq ct$ for some $c > 0$ and small $t > 0$, for any non-zero $\mu \in \mathcal{M}_F(E)$. Hence, if $d_s < 2$, then the maximum on the RHS of (2.2) is attained by $\|(\mu q_t)\|_{\infty,\nu}$ for small $t > 0$.

Suppose that $d_s \geq 2$, and that Theorem 2.3 held with $2 - d_s/2$ on the RHS of (2.2) and (2.3) replaced by some $a' > 2 - d_s/2$ (simultaneously for all $n \in \mathbb{N}$ and $\mu \in \mathcal{M}_F(E)$). Then take $\mu \in \mathcal{M}_F(E)$ with a density w.r.t. ν that is bounded by 1 and positive everywhere. Such a density exists; by our assumption on ν it can be chosen as $g = \sum_{N=1}^{\infty} a_N \mathbb{1}_{B(x_0,N)}$ for some $x_0 \in E$ and appropriate constants $a_N > 0$, $N \in \mathbb{N}$. Then, (2.3) holds with RHS equal to $c_5\nu(f)^n t^{na'}$. This would allow for a proof of (4.5) with $a \in (2 - d_s/2, a')$ and some fixed $\delta > 0$, and thus imply β'-Hölder continuity of $L_\delta(x_0, \cdot)$ for all $\beta' \in (2 - d_s/2, a)$ and some fixed $x_0 \in E$, see proof of Theorem 2.1 (iii). By Theorem 2.1 (iv), this would imply that $L_\delta(x_0, \cdot) \equiv 0$ $\mathbb{P}^\mu$-a.s., which is a contradiction of

$$
\begin{aligned}
\mathbb{E}^\mu\left[L_\delta(x_0, t)\right] = \lim_{h\to 0} \mathbb{E}^\mu\left[\int_\delta^t X_s(p_h(x_0, \cdot))\,\mathrm{d}s\right] &= \lim_{h\to 0} \int_\delta^t \mu(p_{s+h}(x_0, \cdot))\,\mathrm{d}s \\
&= \int_\delta^t \mu(p_s(x_0, \cdot))\,\mathrm{d}s \\
&= \int_\delta^t \int p_s(x_0, y) g(y) \nu(\mathrm{d}y)\,\mathrm{d}s \\
&> 0,
\end{aligned}
$$

for any $t > \delta$, where we used Theorem 2.1 (i), Tonelli's theorem, (A.1), dominated convergence, and the fact that $g > 0$ everywhere. $\square$

Lemma 4.4 is an immediate consequence of Theorem 2.10 and the following lemma, for the application of which we recall that, $\mathbb{P}^\mu$-a.s., $\sup_{t\geq 0} X_t(1)$ is finite since $(X_t(1))$ is continuous and eventually absorbed at zero.

Lemma 4.5. *Let $\lambda \in \mathcal{M}_F(E)$, and suppose there are $a > d_f - \beta$ and $b, c, r_1 > 0$ with*

$$
\sup_{x\in E} \lambda(B(x, r)) \leq c r^a \left(\log \frac{1}{r}\right)^b, \quad 0 < r < r_1. \tag{4.15}
$$

Then there exist $c_{21}, h_0 > 0$ such that

$$
\|(\lambda q_h)\|_\infty \leq c_{21} h^{1+(a-d_f)/\beta} \log\left(\frac{1}{h}\right)^{b+a/(\beta\gamma)}, \quad 0 < h < h_0,
$$

where c_{21} does not depend on r_1, and on λ only through a, b, and c. Furthermore, if (4.15) *holds for a family $(\lambda_i)_{i\in I}$ of measures with $\sup_{i\in I} \lambda_i(1) < \infty$, then h_0 can be chosen for all λ_i simultaneously.*

The proof of Lemma 4.5 is straight-forward and technical, and we hide it in Appendix B. We close with a proof of Theorem 2.1 (ii).

Proof of Theorem 2.1 (ii). For $N \in \mathbb{N}$ and some fixed $x_0 \in E$, put

$$
C_N := \{f \in C_+(E) \colon f = 0 \text{ on } E \setminus B(x_0, N)\},
$$

and $f_N := f \wedge N$ for $f \in C_+(E)$, where $C_+(E)$ is the set of non-negative continuous functions on E. Then, for $t \geq \delta$ and $N \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}^\mu\left[\sup_{f\in C_N}\left|\int_E f_N(x)(L_\delta(t, x) - L_\delta^h(t, x))\nu(\mathrm{d}x)\right|\right] &\leq N \int_{B(x_0,N)} \mathbb{E}^\mu\left[\left|L_\delta(t, x) - L_\delta^h(t, x)\right|\right] \nu(\mathrm{d}x) \\
&\leq N\nu(B(x_0, N)) \sup_{x\in E} \left\|L_\delta(t, x) - L_\delta^h(t, x)\right\|_{L^2} \\
&\xrightarrow{h\to 0} 0.
\end{aligned}
$$

In particular, there exists $h_n \longrightarrow 0$ such that, off a $\mathbb{P}^\mu$-null set, $\int_E f_N(x) L_\delta^{h_n}(t,x)\nu(\mathrm{d}x) \longrightarrow \int_E f_N(x) L_\delta(t,x)\nu(\mathrm{d}x)$ as $n \to \infty$ for all $N \in \mathbb{N}$ and $f \in C_N$. Off this null set, for $N \in \mathbb{N}$, $N > 1/\delta$, and $f \in C_N$,

$$\begin{aligned}
\int_E f_N(x) L_\delta(t,x)\nu(\mathrm{d}x) &= \lim_{n\to\infty} \int_E f_N(x) \left(\int_\delta^t X_s(p_{h_n}(x,\cdot)\,\mathrm{d}s \right) \nu(\mathrm{d}x) \\
&= \lim_{n\to\infty} \int_\delta^t \mathrm{d}s \int_E X_s(\mathrm{d}y) \left(\int_E p_{h_n}(y,x) f_N(x)\nu(\mathrm{d}x) \right) \\
&= \lim_{n\to\infty} \int_\delta^t X_s(P_{h_n} f_N)\,\mathrm{d}s \\
&= \int_\delta^t X_s(f_N)\,\mathrm{d}s,
\end{aligned}$$

where we could apply dominated convergence in the last step because f_N is continuous and bounded and so $P_{h_n} f_N \longrightarrow f_N$ pointwise, and $\left\| P_{h_n} f_N \right\|_\infty \le N$ for all $n \in \mathbb{N}$. If $f\colon E \to \mathbb{R}$ is continuous, non-negative, and compactly supported, write $f \in C_{c+}(E)$, then $f \in C_N$ for large $N \in \mathbb{N}$, and $0 \le f_N \uparrow f$ as $N \to \infty$. Hence by monotone convergence, $\mathbb{P}^\mu$-a.s., for all $f \in C_{c+}(E)$,

$$\begin{aligned}
\int_E f(x) L_\delta(t,x)\nu(\mathrm{d}x) &= \lim_{N\to\infty} \int_E f_N(x) L_\delta(t,x)\nu(dx) \\
&= \lim_{N\to\infty} \int_\delta^t X_s(f_N)\,\mathrm{d}s \\
&= \int_\delta^t X_s(f)\,\mathrm{d}s.
\end{aligned}$$

Finally, both the LHS and the RHS above are $\mathbb{P}^\mu$-a.s. continuous in t (LHS by monotone convergence and Theorem 2.1 (iii); RHS by dominated convergence), so we may choose a single set of $\mathbb{P}^\mu$-probability one on which the claim holds for all $t \ge \delta$ and all $f \in C_{c+}(E)$. A monotone class argument finishes the proof. □

Theorem 2.1 (iv) will be proved at the end of next section. The idea is that a stronger Hölder continuity of $L_\delta(\cdot, x_0)$ would imply a lower bound on $\dim_H \mathcal{T}(x_0)$ that contradicts the upper bound in Theorem 2.5.

5. Dimension Results

We begin by proving our range result Theorem 2.8. It can be understood as a modern formulation of Tribe's [36] proof; instead of using non-standard techniques to apply the modulus of continuity of linear Brownian motions to the associated superprocess, we use the modulus of continuity of the historical process, Theorem 3.3.

Proof of Theorem 2.8. The lower bound on $\mathrm{S}(X_t)$ follows immediately from Theorem 2.10 (see, for example, [19, page 61]). For the upper bound, fix $\mu \in \mathcal{M}_F(E)$ and let $m \in \mathcal{M}_F^0(E)$ with $\pi_0(m) = \mu$, so that the distribution of (X_t) under $\mathbb{P}^\mu$ is the same as that of $(\pi_t(H_t))$ under $\mathbb{Q}_{0,m}$. We work with the latter, and slightly abuse notation by writing $X_t := \pi_t(H_t)$ for the duration of the proof.

First note that $\dim_H \mathcal{R}(I) \le \dim_H E = d_f$ for any $I \subset (0,\infty)$ is trivial. We show that for fixed but arbitrary $N \in \mathbb{N}$ and $\varepsilon > 0$, $\mathbb{Q}_{0,m}$-a.s.,

$$\dim_H \mathcal{R}(I) \le \beta\,(1 + \dim_H I + 2\varepsilon)\,,$$

for any $I \subset [1/N, N \wedge \tau_N]$, where $\tau_N = \inf\{t \ge 0\colon X_t(1) \ge N\}$. This implies $\dim_H \mathcal{R}(I) \le \beta(1 + \dim_H I)$ $\mathbb{Q}_{0,m}$-a.s. by countable stability of Hausdorff dimension, and the fact that $\tau_N \to \infty$ $\mathbb{Q}_{0,m}$-a.s. as $N \to \infty$ because $(X_t(1))$ is a continuous path that is $\mathbb{Q}_{0,m}$-a.s. eventually absorbed at zero.

For $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$, put $t_{nj} := j2^{-n}$ and $I_{nj} := [t_{nj}, t_{n,j+1}]$, and denote for $j \in \mathbb{N}$ by M_{nj} and $y_{nj}^{(i)}$, $i \in \mathbb{N}$, the respectively $\mathbb{N}_0$- and $C(E)$-valued random variables from Theorem 3.1 for which

$$\mathrm{S}(r_{t_{n,j-1}}(H_{t_{nj}})) = \left\{ y_{nj}^{(1)}, \ldots, y_{nj}^{(M_{nj})} \right\}, \quad \mathbb{Q}_{0,m}\text{-a.s.},$$

and, given $\mathcal{F}^H_{t_{n,j-1}}$, $(y^{(i)}_{nj})_{i=1}^{M_{nj}}$ follows the distribution of a Poisson point process on $C(E)$ with intensity measure $\frac{2^{n+1}}{\gamma_b} H_{t_{n,j-1}}$. In particular, $M_{nj} \sim \mathrm{Poi}\left(\frac{2^{n+1}}{\gamma_b} H_{t_{n,j+1}}(1)\right)$, so by a Poisson tail bound (see Lemma B.3),

$$\begin{aligned}
\mathbb{Q}_{0,m}\left(\exists j \in \mathbb{N}\colon t_{nj} \le N \wedge \tau_N, M_{nj} \ge \frac{N2^{n+2}}{\gamma_b}\right) &\le \sum_{j=1}^{N2^n} \mathbb{Q}_{0,m}\left(t_{n,j-1} \le \tau_N, M_{nj} \ge \frac{N2^{n+2}}{\gamma_b}\right) \\
&= \sum_{j=1}^{N2^n} \mathbb{Q}_{0,m}\left(\mathbb{1}_{\{t_{n,j-1} \le \tau_N\}} \mathbb{Q}_{0,m}\left(M_{nj} \ge 2 \cdot \frac{2^{n+1}N}{\gamma_b} \,\middle|\, \mathcal{F}^H_{t_{n,j-1}}\right)\right) \\
&\le \sum_{j=1}^{N2^n} \mathbb{Q}_{0,m}\left(\mathbb{1}_{\{t_{n,j-1} \le \tau_N\}} \exp\left(-\frac{3}{16}\frac{2^{n+2}N}{\gamma_b}\right)\right) \\
&\le N2^n \exp\left(-3 \cdot 2^{n-2} N/\gamma_b\right),
\end{aligned}$$

where we used that $t_{n,j-1} \le \tau_N$ implies $H_{t_{n,j-1}}(1) = X_{t_{n,j-1}}(1) \le N$. This bound is summable over n, hence outside some null set there exists $n_0 = n_0(\omega) \in \mathbb{N}$ such that $M_{nj} \le \frac{N2^{n+2}}{\gamma_b}$ for all $n \ge n_0$ and $j \in \mathbb{N}$ with $t_{nj} \le N \wedge \tau_N$. Now let $c > 0$ be so large that, omitting another null set, we find $\delta = \delta(c, \omega) > 0$ according to Theorem 3.3 and Corollary 3.4 (ii). Then, $\mathbb{Q}_{0,m}$-a.s. if $2^{-n} < \delta$ and $j \in \mathbb{N}$, applying Corollary 3.4 (ii), Corollary 3.4 (i), and Theorem 3.3 in that order,

$$\begin{aligned}
\mathcal{R}(I_{nj}) = \bigcup_{s \in I_{nj}} \mathrm{S}(X_s) &\subset \mathrm{S}(X_{t_{nj}})^{ch(2^{-n})} \\
&= \pi_{t_{nj}}(\mathrm{S}(H_{t_{nj}}))^{ch(2^{-n})} \\
&\subset \pi_{t_{n,j-1}}(\mathrm{S}(H_{t_{nj}}))^{2ch(2^{-n})} \\
&= \pi_{t_{n,j-1}}(\mathrm{S}(r_{t_{n,j-1}}(H_{t_{nj}})))^{2ch(2^{-n})} \\
&= \bigcup_{i=1}^{M_{nj}} \underbrace{B(y^{(i)}_{nj}(t_{n,j-1}), 2ch(2^{-n}))}_{=:B^i_{nj}}.
\end{aligned} \tag{5.1}$$

Now fix ω outside the excluded null sets and let $I \subset [N^{-1}, N \wedge \tau_N]$ with Hausdorff dimension $d_I := \dim_H I$. Then, $\mathcal{H}^{d_I+\varepsilon}(I) = 0$ (recall that $\mathcal{H}^s\colon \mathcal{P}(E) \to [0,\infty]$ for $s > 0$ denotes the s-dimensional Hausdorff outer measure). Hence, there are for all $n \in \mathbb{N}$ dyadic 2^{-n}-covers $C_n = (I_{kj}\colon k \ge n, j \in J^n_k)$ of I, where $J^n_k \subset \left\{1, \dots, N2^k\right\}$, $k \ge n$, such that

$$0 = \lim_{n\to\infty} \sum_{I \in C_n} |I|^{d_I+\varepsilon} = \lim_{n\to\infty} \sum_{k=n}^{\infty} \left|J^n_k\right| 2^{-k(d_I+\varepsilon)},$$

where by $|A|$ for a subset A of a metric space we denote its diameter. Then, if $2^{-n} < \delta$, the family

$$\mathcal{B}_n := \left(B^i_{kj}\colon k \ge n, j \in J^n_k, i \le M_{kj}\right)$$

is a cover of $\mathcal{R}(I)$ with maximal diameter $2ch(2^{-n})$, which tends to zero as $n \to \infty$. Further recall that $h(2^{-k}) =$

$2^{-k/\beta}(k\log 2)^{1/(\gamma\beta)}$. Hence,

$$\begin{aligned}
\mathcal{H}^{\beta(1+d_I+2\varepsilon)}(\mathcal{R}(I)) &\le \varliminf_{n\to\infty} \sum_{B\in\mathcal{B}_n} |B|^{\beta(1+d_I+2\varepsilon)} \\
&= \varliminf_{n\to\infty} \sum_{k=n}^{\infty} \sum_{j\in J_k^n} M_{kj} h(2^{-k})^{\beta(1+d_I+2\varepsilon)} \\
&\le \varliminf_{n\to\infty} \sum_{k=n}^{\infty} \left|J_k^n\right| N 2^{k+1} 2^{-k(1+d_I+2\varepsilon)} (k\log 2)^{(1+d_I+2\varepsilon)/\gamma} \\
&\le c_{22} \underbrace{\Big(\sup_{k\in\mathbb{N}} (k\log 2)^{(1+d_I+2\varepsilon)/\gamma} 2^{-k\varepsilon} \Big)}_{<\infty} \cdot \underbrace{\varliminf_{n\to\infty} \sum_{k=n}^{\infty} \left|J_k^n\right| 2^{-k(d_I+\varepsilon)}}_{=0} \\
&= 0,
\end{aligned}$$

where $c_{22} = 2N(\log 2)^{(1+d_I+2\varepsilon)/\gamma}$, which implies $\dim_H \mathcal{R}(I) \le \beta(1+d_I+2\varepsilon)$. □

Recall the moment formula for historical processes [33, Equation (II.8.5)]: If $m \in \mathcal{M}_F^0(C)$, and $f\colon C(E) \to \mathbb{R}$ is bounded and measurable, then

$$\mathbb{Q}_{0,m}(H_t(f)) = \int_{C(E)} \mathbb{P}^{y(0)}\left(f(Y^t)\right) m(\mathrm{d}y), \quad t \ge 0, \tag{5.2}$$

where we write $\mathbb{P}(g) := \int g \,\mathrm{d}\mathbb{P}$ for the expectation of a random variable g w.r.t. a probability measure $\mathbb{P}$.

Proof of Theorem 2.5, Upper Bound. Fix $\mu \in \mathcal{M}_F(E)$. Let $m \in \mathcal{M}_F^0(E)$ with $\pi_0(m) = \mu$, so that the distribution of (X_t) under $\mathbb{P}^\mu$ is the same as that of $(\pi_t(H_t))$ under $\mathbb{Q}_{0,m}$. We work with the latter and, with a slight abuse of notation, write $X_t := \pi_t(H_t)$, $t \ge 0$, for the duration of the proof.

Let $A \subset E$ and put $d_A := \dim_H A$. By countable stability of Hausdorff dimension, it suffices to show that for fixed but arbitrary $N \in \mathbb{N}$ and $\varepsilon > 0$, $\mathbb{Q}_{0,m}$-a.s.,

$$\dim_H\left(\mathcal{T}(A) \cap [N^{-1}, N]\right) \le D + 2\varepsilon, \quad \text{where} \quad D := \left(2 - \frac{d_s}{2} + \frac{d_A}{\beta}\right)_+ = \left(2 - \frac{d_f - d_A}{\beta}\right)_+.$$

Since $\mathcal{H}^{d_A+\beta\varepsilon}(A) = 0$, there are covers C_n of A, $n \in \mathbb{N}$, comprised of balls with dyadic diameters no larger than 2^{-n}, such that

$$0 = \lim_{n\to\infty} \sum_{C\in C_n} |C|^{d_A+\beta\varepsilon} = \lim_{n\to\infty} \sum_{k\ge n} m_k^n 2^{-k(d_A+\beta\varepsilon)}, \tag{5.3}$$

where

$$m_k^n := \left|\left\{C \in C_n \colon |C| = 2^{-k}\right\}\right|, \quad n \in \mathbb{N},\ k \ge n, \tag{5.4}$$

denotes the number of balls in C_n with diameter 2^{-k}.

For $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$, put $t_{nj} := j2^{-n\beta}$ and $I_{nj} := [t_{nj}, t_{n,j+1}]$. Denote for $n, j \in \mathbb{N}$ by M_{nj} and $y_{nj}^{(i)}$, $i \in \mathbb{N}$, the respectively $\mathbb{N}_0$- and $C(E)$-valued random variables from Theorem 3.1 for which

$$\mathrm{S}(r_{t_{n,j-1}}(H_{t_{nj}})) = \left\{y_{nj}^{(1)}, \ldots, y_{nj}^{(M_{nj})}\right\}, \quad \mathbb{Q}_{0,m}\text{-a.s.},$$

where, given $\mathcal{F}^H_{t_{n,j-1}}$, $(y_{nj}^{(i)})_{i=1}^{M_{nj}}$ follows the distribution of a Poisson point process on $C(E)$ with intensity measure $\frac{2^{n\beta+1}}{\gamma_b} H_{t_{n,j-1}}$. With the same argument employed in (5.1), we find $c > 0$ such that $\mathbb{Q}_{0,m}$-a.s. there is $\delta = \delta(c,\omega) > 0$ such that if $2^{-n\beta} < \delta$ and $j \in \mathbb{N}$, then

$$\bigcup_{s\in I_{nj}} \mathrm{S}(X_s) \subset \bigcup_{i=1}^{M_{nj}} B(y_{nj}^{(i)}(t_{n,j-1}), r_n) =: B_{nj}, \quad \text{where} \quad r_n := 2ch(2^{-n\beta}) = c_{23} 2^{-n} n^{1/(\beta\gamma)},$$

and $c_{23} = 2c(\beta \log 2)^{1/(\beta\gamma)}$. Thus if $2^{-n\beta} < \delta$, then a cover of $\mathcal{T}(A) \cap [N^{-1}, N]$ with maximal diameter $2^{-n\beta}$ is given by

$$\mathcal{B}_n := \bigcup_{k \geq n} \bigcup_{\substack{C \in C_n \\ |C| = 2^{-k}}} \left\{ I_{kj} \colon j \in J_k, B_{kj} \cap C \neq \emptyset \right\},$$

where $J_k = \left\{ j \in \mathbb{N} \colon I_{kj} \cap [N^{-1}, N] \neq \emptyset \right\} = \left\{ \lfloor N^{-1} 2^{-n\beta} \rfloor, \ldots, \lfloor N 2^{-n\beta} \rfloor \right\}$. Then,

$$\begin{aligned} \mathbb{Q}_{0,m}\left(\mathcal{H}^{D+2\varepsilon}(\mathcal{T}(A) \cap [N^{-1}, N]) \right) &\leq \mathbb{Q}_{0,m}\left(\varliminf_{n \to \infty} \sum_{B \in \mathcal{B}_n} |B|^{D+2\varepsilon} \right) \\ &= \mathbb{Q}_{0,m}\left(\varliminf_{n \to \infty} \sum_{k \geq n} \sum_{\substack{C \in C_n \\ |C| = 2^{-k}}} \sum_{j \in J_k} \left| I_{kj} \right|^{D+2\varepsilon} \mathbb{1}_{\{B_{kj} \cap C \neq \emptyset\}} \right) \\ &\leq \varliminf_{n \to \infty} \sum_{k \geq n} \sum_{\substack{C \in C_n \\ |C| = 2^{-k}}} \sum_{j \in J_k} 2^{-k\beta(D+2\varepsilon)} \mathbb{Q}_{0,m}\left(B_{kj} \cap C \neq \emptyset \right), \end{aligned} \tag{5.5}$$

where we used Fatou's lemma in the last step. Let $n \in \mathbb{N}$ and $C \in C_n$, so $C = B(x_0, 2^{-k-1})$ for some $x_0 \in E$ and $k \geq n$. Then, for $j \in J_k$, $B_{kj} \cap C \neq \emptyset$ if and only if $B(y_{kj}^{(i)}(t_{k,j-1}), r_k) \cap C \neq \emptyset$ for some $i \leq M_{nk}$, if and only if $\rho(x_0, y_{kj}^{(i)}(t_{k,j-1})) < 2^{-k-1} + r_k$ for some $i \leq M_{nk}$. Thus if k is so large that $2^{-k-1} < r_k$, then

$$\mathbb{Q}_{0,m}\left(B_{kj} \cap C \neq \emptyset \right) \leq \mathbb{Q}_{0,m}\left(\exists i \leq M_{kj} \colon \rho(x_0, y_{kj}^{(i)}(t_{k,j-1})) < 2r_k \right), \tag{5.6}$$

which we evaluate by conditioning on $\mathcal{F}^H_{t_{k,j-1}}$. To that end, assume that for some $\theta \in \mathcal{M}_F(C) \setminus \{0\}$, $\eta = (y_i)_{i=1}^M$ is a Poisson point process with intensity measure $\frac{2^{k\beta+1}}{\gamma_b}\theta$. Then, for any $K, t > 0$,

$$\begin{aligned} \mathbb{P}\left(\exists i \leq M \colon \rho(x_0, y_i(t)) < K \right) &= 1 - \mathbb{P}\Big(\eta(y \colon \rho(x_0, y(t)) < K) = 0 \Big) \\ &= 1 - \exp\left(-\frac{2^{k\beta+1}}{\gamma_b} \theta(y \colon \rho(x_0, y(t)) < K) \right) \\ &\leq \frac{2^{k\beta+1}}{\gamma_b} \theta(y \colon \rho(x_0, y(t)) < K). \end{aligned}$$

This clearly holds also if $\theta = 0$ (then LHS = RHS = 0). Applying this and (5.2) to (5.6) yields

$$\begin{aligned} \mathbb{Q}_{0,m}\left(B_{kj} \cap C \neq \varnothing \right) &= \mathbb{Q}_{0,m}\left(\mathbb{Q}_{0,m}\left(\exists i \leq M_{kj} \colon \rho\left(x_0, y_{kj}^{(i)}(t_{k,j-1}) \right) < 2r_k \,\middle|\, \mathcal{F}^H_{t_{k,j-1}} \right) \right) \\ &\leq \frac{2^{k\beta+1}}{\gamma_b} \mathbb{Q}_{0,m}\left(H_{t_{k,j-1}}(y \colon \rho\left(x_0, y(t_{k,j-1}) \right) < 2r_k) \right) \\ &= \frac{2^{k\beta+1}}{\gamma_b} \int_{C(E)} \mathbb{P}^{y(0)}\left(\rho\left(x_0, Y(t_{k,j-1}) \right) < 2r_k \right) m(\mathrm{d}y) \\ &\leq \frac{2^{k\beta+1}}{\gamma_b} m(1) \left\| p_{t_{k,j-1}} \right\|_\infty \nu(B(x_0, 2r_k)). \end{aligned}$$

Now if k is so large that $\inf_{j \in J_k} t_{k,j} = \lfloor N^{-1} 2^{k\beta} \rfloor 2^{-k\beta} \geq (2N)^{-1}$, say, then

$$\mathbb{Q}_{0,m}\left(B_{kj} \cap C \neq \emptyset \right) \leq \frac{2^{k\beta+1}}{\gamma_b} m(1) \left(\sup_{t \geq (2N)^{-1}} \|p_t\|_\infty \right) c_2 (2r_k)^{d_f} = c_{24} 2^{k(\beta - d_f)} k^{d_f/(\beta\gamma)},$$

where $c_{24} = 2m(1)\gamma_b^{-1}(\sup_{t\geq(2N)^{-1}}\|p_t\|_\infty)c_2c_{23}^{d_f}$ is finite because $\|p_t\|_\infty \leq \left\|p_{t\wedge t_0}\right\|_\infty \leq c_3(t\wedge t_0)^{-d_f/\beta}$ for all $t > 0$ by (1.4) and (1.2). Plug this bound into (5.5), and use (5.4) and $|J_k| \leq N2^{k\beta}$, to obtain

$$\begin{aligned}
\mathbb{Q}_{0,m}\left(\mathcal{H}^{D+2\varepsilon}(\mathcal{T}(A)\cap[N^{-1},N])\right) &\leq c_{24}\varliminf_{n\to\infty}\sum_{k\geq n}\sum_{\substack{C\in\mathcal{C}_n\\|C|=2^{-k}}}\sum_{j\in J_k}2^{-k\beta(D+2\varepsilon)}2^{k(\beta-d_f)}k^{d_f/(\beta\gamma)}\\
&\leq c_{24}N\varliminf_{n\to\infty}\sum_{k\geq n}m_k^n k^{d_f/(\beta\gamma)}2^{k\beta(2-d_f/\beta-D-2\varepsilon)}\\
&\leq c_{24}N\varliminf_{n\to\infty}\sum_{k\geq n}m_k^n k^{d_f/(\beta\gamma)}2^{-k(d_A+2\beta\varepsilon)}\\
&\leq c_{24}N\underbrace{\left(\sup_{k\in\mathbb{N}}k^{d_f/(\beta\gamma)}2^{-k\beta\varepsilon}\right)}_{<\infty}\underbrace{\varliminf_{n\to\infty}\sum_{k\geq n}m_k^n 2^{-k(d_A+\beta\varepsilon)}}_{=0\text{ by (5.3)}}\\
&= 0,
\end{aligned}$$

where we used in the third step that $D \geq 2-\frac{d_f-d_A}{\beta}$, so $2-\frac{d_f}{\beta}-D \leq -\frac{d_A}{\beta}$. This implies that $\mathcal{H}^{D+2\varepsilon}(\mathcal{T}(A)\cap[N^{-1},N]) = 0$ $\mathbb{Q}_{0,m}$-a.s. and finishes the proof. □

To prove the lower bounds in Theorem 2.5, we use an idea similar to Serlet's [34] and apply what is sometimes called the *energy method*: If γ is a finite non-zero Borel measure on $\mathbb{R}$ with $\iint|t-s|^{-\theta}\gamma(\mathrm{d}s)\gamma(\mathrm{d}t) < \infty$ for some $\theta\in(0,1]$, then $\dim_H \mathrm{S}(\gamma)\geq\theta$ (see e.g. [13, Theorem 4.13]). For $\lambda\in\mathcal{M}_F(E)$ and $s>0$, put $p_s^\lambda(\cdot) := \int_E p_s(x,\cdot)\lambda(\mathrm{d}x)$.

Proposition 5.1. *Let $\mu\in\mathcal{M}_F(E)$ and $\delta>0$, and suppose that $\lambda\in\mathcal{M}_F(E)$ is non-zero and satisfies $\lambda(B(x,r))\leq c_{25}r^d$ for all $x\in E$ and $r\in(0,r_\lambda)$, for some $d\in[0,d_f]$ with $(d_f-d)/\beta<2$ and some $c_{25}, r_\lambda>0$. Then there is a random Borel measure γ on $[\delta,\infty)$ such that the following hold.*

(i) There is a sequence $h_k\to 0$ such that $\mathbb{P}^\mu$-a.s. for all rational $\delta\leq s<t$,

$$\gamma((s,t]) = \lim_{k\to\infty}\int_s^t X_r(p_{h_k}^\lambda)\,\mathrm{d}r.$$

(ii) For $\delta\leq s<t$,

$$\mathbb{E}^\mu\left[\gamma((s,t])\right] = \int_s^t \mu(p_r^\lambda)\,\mathrm{d}r.$$

(iii) If λ has compact support, then $\mathbb{P}^\mu$-a.s. $\mathrm{S}(\gamma)\subset\mathcal{T}(\mathrm{S}(\lambda))\cup J$ for a random countable set J.

(iv) If $\theta\in\left(0,1\wedge\left(2-\frac{d_f-d}{\beta}\right)\right)$ and $T>0$, then

$$\mathbb{E}^\mu\left[\int_\delta^T\int_\delta^T|t-s|^{-\theta}\,\gamma(\mathrm{d}s)\gamma(\mathrm{d}t)\right]<\infty.$$

In particular, if λ has compact support then $\mathbb{P}^\mu$-a.s. $\gamma\neq 0$ implies $\dim_H\mathcal{T}(\mathrm{S}(\lambda))\geq 1\wedge\left(2-\frac{d_f-d}{\beta}\right)$.

The random set J in (iii) is the set of discontinuities of the support process of X, which we briefly introduce. Denote by $\mathcal{K}(E)$ the set of non-empty compact subsets of E, which is complete and separable [2, Theorems 15.3 and 15.6] when endowed with the Hausdorff metric

$$d(F,K) = \left(\sup_{x\in F}d(x,K)\vee\sup_{x\in K}d(x,F)\right)\wedge 1,\qquad F,K\in\mathcal{K}(E). \tag{5.7}$$

In other words, $d(F,K)<\varepsilon$ for $\varepsilon\in(0,1)$ if and only if $K\subset F^\varepsilon$ and $F\subset K^\varepsilon$ (recall the meaning of K^ε from (3.3)). Since the mapping $\{\lambda\in\mathcal{M}_F(E)\colon \mathrm{S}(\lambda)\in\mathcal{K}(E)\}\to\mathcal{K}(E);\ \lambda\mapsto\mathrm{S}(\lambda)$ is Borel measurable [6, Theorem 4.4.1], and $\mathrm{S}(X_t)\in\mathcal{K}(E)$ for all $t>0$ a.s., the *support process* $(\mathrm{S}(X_t))_{t\geq0}$ of X is (after modification on a null set) a $\mathcal{K}(E)$-valued process.

Lemma 5.2. *For $\mu \in \mathcal{M}_F(E)$, $\mathbb{P}^\mu$-a.s., $(\mathrm{S}(X_t))_{t\geq 0}$ is right-continuous with left limits, hence has only countably many discontinuities.*

Proof. Right-continuity and the existence of left limits are respectively proved in [7, Theorem 1.2] and [30, Lemma 4.1] for superBrownian motion, but the proofs work in generality as long as the assertion of Corollary 3.4 (ii) holds. The fact that a right-continuous path with left limits in a Polish space has only countably many discontinuities is standard. □

Proof of Proposition 5.1. Recall that $p_s^\lambda(\cdot) = \int_E p_s(x,\cdot)\lambda(\mathrm{d}x)$ for $s > 0$, and put

$$L^h(t) := \int_\delta^t X_r(p_h^\lambda)\,\mathrm{d}r, \quad h > 0.$$

We can then repeat the exact same arguments used to show existence of local times (see Propositions 4.1 and 4.2, and (4.4)) to prove that there are random variables $(L(t))_{t\geq\delta}$ with

$$\sup_{\delta\leq t\leq T} \mathbb{E}^\mu\left[\left|L^h(t) - L(t)\right|^2\right] \to 0, \qquad T > 0. \tag{5.8}$$

When copying the proof,

(i) Replace p_s^x by p_s^λ, $q_{s,t}(x,\cdot)$ by $q_{s,t}^\lambda(\cdot) := \int_s^t p_r^\lambda(\cdot)\,\mathrm{d}r$, q_s^x by $q_s^\lambda := q_{0,s}^\lambda$, for $x \in E$, $0 \leq s \leq t$. In particular, $(\mu q_{s,t})(x)$ is replaced by $\mu(q_{s,t}^\lambda)$. Also note that $P_s p_r^\lambda = p_{s+r}^\lambda$ and $P_s q_{a,b}^\lambda = q_{a+s,b+s}^\lambda$ for $r, s > 0$, $0 \leq a < b$, by (1.4) and Tonelli's theorem.

(ii) Discard all suprema taken over $x \in E$ (in particular, $\|(\mu q_{s,t})\|_\infty$ is replaced by $\mu(q_{s,t}^\lambda)$),

(iii) In place of $\|p_s\|_\infty \leq c_3(s\wedge t_0)^{-d_f/\beta}$, use the bound $\left\|p_s^\lambda\right\|_\infty \leq c_{27}(s\wedge s_0)^{-\alpha}$ for constants $\alpha \in (0,2)$, $s_0, c_{27} > 0$, see Lemma 5.3 below.

(iv) In (4.3), the bound

$$\int p_s^x(z)p_{s'}^x(z)\nu(\mathrm{d}z) = p_{s+s'}(x,x) \leq c_3(s+s')^{-d_f/\beta}, \quad s,s' \in (0,t_0/2),$$

can be replaced by

$$\int p_s^\lambda(z)p_{s'}^\lambda(z)\nu(\mathrm{d}z) = \int\int \lambda(\mathrm{d}x)\lambda(\mathrm{d}y)p_{s+s'}(x,y) \leq \lambda(1)\left\|p_{s+s'}^\lambda\right\|_\infty \leq c_{25}\lambda(1)(s+s')^{-\alpha}, \quad s,s,' \in (0,s_0/2).$$

By a diagonal argument, we then find a sequence $h_k \to 0$ such that $L^{h_k}(q) \to L(q)$ holds $\mathbb{P}^\mu$-a.s. simultaneously for all $q \in \mathbb{Q}\cap[\delta,\infty)$. In particular, $\mathbb{P}^\mu$-a.s. L is increasing when restricted to rationals.

Define a random set function γ by $\gamma((s,t]) = L(t) - L(s)$ for rational $\delta \leq s < t$. To show that it extends uniquely to a random Borel measure on $[\delta,\infty)$, it suffices to prove that $\mathbb{P}^\mu$-a.s., if $s_n, s, u_n, u \in [\delta,\infty)$, $n \in \mathbb{N}$, are rational with $s_n \downarrow s$ and $u_n \downarrow u$, then $\gamma((s_n,u]) \to \gamma((s,u])$, and $\gamma((s,u_n]) \to \gamma((s,u])$ (see e.g. [9, Lemma 5.1]). By monotonicity of L on rationals, it suffices to show that for any given $t \in [\delta,\infty)\cap\mathbb{Q}$, there $\mathbb{P}^\mu$-a.s. exists a rational sequence $t_n \downarrow t$ with $L(t_n) \to L(t)$, and similarly for left-sided limits. So let $t_n, t \geq \delta$ be rational numbers with $t_n \downarrow t$. Then,

$$\begin{aligned}
\mathbb{E}^\mu\left[|L(t_n) - L(t)|\right] &= \mathbb{E}^\mu\left[\lim_{k\to\infty}\int_t^{t_n} X_r(p_{h_k}^\lambda)\,\mathrm{d}r\right] \\
&\leq \varliminf_{k\to\infty}\int_t^{t_n} \mathbb{E}^\mu\left[X_r(p_{h_k}^\lambda)\right]\mathrm{d}r \\
&= \varliminf_{k\to\infty}\int_t^{t_n} \mu(p_{r+h_k}^\lambda)\,\mathrm{d}r \\
&\leq \mu(1)(t_n - t)\sup_{s\geq\delta}\left\|p_s^\lambda\right\|_\infty,
\end{aligned}$$

which tends to zero as $n \to \infty$. In particular, there is a subsequence $(t_{k(n)})$ of (t_n) such that $L(t_{k(n)}) \to L(t)$ $\mathbb{P}^\mu$-a.s. An analogous argument works for left-sided limits, so we conclude the existence of a random measure γ on $[\delta, \infty)$ with $\gamma((s,t]) = L(t) - L(s) = \lim_{k\to\infty} \int X_r(p^\lambda_{h_k})\,\mathrm{d}r$ for rational $\delta < s \leq t$. Note that $\gamma((s,t])$ is measurable (as a function from the underlying probability space to the reals) for rational $\delta \leq s < t$, hence by a monotone class argument, $\gamma(B)$ is measurable for all Borel sets $B \subset [\delta, \infty)$. This finishes the proof of (i).

If $\delta \leq s < t$ are rational, then by definition of γ, and the fact that $L^h(r) \longrightarrow L(r)$ in L^2 and thus in L^1 for all $r \geq \delta$ by (5.8),

$$\begin{aligned}\mathbb{E}^\mu\left[\gamma((s,t])\right] = \mathbb{E}^\mu\left[L(t) - L(s)\right] = \lim_{h\to 0} \mathbb{E}^\mu\left[L^h(t) - L^h(s)\right] &= \lim_{h\to\ 0} \int_s^t \mathbb{E}^\mu\left[X_r(p^\lambda_h)\right]\mathrm{d}r \\ &\overset{(A.1)}{=} \lim_{h\to 0} \int_s^t \mu(p^\lambda_{r+h})\,\mathrm{d}r \\ &= \int_s^t \mu(p^\lambda_r)\,\mathrm{d}r,\end{aligned}$$

where we used dominated convergence in the final step. The claim for general t and s follows by approximation through rationals and monotone convergence.

If $\delta \leq t \in \mathcal{T}(\mathrm{S}(\lambda))^c$, then there is an $\varepsilon > 0$ such that $\mathrm{S}(X_t)^\varepsilon \cap \mathrm{S}(\lambda) = \emptyset$ (both are compact sets). If $s \mapsto \mathrm{S}(X_s)$ is continuous at t (recall this is w.r.t. the metric defined in (5.7)), then there are rational $t_0 < t < t_1$ such that $\bigcup_{t_0<r<t_1} \mathrm{S}(X_r) \subset \mathrm{S}(X_t)^{\varepsilon/2}$, so $\rho(x,y) \geq \varepsilon/2$ for all $y \in \mathrm{S}(\lambda)$ and $x \in \mathrm{S}(X_r)$, $r \in (t_0, t_1)$. Thus, $\mathbb{P}^\mu$-a.s.,

$$\begin{aligned}\gamma((t_0,t_1]) &= \lim_{k\to\infty} \int_{t_0}^{t_1} X_r(p^\lambda_{h_k})\,\mathrm{d}r \\ &= \lim_{k\to\infty} \int_{t_0}^{t_1} \mathrm{d}r \int_E X_r(\mathrm{d}x) \int_E \lambda(\mathrm{d}y) p_{h_k}(x,y) \\ &\leq (t_1 - t_0)(\sup_{s\geq 0} X_s(1))\lambda(1) \varlimsup_{k\to\infty} \Big(\sup_{\rho(x,y)\geq\varepsilon/2} p_{h_k}(x,y)\Big) \\ &= 0,\end{aligned}$$

so $t \in \mathrm{S}(\gamma)^c$. This implies that, $\mathbb{P}^\mu$-a.s., $\mathrm{S}(\gamma) \subset \mathcal{T}(\mathrm{S}(\lambda)) \cup J$ for the set J of discontinuities of $(\mathrm{S}(X_s))_{s\geq 0}$, which is countable by Lemma 5.2.

To prove (iv), first note that (i) implies the $\mathbb{P}^\mu$-almost sure weak convergence of the random measures γ_k, $k \in \mathbb{N}$, on $[\delta, \infty)$ defined by $\gamma_k(\mathrm{d}t) := X_t(p^\lambda_{h_k})\,\mathrm{d}t$ to γ as $k \to \infty$. Hence also $\gamma_k \otimes \gamma_k \to \gamma \otimes \gamma$ weakly as $k \to \infty$, so for any measurable $f\colon [\delta,\infty) \times [\delta,\infty) \to [0,\infty]$ we have $\iint f\,\mathrm{d}(\gamma\otimes\gamma) \leq \varliminf_{k\to\infty} \iint f\,\mathrm{d}(\gamma_k \otimes \gamma_k)$. In particular,

$$\begin{aligned}\mathbb{E}^\mu\left[\int_\delta^T\int_\delta^T |t-s|^{-\theta}\,\gamma(\mathrm{d}t)\gamma(\mathrm{d}s)\right] &\leq \varliminf_{k\to\infty} \mathbb{E}^\mu\left[\int_\delta^T\int_\delta^T |t-s|^{-\theta}\,\gamma_k(\mathrm{d}t)\gamma_k(\mathrm{d}s)\right] \\ &= 2\varliminf_{k\to\infty} \int_\delta^T \mathrm{d}t \int_\delta^t \mathrm{d}s\,|t-s|^{-\theta}\,\mathbb{E}^\mu\left[X_t(p^\lambda_{h_k})X_s(p^\lambda_{h_k})\right] \\ &= 2\varliminf_{k\to\infty} \int_{\delta/2}^{T-\delta/2} \mathrm{d}t \int_{\delta/2}^t \mathrm{d}s\,|t-s|^{-\theta}\,\mathbb{E}^\mu\left[\mathbb{E}^{X_{\delta/2}}\left[X_t(p^\lambda_{h_k})X_s(p^\lambda_{h_k})\right]\right],\end{aligned} \tag{5.9}$$

where we also applied Fatou's lemma, Tonelli's theorem, and the Markov property. Now for any $h > 0$ and $\delta/2 < s < t$, by (A.2),

$$\begin{aligned}\mathbb{E}^{X_{\delta/2}}\left[X_t(p^\lambda_h)X_s(p^\lambda_h)\right] &= X_{\delta/2}(p^\lambda_{t+h})X_{\delta/2}(p^\lambda_{s+h}) + \int_0^s X_{\delta/2}\left(T_r(p^\lambda_{t-r+h}p^\lambda_{s-r+h})\right)\mathrm{d}r \\ &\leq X_{\delta/2}(1)^2\lambda(1)^2(\sup_{u\geq\delta/2}\|p_u\|_\infty)^2 + \int_0^s X_{\delta/2}\left(T_{s-r}(p^\lambda_{t-s+r+h}p^\lambda_{r+h})\right)\mathrm{d}r.\end{aligned}$$

Thus and by (A.1),

$$\mathbb{E}^{\mu}\left[\mathbb{E}^{X_{\delta/2}}\left[X_t(p_h^\lambda)X_s(p_h^\lambda)\right]\right] \le \lambda(1)^2(\sup_{u\ge\delta/2}\|p_u\|_\infty)^2\mathbb{E}^{\mu}\left[X_{\delta/2}(1)^2\right] + \int_0^s \mathbb{E}^{\mu}\left[X_{\delta/2}\left(T_{s-r}(p^\lambda_{t-s+r+h}p^\lambda_{r+h})\right)\right]$$
$$= \lambda(1)^2(\sup_{u\ge\delta/2}\|p_u\|_\infty)^2\left(\mu(1)^2+(\delta/2)^2\right) + \underbrace{\int_0^s \mu\left(T_{s-r+\delta/2}(p^\lambda_{t-s+r+h}p^\lambda_{r+h})\right)\mathrm{d}r}_{(\star)},$$

where we used that $\mathbb{E}^{\mu}\left[X_u(1)^2\right] = \mu(1)^2 + u$ for $u \ge 0$, which is an immediate consequence of (A.2). The first term is independent of t, s, and h, and since $\theta < 1$, its contribution to the integral on the RHS of (5.9) is finite and bounded uniformly in k.

We can rewrite and upper bound the second term by

$$\begin{aligned}(\star) &= \int_0^s \mathrm{d}r \int \mu(\mathrm{d}x)\int \nu(\mathrm{d}y)p_{s-r+\delta/2}(x,y)p^\lambda_{t-s+r+h}(y)p^\lambda_{r+h}(y)\\
&\le \mu(1)(\sup_{u\ge\delta/2}\|p_u\|_\infty)\int_0^s \mathrm{d}r\int \nu(\mathrm{d}y)p^\lambda_{t-s+r+h}(y)p^\lambda_{r+h}(y)\\
&\le \mu(1)(\sup_{u\ge\delta/2}\|p_u\|_\infty)\int_0^{T+1} \mathrm{d}r\int \nu(\mathrm{d}y)p^\lambda_{t-s+r}(y)p^\lambda_{r}(y)\\
&= \mu(1)(\sup_{u\ge\delta/2}\|p_u\|_\infty)\int_0^{T+1} \mathrm{d}r\iint \lambda(\mathrm{d}x)\lambda(\mathrm{d}y)p_{t-s+2r}(x,y).\end{aligned}$$

We claim that for any $\eta \in (0,1)$, there is $c_{26} > 0$ such that for all $u > 0$,

$$\iint p_u(x,y)\lambda(\mathrm{d}x)\lambda(\mathrm{d}y) \le c_{26}(1 \vee u^{-(d_f-\eta d)/\beta}). \tag{5.10}$$

If this has been shown, we can conclude that the contribution of $(\star)$ to the integral on the RHS of (5.9) is no larger than

$$\mu(1)(\sup_{u\ge\delta/2}\|p_u\|_\infty)c_{26}\int_0^T \mathrm{d}t\int_0^t \mathrm{d}s\,|t-s|^{-\theta}\int_0^{T+1}\mathrm{d}r\left[1\vee(t-s+2r)^{-(d_f-\eta d)/\beta}\right]$$
$$= \mu(1)(\sup_{u\ge\delta/2}\|p_u\|_\infty)c_{26}\int_0^T \mathrm{d}t\int_0^t \mathrm{d}s\int_0^{T+1}\mathrm{d}r\,s^{-\theta}\left[1\vee(s+2r)^{-(d_f-\eta d)/\beta}\right],$$

where we substituted $s \to t-s$. This is independent of h, and finite if $\theta < 2 - d_f/\beta + \eta d/\beta$, which can be arranged by choosing η sufficiently close to 1. This proves (iv). In particular, $\int_\delta^T\int_\delta^T |t-s|^{-\theta}\,\gamma(\mathrm{d}s)\gamma(\mathrm{d}t) < \infty$ $\mathbb{P}^\mu$-a.s., so by the energy method $\dim_H \mathrm{S}(\gamma\restriction_{[\delta,T]}) \ge \theta$ whenever $\gamma\restriction_{[\delta,T]} \ne 0$, $\mathbb{P}^\mu$-a.s. If $\gamma \ne 0$ then this is the case for sufficiently large T, hence in that case by (iii),

$$\dim_H \mathcal{T}(\mathrm{S}(\lambda)) = \dim_H(\mathcal{T}(\mathrm{S}(\lambda))\cup J) \ge \dim_H \mathrm{S}(\gamma) = \sup_{T>0}\dim_H \mathrm{S}(\gamma\restriction_{[\delta,T]}) \ge \theta$$

for all $\theta \in (0, 1 \wedge (2-(d_f-d)/\beta)$.

It remains to prove the claim surrounding (5.10). Let $\eta \in (0,1)$ and $u \in (0,t_0)$. For an $R = R(\eta,u) \in (0,r_\lambda)$ that we have yet to choose, we can bound

$$\begin{aligned}\iint p_u(x,y)\lambda(\mathrm{d}x)\lambda(\mathrm{d}y) &\le \int c_3u^{-d_f/\beta}\lambda(B(x,R))\lambda(\mathrm{d}x) + c_3\lambda(1)^2u^{-d_f/\beta}\exp\left(-c_4R^{\beta\gamma}u^{-\gamma}\right)\\
&\le c_3\lambda(1)(c_{25}+\lambda(1))u^{-d_f/\beta}\left(R^d + \exp\left(-c_4R^{\beta\gamma}u^{-\gamma}\right)\right).\end{aligned} \tag{5.11}$$

Now choose $R = u^{\eta/\beta}$ (for u so small that $R < r_\lambda$), then

$$R^d + \exp\left(-c_4R^{\beta\gamma}u^{-\gamma}\right) = u^{\eta d/\beta} + \exp\left(-c_4u^{-\gamma(1-\eta)}\right) \le 2u^{\eta d/\beta}$$

for sufficiently small u. Combining with (5.11) yields the claim for $u \in (0, u_0)$ for some $u_0 > 0$. If $u > u_0$, then

$$\iint p_u(x,y)\lambda(\mathrm{d}x)\lambda(\mathrm{d}y) \le \lambda(1)^2 (\sup_{u\ge u_0} \|p_u\|_\infty).$$

This proves the claim with $c_{26} := \lambda(1)^2(\sup_{u\ge u_0}\|p_u\|_\infty) \vee 2c_3\lambda(1)(c_{25}+\lambda(1))$. □

Lemma 5.3. *Under the assumptions of Proposition 5.1, there exist $\alpha \in (0,2)$, $s_0 > 0$, and $c_{27} > 0$ such that*

$$\left\|p_s^\lambda\right\|_\infty \le c_{27}(s \wedge s_0)^{-\alpha}, \quad s > 0.$$

Proof. Recall that $p_s(x,y) \le \overline{p}_s(\rho(x,y))$ for $x,y \in E$ and $s \in (0,t_0)$, where

$$\overline{p}_s(r) := c_3 s^{-d_f/\beta} \exp\left(-c_4 (r^\beta/s)^\gamma\right), \quad s, r > 0.$$

Since $\overline{p}_s(\cdot)$ is decreasing, we have in fact that $\sup_{\rho(x,y)\ge r} p_s(x,y) \le \overline{p}_s(r)$ for all $r > 0$ and $s \in (0,t_0)$. Now fix an $\varepsilon > 0$ that we have yet to specify, and put $R := R(s) := s^{(1-\varepsilon)/\beta}$ for $s > 0$, so that for any fixed $a > 0$,

$$\overline{p}_s(aR(s)) = c_3 s^{-d_f/\beta}\exp\left(-c_4 a^{\beta\gamma} s^{-\varepsilon\gamma}\right) \le 1$$

for sufficiently small s. Then, again for sufficiently small s, and any $x \in E$,

$$\begin{aligned}
p_s^\lambda(x) &= \int p_s(x,y)\lambda(\mathrm{d}y) \\
&\le \lambda(1)\overline{p}_s(R) + \sum_{n=0}^\infty \int_{B(x,R2^{-n})\setminus B(x,R2^{-(n+1)})} p_s(x,y)\lambda(\mathrm{d}y) \\
&\le \lambda(1)\overline{p}_s(R) + \sum_{n=0}^\infty \left(\lambda(B(x,R2^{-n})) - \lambda(B(x,R2^{-(n+1)}))\right)\overline{p}_s(R2^{-(n+1)}) \\
&\le \lambda(1) + \lambda(B(x,R))\overline{p}_s(R/2) + \sum_{n=1}^\infty \lambda(B(x,R2^{-n}))\left(\overline{p}_s(R2^{-(n+1)}) - \overline{p}_s(R2^{-n})\right) \qquad (5.12)\\
&\le 2\lambda(1) + \sum_{n=1}^\infty c_{25}(R2^{-n})^d c_3 s^{-d_f/\beta} \underbrace{\exp\left(-c_4 R^{\beta\gamma}2^{-(n+1)\beta\gamma}s^{-\gamma}\right)}_{\le 1} \underbrace{\left[1-\exp\left(-c_4R^{\beta\gamma}2^{-(n+1)\beta\gamma}s^{-\gamma}\right)\right]}_{\le c_4R^{\beta\gamma}2^{-(n+1)\beta\gamma}s^{-\gamma}} \\
&\le 2\lambda(1) + c_{25}c_3R^d s^{-d_f/\beta}R^{\beta\gamma}s^{-\gamma}\sum_{n=1}^\infty 2^{-nd}2^{-(n+1)\beta\gamma} \\
&\le c_{27}s^{(d-d_f)/\beta-\varepsilon(\gamma+d/\beta)},
\end{aligned}$$

where $c_{27} = 2\lambda(1) + c_3c_{25}2^{-\beta\gamma}\sum_{n=1}^\infty 2^{-n(d+\beta\gamma)}$. Since $(d_f - d)/\beta < 2$, we can choose $\varepsilon > 0$ so small that $\alpha := (d_f-d)/\beta + \varepsilon(\gamma + d/\beta) < 2$. Say (5.12) holds for $s \in (0, s_0]$. Then if $s > s_0$, by (1.4),

$$\int p_s(x,x_0)\lambda(\mathrm{d}x) = \int \nu(\mathrm{d}y)p_{s-s_0}(y,x_0)\left(\int p_{s_0}(x,y)\lambda(\mathrm{d}x)\right) \le c_{27}s_0^{-\alpha}\left(\int p_{s-s_0}(y,x_0)\nu(\mathrm{d}y)\right) = c_{27}s_0^{-\alpha}.$$

□

The lower bounds in Theorem 2.5 are now a consequence of Proposition 5.1 and Frostman's lemma.

Proof of Theorem 2.5, Lower Bounds. Let $A \subset E$ be analytic with Hausdorff dimension $d_A := \dim_H A$ such that $d_s/2 - d_A/\beta = (d_f - d_A)/\beta < 2$ (otherwise the claim is trivial). We prove that, if $s \in (0, d_A]$ with $\mathcal{H}^s(A) > 0$, then $\dim_H \mathcal{T}(A) \ge 1 \wedge (2 - (d_f - s)/\beta) = (2 - d_s/2 + s/\beta)$ with positive probability. Since $\mathcal{H}^s(A) = \infty$ for all $s \in (0,d)$, this implies both (ii) and (iii).

If $s \in (0, d_A]$ with $\mathcal{H}^s(A) > 0$, then by [21, Corollary 7] there exists a compact subset $K \subset A$ with $\mathcal{H}^s(K) \in (0, \infty)$, and by a version of Frostman's lemma (e.g. [29, Theorem 8.17]), $\lambda := \mathcal{H}^s(K \cap \cdot)$ satisfies the assumptions of Proposition 5.1. Then, for $\delta > 0$ which we choose later, denote by γ the random measure on $[\delta, \infty)$ whose existence is asserted by Proposition 5.1. Then, $\mathbb{P}^\mu$-a.s. if $\gamma \neq 0$,

$$\dim_H \mathcal{T}(A) \geq \dim_H \mathcal{T}(\mathrm{S}(\lambda)) \geq 1 \wedge \left(2 - \frac{d_f - s}{\beta}\right).$$

It remains to show that $\gamma \neq 0$ with positive probability. By assumption, we can choose $\delta > 0$ such that $\int_\delta^\infty \mathrm{d}t \int \mu(\mathrm{d}x) p_t(x, y) > 0$ for all $y \in A$, so Proposition 5.1(ii) yields (with monotone convergence)

$$\mathbb{E}^\mu \left[\gamma((\delta, \infty))\right] = \int_\delta^\infty \mu(p_r^\lambda)\, \mathrm{d}r = \int \underbrace{\left(\int_\delta^\infty \mathrm{d}t \int \mu(\mathrm{d}x) p_t(x, y)\right)}_{>0 \text{ for all } y \in A} \lambda(\mathrm{d}y) > 0.$$

□

The lower bound in the singleton case, Theorem 2.7, follows from the Hölder continuity of the local times and the following fact, which is an elementary exercise in Hausdorff measures.

Lemma 5.4. *Let $F \subset \mathbb{R}$ be closed, $I \supset F$ an interval, and suppose there exists an increasing, non-constant function $f: I \to \mathbb{R}$ which is locally β'-Hölder continuous for some $\beta' \in (0, 1)$, and satisfies*

$$\forall a \leq b,\ a, b \in \mathbb{Q},\ [a, b] \subset I \setminus F:\ f(b) - f(a) = 0.$$

Then $\mathcal{H}^{\beta'}(F) > 0$, in particular $\dim_H F \geq \beta'$.

Proofs of Theorem 2.7 and Theorem 2.1 (iv). Fix $\mu \in \mathcal{M}_F(E)$ and $x_0 \in E$, put $\mathcal{T} := \mathcal{T}(x_0)$ and $d(t) := \rho(x_0, \mathrm{S}(X_t))$ for $t \geq 0$, so that $\mathcal{T} = \{t \geq 0 : d(t) = 0\}$. Consider an event Ω' of $\mathbb{P}^\mu$-probability one on which the assertion of Corollary 3.4 (ii) holds (with some fixed, sufficiently large $c > 0$). Put

$$J := \left\{t \geq 0 : d(t) > 0 \text{ and } \varliminf_{s \to t} d(s) = 0\right\},$$

$$\mathcal{T}' := \mathcal{T} \cup J = \left\{t \geq 0 : \varliminf_{s \to t} d(s) = 0\right\}.$$

Then $\mathcal{T}'$ is closed and $\dim_H \mathcal{T}' = \dim_H \mathcal{T}$ because J is countable. Indeed, for every $t \in J$, Corollary 3.4 (ii) implies that there exists $\varepsilon > 0$ (depending on t and $\omega \in \Omega'$) such that $(t, t + \varepsilon) \cap J = \varnothing$. Now suppose that $\delta > 0$ and $t > \delta$ with $L_\delta(t, x_0) > 0$. Then $L_\delta(\cdot, x_0): [\delta, \infty) \to \mathbb{R}$ is a non-constant, increasing function, and whenever $a, b \in \mathbb{Q}$, $[a, b] \subset [\delta, \infty) \setminus \mathcal{T}'$, there is $r = r(a, b, \omega) > 0$ with $d(s) \geq r$ for all $s \in [a, b]$. Indeed, otherwise there would be $(s_n) \in [a, b]^\mathbb{N}$ with $d(s_n) \longrightarrow 0$, say $s_n \longrightarrow s \in [a, b]$ by passing to a subsequence; but this would imply $s \in J$, which contradicts $[a, b] \cap J \subset [a, b] \cap \mathcal{T}' = \varnothing$. Omitting another null set, we can assume that $K := \sup_{s \geq 0} X_s(1) < \infty$, and, by Theorem 2.1 (i), that there is $\varepsilon_n \downarrow 0$ with $L_\delta(t, x_0) = \lim_{n \to \infty} \int_\delta^t X_s(p_{\varepsilon_n}(x_0, \cdot))\, \mathrm{d}s$ for all $t \in \mathbb{Q}_{\geq \delta}$. Then,

$$\begin{aligned}
L_\delta(b, x_0) - L_\delta(a, x_0) &= \lim_{n \to \infty} \int_a^b X_s(p_{\varepsilon_n}(x_0, \cdot))\, \mathrm{d}s \\
&\leq \varliminf_{n \to \infty} \left(K(b - a) \sup_{\rho(x, x_0) \geq r} \left|p_{\varepsilon_n}(x_0, x)\right|\right) \\
&\leq K(b - a) \varliminf_{n \to \infty} \left(c_3 \varepsilon_n^{-d_f/\beta} \exp\left(-c_4 r^{\beta\gamma} \varepsilon_n^{-\gamma}\right)\right) \\
&= 0.
\end{aligned}$$

Furthermore, $L_\delta(\cdot, x_0)$ is locally β'-Hölder continuous for every $\beta' < 1 \wedge (2 - d_s/2)$, so Lemma 5.4 implies $\dim_H \mathcal{T} = \dim_H \mathcal{T}' \geq \beta'$ a.s. for all such β'.

Now suppose $d_s \geq 2$ (otherwise $2 - d_s/2 > 1$ and Theorem 2.1 (iv) is trivial). By omitting another null set, assume that $\dim_H \mathcal{T} \leq (2 - d_s/2)$. Then, if there was a $\delta > 0$, $t > s \geq \delta$, and $\beta' > (2 - d_s/2)$ such that $L_\delta(\cdot, x_0)$ is β'-Hölder continuous and not constant on $[s, t]$, then Lemma 5.4 implies by the same argument we just used that $\dim_H \mathcal{T} = \dim_H \mathcal{T}' \geq \dim_H(\mathcal{T}' \cap [s,t]) \geq \beta' > 2 - d_s/2$, a contradiction of Theorem 2.5 (i). □

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Appendix A. Dynkin's Moment Formula

The purpose of this section is to derive (4.6) from Dynkin's moment formula [11, Theorem 1.7], to which end we introduce some notation. Denote by $\mathbb{D}_\Lambda$, for finite $\varnothing \neq \Lambda \subset \mathbb{N}$, the set of equivalence classes of connected, complete binary trees (complete means that every inner node has two children) with leaf set Λ, where two trees are called equivalent if they can be transformed into one another by a sequence of flips, that is, by swapping the left and right children of a number of nodes at arbitrary levels of the tree. Write $|D| := |\Lambda|$ for $D \in \mathbb{D}_\Lambda$. Call an equivalence class of a (possibly unconnected) complete binary tree a *forest*, and denote by $\mathbb{G}_\Lambda$ the set of forests with leaves Λ. For $G \in \mathbb{G}_\Lambda$, write $G = \{D_i\}_{i=1}^m$ if G has $m \in \mathbb{N}$ connected components $D_1, \ldots, D_m$. If $D \in \mathbb{D}_{\Lambda_1}, D' \in \mathbb{D}_{\Lambda_2}$, and $\Lambda_1 \cap \Lambda_2 = \emptyset$, denote by $D \vee D' \in \mathbb{D}_{\Lambda_1 \cup \Lambda_2}$ the tree obtained by taking a new root vertex and making the roots of D_1 and D_2 its children – the two ways of doing so result in equivalent trees. For $D \in \mathbb{D}_\Lambda$ with $|\Lambda| \geq 2$, there is a decomposition $D = D_1 \vee D_2$ (unique up to swapping D_1 and D_2), where $D_1 \in \mathbb{D}_{\Lambda_1}$, $D_2 \in \mathbb{D}_{\Lambda_2}$, $\Lambda = \Lambda_1 \cup \Lambda_2$, are the two trees that remain after removing the root in D. Write $\mathbb{D} := \bigcup_{\varnothing \neq \Lambda \subset \mathbb{N} \text{ finite}} \mathbb{D}_\Lambda$. If $t_i \in \mathbb{R}$, $f_i \colon E \to \mathbb{R}$, $i \in \mathbb{N}$, write $t_\Lambda := (t_i)_{i \in \Lambda}$, and $f_\Lambda := (f_i)_{i \in \Lambda}$. If we apply operations to either object we mean it in an entry-wise sense, that is, $t_\Lambda + t := (t_i + t)_{i \in \Lambda}$ for $t \in \mathbb{R}$, $\bigwedge t_\Lambda := \bigwedge_{i \in \Lambda} t_i$, $|f|_\Lambda := (|f_i|)_{i \in \Lambda}$, and so on. Finally, write $\mathbb{D}_n := \mathbb{D}_{\{1,\ldots,n\}}$ and $\mathbb{G}_n := \mathbb{G}_{\{1,\ldots,n\}}$ for $n \in \mathbb{N}$.

Definition A.1. *Let $t_i \geq 0$, $i \in \mathbb{N}$, and either $f_i \in B_+(E)$ for all $i \in \mathbb{N}$, or $f_i \in B_b(E)$ for all $i \in \mathbb{N}$. Recursively define, for a tree $D \in \mathbb{D}_\Lambda$, $\varnothing \neq \Lambda \subset \mathbb{N}$, a function $\overline{D}(t_\Lambda; f_\Lambda) \colon E \to \mathbb{R}$ by setting $\overline{D}(t; f) := P_t f$ if $t \geq 0$ and $|D| = 1$, and*

$$\overline{D}(t_\Lambda; f_\Lambda) := \gamma_b \int_0^{\bigwedge t_\Lambda} P_r\Big(\overline{D_1}(t_{\Lambda_1} - r; f_{\Lambda_1})\overline{D_2}(t_{\Lambda_2} - r; f_{\Lambda_2})\Big)\,\mathrm{d}r,$$

if $D = D_1 \vee D_2$, and $D_1 \in \mathbb{D}_{\Lambda_1}$, $D_2 \in \mathbb{D}_{\Lambda_2}$. It is clear inductively that $\overline{D}(t_\Lambda; f_\Lambda) \in B_+(E)$ (resp. $B_b(E)$). If $f_i = f$ for all $i \in \Lambda$, write $D(t_\Lambda; f) := D(t_\Lambda; f_\Lambda)$, analogously if all $t_i, i \in \Lambda$, are equal.

Proposition A.2 (Dynkin's Expression for Moments)**.** *Let $\mu \in \mathcal{M}_F(E)$, $n \in \mathbb{N}$, $s_1, \ldots, s_n \geq 0$, and either $f_1, \ldots, f_n \in B_+(E)$ or $f_1, \ldots, f_n \in B_b(E)$. Then,*

$$\mathbb{E}^\mu\left[\prod_{i=1}^n X_{s_i}(f_i)\right] = \sum_{\{D_i\}_{i=1}^m \in \mathbb{G}_n} \prod_{i=1}^m \mu\Big(\overline{D_i}(s_{\Lambda_i}; f_{\Lambda_i})\Big),$$

where, for $m \in \mathbb{N}$, and $\{D_i\}_{i=1}^m \in \mathbb{G}_n$, $\Lambda_i \subset [n]$ is the set of leaves of D_i.

Proof. Dynkin [11, Theorem 1.7] proves this if $f_i \in B_+(E)$ for $i \in [n]$, from which the case where $f_i \in B_b(E)$ follows by splitting into positive and negative parts. □

Corollary A.3. *Let $\mu \in \mathcal{M}_F(E)$, and $f, g \in B_+(E) \cup B_b(E)$. Then, for $t, s \geq 0$,*

$$\mathbb{E}^\mu\left[X_t(f)\right] = \mu\left(P_t f\right), \tag{A.1}$$

$$\mathbb{E}^\mu\left[X_t(f)X_s(g)\right] = \mu\left(P_t f\right)\mu\left(P_s g\right) + \gamma_b \int_0^{t\wedge s} \mu\left(P_r((P_{t-r}f)(P_{s-r}g))\right)\mathrm{d}s. \tag{A.2}$$

We now use Proposition A.2 to derive formula (4.6).

Definition A.4. *Let $f \in B_+(E)$ (resp. $B_b(E)$), and $t \geq 0$. Recursively define, for $D \in \mathbb{D}$, a function $D(t; f)\colon E \to \mathbb{R}$ by setting $D(t; f) := \int_0^t P_r f \,\mathrm{d}r$ if $|D| = 1$, and*

$$D(t; f) := \gamma_b \int_0^t P_r\Big(D_1(t-r; f) D_2(t-r; f)\Big)\,\mathrm{d}r,$$

if $D = D_1 \vee D_2$. Again, $D(t; f) \in B_+(E)$ (resp. $B_b(E)$).

Proposition A.5. *Suppose that $f \in B_+(E) \cup B_b(E)$, and $t \geq 0$. Then, for any $\mu \in \mathcal{M}_F(E)$,*

$$\mathbb{E}^\mu\left[\left(\int_0^t X_r(f)\,\mathrm{d}r\right)^n\right] = \sum_{\{D_i\}_{i=1}^m \in \mathbb{G}_n} \prod_{i=1}^m \mu\Big(D_i(t; f)\Big).$$

Proof. Abbreviate $\int_0^t \mathrm{d}s_\Lambda := \int_0^t \mathrm{d}s_{\lambda_1} \dots \int_0^t \mathrm{d}s_{\lambda_m}$ for $\Lambda = \{\lambda_1, \dots, \lambda_m\} \subset \mathbb{N}$, $m \in \mathbb{N}$. By Proposition A.2 and Fubini-Tonelli,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t X_r(f)\,\mathrm{d}r\right)^n\right] &= \int_0^t \mathrm{d}s_1 \dots \int_0^t \mathrm{d}s_n\, \mathbb{E}\left[\prod_{i=1}^n X_{s_i}(f)\right] \\
&= \sum_{\{D_j\}_{j=1}^m \in \mathbb{G}_n} \int_0^t \mathrm{d}s_1 \dots \int_0^t \mathrm{d}s_n \prod_{j=1}^m \mu\Big(\overline{D_j}(s_{\Lambda_j}; f)\Big) \\
&= \sum_{\{D_j\}_{j=1}^m \in \mathbb{G}_n} \prod_{j=1}^m \mu\left(\int_0^t \mathrm{d}s_{\Lambda_j}\, \overline{D_j}(s_{\Lambda_j}; f)\right).
\end{aligned}$$

It thus suffices to show that, for any $\varnothing \neq \Lambda \subset \mathbb{N}$, $D \in \mathbb{D}_\Lambda$,

$$\int_0^t \mathrm{d}s_\Lambda \overline{D}(s_\Lambda; f) = D(t; f), \tag{A.3}$$

which we prove by induction. If $|D| = 1$, then $D(t; f) = \int_0^t P_r f\,\mathrm{d}r = \int_0^t \overline{D}(r; f)\,\mathrm{d}r$. Now suppose $D = D_1 \vee D_2 \in \mathbb{D}_\Lambda$, and that (A.3) holds for $D_1 \in \mathbb{D}_{\Lambda_1}$ and $D_2 \in \mathbb{D}_{\Lambda_2}$, where $\Lambda = \Lambda_1 \cup \Lambda_2$. Then,

$$\begin{aligned}
\int_0^t \mathrm{d}s_\Lambda\, \overline{D}(s_\Lambda; f) &= \gamma_b \int_0^t \mathrm{d}s_\Lambda \int_0^{\wedge s_\Lambda} \mathrm{d}r\, P_r\Big(\overline{D_1}(s_{\Lambda_1} - r; f)\overline{D_2}(s_{\Lambda_2} - r; f)\Big) \\
&= \gamma_b \int_0^t \mathrm{d}r\, P_r\left(\int_r^t \mathrm{d}s_\Lambda\, \overline{D_1}(s_{\Lambda_1} - r; f)\overline{D_2}(s_{\Lambda_2} - r; f)\right) \\
&= \gamma_b \int_0^t \mathrm{d}r\, P_r\left(\int_0^{t-r} \mathrm{d}s_\Lambda\, \overline{D_1}\,(s_{\Lambda_1}; f)\,\overline{D_2}\,(s_{\Lambda_2}; f)\right) \\
&= \gamma_b \int_0^t \mathrm{d}r\, P_r\Big(D_1(t-r; f) D_2(t-r; f)\Big) \\
&= D(t; f).
\end{aligned}$$

□

Lemma A.6. *If $f \in B_+(E)$ and $D \in \mathbb{D}$, then $D(\cdot, f)$ is increasing (pointwise on E).*

Proof. If $|D| = 1$, then $D(t; f) = \int_0^t P_r f \, \mathrm{d}r$ for $t \geq 0$ and the claim follows because $f \geq 0$. If $D = D_1 \vee D_2$, and the claim holds for D_1, D_2, then, whenever $0 \leq s \leq t$,

$$\begin{aligned} D(s; f) &= \gamma_b \int_0^s P_r\Big(D_1(s-r; f) D_2(s-r; f)\Big) \, \mathrm{d}r \\ &\leq \gamma_b \int_0^s P_r\Big(D_1(t-r; f) D_2(t-r; f)\Big) \, \mathrm{d}r \\ &\leq \gamma_b \int_0^t P_r\Big(D_1(t-r; f) D_2(t-r; f)\Big) \, \mathrm{d}r \\ &= D(t; f). \end{aligned}$$

□

Appendix B. Proofs of Technical Lemmas

Lemma B.1. *There are $c_1, r_0 > 0$ such that $\nu(B(x, r)) \geq c_1 r^{d_f}$ for all $r \in (0, r_0)$. If $t_0 = \infty$ or* $\mathrm{diam}(E) < \infty$*, then* $r_0 = \mathrm{diam}(E)$.

Proof. Recall Lemma 3.2, and let $c > 0$ so small that $c_7 \mathrm{e}^{-c_8 c^{-\gamma}} \leq 1/2$, and put $r_0 := (t_0/c)^{1/\beta} \wedge \mathrm{diam}(E)$. For $r \in (0, r_0)$, put $t := t(r) := c r^\beta \in (0, t_0)$, then by Lemma 3.2 and (1.1),

$$1/2 \leq 1 - c_7 \mathrm{e}^{-c_8 r^{\beta\gamma} t^{-\gamma}} \leq \mathbb{P}^x\left(\rho(x, Y_t) \leq r\right) = \int_{B(x,r)} p_t(x, y) \nu(\mathrm{d}y) \leq c_3 t^{-d_f/\beta} \nu(B(x, r)) = c_3 c^{-d_f/\beta} r^{-d_f} \nu(B(x, r)),$$

which implies the claim with $c_1 = c^{d_f/\beta}/(2c_3)$. If $t_0 = \infty$ then $r_0 = \mathrm{diam}(E)$ by definition, and if $\mathrm{diam}(E) < \infty$ then we can arrange $r_0 = \mathrm{diam}(E)$ by adjusting c_1 to be sufficiently small. □

Recall that $C(E)$ denotes the space of continuous paths $[0, \infty) \to E$ with the topology of uniform convergence on compacts, and $\mathcal{M}_1(C(E))$ is the space of Borel probability measures on $C(E)$, equipped with the topology of weak convergence.

Lemma B.2. *If the assertion of Lemma 3.2 holds, then $E \to \mathcal{M}_1(C(E))$; $x \mapsto \mathbb{P}^x$ is continuous.*

Proof. Suppose $x_n, x \in E$, $n \in \mathbb{N}$, and $x_n \to x$, then we have to show that $\mathbb{P}^{x_n} \longrightarrow \mathbb{P}^x$. Since Y is Feller, it suffices to show that $(\mathbb{P}^{x_n})_{n \in \mathbb{N}}$ is tight, which is equivalent to

(i) $(\mathbb{P}^{x_n}(Y_0 \in \cdot))_{n \in \mathbb{N}} \subset \mathcal{M}_1(E)$ is tight,

(ii) For every $T > 0$ and $\varepsilon > 0$, $\sup_{n \in \mathbb{N}} \mathbb{P}^{x_n}(w(Y, \delta, T) \geq \varepsilon) \longrightarrow 0$ as $\delta \to 0$,

where $w(y, \delta, T) = \sup_{0 \leq s, t \leq T, |t-s| < \delta} \rho(y(t), y(s))$ for $y \in C(E)$. We have $\mathbb{P}^{x_n}(Y_0 = x_n) = 1$ for $n \in \mathbb{N}$ and (x_n) is bounded, so (i) is clear. For (ii), we use entropy numbers: Fix $\varepsilon > 0$, and define stopping times $(T_j)_{j \in \mathbb{N}_0}$ by $T_0 := 0$ and

$$\begin{aligned} T_j &:= \inf\left\{t > T_{j-1} \colon \rho\left(Y(T_{j-1}), Y(t)\right) \geq \varepsilon\right\}, \\ \mathcal{N}(\varepsilon, T) &:= \sup\left\{k \in \mathbb{N}_0 \colon T_k \leq T\right\}, \\ \delta(\varepsilon, T) &:= \inf\left\{T_j - T_{j-1} \colon 1 \leq j \leq \mathcal{N}(\varepsilon, T)\right\}. \end{aligned}$$

It follows from the definitions that, for $n \in \mathbb{N}$, and every $k \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}^{x_n}\big(w(Y, \delta, T) > 2\varepsilon\big) &\leq \mathbb{P}^{x_n}\left(\delta(\varepsilon, T) < \delta\right) \\ &\leq \underbrace{\mathbb{P}^{x_n}\left(\mathcal{N}(\varepsilon, T) > k\right)}_{\leq \mathbb{P}^{x_n}(T_k \leq T)} + \mathbb{P}^{x_n}\left(\exists j \leq k \colon T_{j-1} < \infty, T_j - T_{j-1} \leq \delta\right) \\ &\leq \sum_{j=1}^k \Big(\mathbb{P}^{x_n}\left(T_{j-1} < \infty, T_j - T_{j-1} \leq T/k\right) + \mathbb{P}^{x_n}\left(T_{j-1} < \infty, T_j - T_{j-1} \leq \delta\right)\Big). \end{aligned} \tag{B.1}$$

If $\eta > 0$ and $x \in E$, then by [3, Equation (3.11)],

$$\mathbb{P}^x(T_1 \leq \eta) \leq c_{28} \exp\left(-c_{29}\varepsilon^{\beta\gamma}\eta^{-\gamma}\right),$$

where $c_{28}, c_{29} > 0$ are constants independent of x, η, ε. Thus, and by the strong Markov property,

$$\mathbb{P}^{x_n}\left(T_{j-1} < \infty, T_j - T_{j-1} \leq \eta\right) = \mathbb{E}^{x_n}\left[\mathbb{1}_{\{T_{j-1}<\infty\}}\mathbb{P}^{Y_{T_{j-1}}}(T_1 \leq \eta)\right] \leq c_{28} \exp\left(-c_{29}\varepsilon^{\beta\gamma}\eta^{-\gamma}\right),$$

for $\eta > 0$ and $j \in \mathbb{N}$. Returning to (B.1),

$$\mathbb{P}^{x_n}(w(Y, \delta, T) > 2\varepsilon) \leq kc_{28}\left(\exp\left(-c_{29}\varepsilon^{\beta\gamma}T^{-\gamma}k^{\gamma}\right) + \exp\left(-c_{29}\varepsilon^{\beta\gamma}\delta^{-\gamma}\right)\right),$$

for every $\delta > 0$, $n \in \mathbb{N}$, and $k \in \mathbb{N}$. Thus,

$$\overline{\lim_{\delta\to 0}}\left(\sup_{n\in\mathbb{N}} \mathbb{P}^{x_n}(w(Y, \delta, T) > 2\varepsilon)\right) \leq kc_{28} \exp\left(-c_{29}\varepsilon^{\beta\gamma}T^{-\gamma}k^{\gamma}\right)$$

holds for all $k \in \mathbb{N}$, and the latter expression vanishes as $k \to \infty$, so (ii) follows. □

Proof of Lemma 4.5. Abbreviate $\xi(r) := r^a\left(\log\frac{1}{r}\right)^b$ for $r > 0$, and fix $K \geq \lambda(1)$. We will track dependence of constants only on r_1 and K, to show that c_{21} is independent of both and h_0 depends on λ only through K. For $h > 0$,

$$\|(\lambda q_h)\|_\infty = \sup_{y\in E}\int_0^h \mathrm{d}s \int_E p_s(x, y)\lambda(\mathrm{d}x) \leq \int_0^h \mathrm{d}s\left(\sup_{y\in E}\int_E p_s(x, y)\lambda(\mathrm{d}x)\right). \tag{B.2}$$

We now claim that there are $c_{30} > 0$ and $s_0 = s_0(r_1, K) > 0$ such that

$$\sup_{y\in E}\left(\int_E p_s(x, y)\lambda(\mathrm{d}x)\right) \leq c_{30}s^{(a-d_f)/\beta}\left(\log\frac{1}{s}\right)^{b+a/(\beta\gamma)}, \quad 0 < s < s_0. \tag{B.3}$$

Assuming this for the moment, and noting the easy fact that, for $u > -1$ and $v > 0$, there is a $c' = c'(u, v) > 0$ such that

$$\int_0^h s^u\left(\log\frac{1}{s}\right)^v \mathrm{d}s \leq c'h^{u+1}\left(\log\frac{1}{h}\right)^v, \quad 0 < h < 1/2,$$

then (B.2) and (B.3) finish the proof with $c_{21} = c_{30}c'((a - d_f)/\beta, b + a/(\beta\gamma))$ and $h_0 = s_0 \wedge 1/2$.

It remains to prove (B.3). Let $0 < s < t_0$ and $y \in E$. For any $0 < r < r_1$, assuming that $\lambda \neq 0$,

$$\begin{aligned}\int_E p_s(x, y)\lambda(\mathrm{d}x) &= \int_{B(y,r)} p_s(x, y)\lambda(\mathrm{d}x) + \int_{E\setminus B(y,r)} p_s(x, y)\lambda(\mathrm{d}x)\\ &\leq c_3s^{-d_f/\beta}\lambda(B(y, r)) + c_3s^{-d_f/\beta}\lambda(1)\exp\left(-c_4r^{\gamma\beta}s^{-\gamma}\right)\\ &\leq c_3Ks^{-d_f/\beta}\Big(\frac{c}{K}\xi(r) + \exp\left(-c_4r^{\gamma\beta}s^{-\gamma}\right)\Big).\end{aligned} \tag{B.4}$$

Choose $r = r(s) := s^{1/\beta}\left(\frac{a}{\beta c_4}\log\frac{1}{s}\right)^{1/(\gamma\beta)}$, so that $\exp\left(-c_4r^{\gamma\beta}s^{-\gamma}\right) = s^{a/\beta}$, and

$$r^a = s^{a/\beta}\left(\frac{a}{\beta c_4}\log\frac{1}{s}\right)^{a/(\beta\gamma)} = c_{31}s^{a/\beta}\left(\log\frac{1}{s}\right)^{a/(\beta\gamma)},$$

where $c_{31} = (a/(\beta c_4))^{a/(\beta\gamma)} > 0$. Note that $r \to 0$ as $s \to 0$, so there is an $s_1 = s_1(r_1) > 0$ such that $r < r_1$ whenever $s < s_1$. Let $s_2 = s_2(K) > 0$ such that $cK^{-1}\beta^{-b}c_{31}\left(\log\frac{1}{s}\right)^{b+a/(\beta\gamma)} \geq 1$ and $r = r(s) \geq s^{1/\beta}$ for $s < s_2$. Then, if

$0 < s < s_0 := s_1 \wedge s_2 \wedge t_0$,

$$
\begin{aligned}
cK^{-1}\xi(r) + \exp\left(-c_4 r^{\gamma\beta} s^{-\gamma}\right) &\leq cK^{-1} r^a \log\left(\frac{1}{s^{1/\beta}}\right)^b + s^{a/\beta} \\
&= cK^{-1}\beta^{-b} c_{31} s^{a/\beta} \left(\log\frac{1}{s}\right)^{b+a/(\beta\gamma)} + s^{a/\beta} \\
&\leq 2cK^{-1}\beta^{-b} c_{31} s^{a/\beta} \left(\log\frac{1}{s}\right)^{b+a/(\beta\gamma)}.
\end{aligned}
$$

Using this in (B.4) finishes the proof with $c_{30} = 2c_3 c_{31} c\beta^{-b}$. □

Lemma B.3. *If $\lambda > 0$ and $t \geq 2\lambda$, then $\mathbb{P}(Poi(\lambda) \geq t) \leq e^{-3/16t}$.*

Proof. By the well-known bound $\mathbb{P}(\mathrm{Poi}(\lambda) \geq \lambda + c) \leq \exp\left(-\frac{c^2}{2(\lambda + c/3)}\right)$,

$$
\mathbb{P}(\mathrm{Poi}(\lambda) \geq t) = \mathbb{P}(\mathrm{Poi}(\lambda) \geq \lambda + (t-\lambda)) \leq \exp\left(-\frac{(t-\lambda)^2}{2(\lambda + (t-\lambda)/3)}\right) = \exp\left(-\frac{(t/2)^2}{4t/3}\right) = \exp\left(-\frac{3}{16}t\right).
$$

□

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