Free Sugihara algebras: a demonstration of the power of duality theory to solve problems in logic and algebra

A Research Case Study by Hilary Priestley
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Sugihara algebras generalise Boolean algebras and provide algebraic semantics for the propositional relevant logic known as R-mingle. These semantics enable logical problems to be translated into algebraic form.

This strategy necessitates working with free Sugihara algebras, which play the role that the Lindenbaum–Tarski algebra does for classical Propositional Calculus. Such algebras are finite but get monstrously large and computational techniques are ruled out.

Duality theory comes to the rescue, by linking algebraic and relational models. To give the flavour:

- A viable strategy is developed for investigating admissible rules for R-mingle: small 'admissibility algebras' take the place of free algebras. For example,

  a 3-generated free algebra of size $2^{44} \cdot 3^{36} \cdot (1 + 2^{-3} \cdot 3^{-4})^6$

  is replaced by

  an algebra with 16 elements.

- A pictorial, order-theoretic, representation is found for finitely generated free Sugihara algebras. This is based on finite trees, and the structures are built in a recursive way which makes them easy to understand.

Below, Oxford mathematician Hilary Priestley reports on this recent work with collaborators Leonardo Cabrer, George Metcalfe and Joshua Wrigley. Leo is a former Oxford postdoc, George gained his first degree from Oxford in Mathematics & Philosophy and is now Professor of Mathematics at the University of Bern, Switzerland, and Joshua is a current MMath undergraduate who undertook a 2019 Summer Project. A [presentation] was given at the conference TACL 2019 (Topology, Algebras and Categories in Logic) in Nice.

The project combines aspects of logic, algebra, ordered structures, enumerative combinatorics and issues of computational complexity. Category theory, by way of dual equivalences, underpins the methodology and hands-on computer programming has pointed the way from special cases to general theory. It illustrates how, in 21st century mathematics and its applications, different techniques can work in partnership.

At the heart of the project are two classes of algebras, *Sugihara algebras* and *Sugihara monoids*. The name honours *Takeo Sugihara*, a Japanese philosopher who in the 1950s explored the logical notion of implication. A century earlier, George Boole had brought algebraic thinking to bear on logical reasoning. (Google acknowledged the influence of Boole's work on modern mathematical logic and on computer science on 2 November, 2015, the 200th anniversary of his birth: it amended its logo for the day and posted a Doodle.)

Loosely, algebraic semantics for a logic enable logical notions to be faithfully recast in algebraic form with, for example, axioms corresponding to algebraic laws (equations). In this sense, Boolean algebras model classical Propositional Calculus. In **CPC** the connectives $\to$ (implication), $\neg$ (negation), $\wedge$ (and) and $\vee$ (or) are closely tied together. The relationships of inter-definability can be weakened in a multitude of ways. The driver here may be philosophical or mathematical curiosity or the quest for models in computer science. In general, propositional logics can be studied syntactically, in terms of a formal language, or semantically, in terms of models, which might be algebras or relational structures. (Valuable information may be obtained by semantic means: for example, the study of modal logics was revolutionised by Kripke's introduction of relational (possible world) semantics.)

The logic R-mingle comes within the ambit of *relevant logics*. Its algebraic language includes symbols $\to$, $\neg$, $\vee$ and $\wedge$ and formulas are constructed in the usual way. The logic is specified by a consequence relation $\vdash$ from finite sets of formulas to formulas. The notion that the consequence relation respects 'relevance' is open to many interpretations but is regarded as less contentious in the presence of the *mingle axiom*: $\phi \to (\phi \to \phi)$. It has long been known that the class $\mathscr{SA}$ of Sugihara algebras provides algebraic semantics for R-mingle. (The class $\mathscr{SM}$ of Sugihara monoids is defined similarly. It models R-mingle afforced with Ackermann's truth constant, interpreted on the integers as 1. There are close affinities between the two scenarios and the same methodologies apply to both.)

Sugihara algebras are built from an algebra $\mathbf{Z}$ which plays a role analogous to that the two-element Boolean algebra $\mathbf{2}$ (the 'truth value algebra') plays for **CPC**. Like $\mathbf{2}$, the algebra $\mathbf{Z}$ has operations $\to$, $\neg$, $\wedge$ and $\vee$, to model logical connectives. But, reflecting the greater generality, $\mathbf{Z}$ is a richer algebra than $\mathbf{2}$. Its universe is $\mathbb{Z}$, the set of integers. Regarding $\mathbb{Z}$ as equipped with its usual total order, $\wedge$ and $\vee$ are min and max (lattice operations). The operation $\neg$ is given by $a \mapsto -a$ and the operation $\to$ by

$$a \to b = \begin{cases} (-a) \vee b & \text{if } a \leqslant b, \\ (-a) \wedge b & \text{otherwise.} \end{cases}$$

The class $\mathscr{SA}$ of Sugihara algebras is then $\mathbb{HSP}(\mathbf{Z})$ (*viz.* homomorphic images of subalgebras of powers of $\mathbf{Z}$, defined as expected). General theory implies that $\mathscr{SA}$ is the variety (equational class) consisting of all algebras of the same type as $\mathbf{Z}$ which satisfy all the laws that hold in $\mathbf{Z}$. It will be important that $\mathscr{SA}$ is locally finite, meaning that any finitely generated algebra is finite.

Now let $\mathbf{Z}_{2n+1} \in \mathbb{S}(\mathbf{Z})$ have universe $\mathbb{Z} \cap [-n, n]$, for $n \geqslant 1$. Let $\mathscr{SA}_{2n+1}$ be the quasivariety $\mathbb{ISP}(\mathbf{Z}_{2n+1})$, the class of isomorphic copies of subalgebras of powers of $\mathbf{Z}_{2n+1}$. In general a quasivariety is characterised by the quasi-identities it satisfies; a quasi-identity takes the form $\Sigma \Rightarrow \varphi \approx \psi$, where Σ is a finite set of terms and φ and ψ are terms. In fact $\mathscr{SA}_{2n+1}$ is a variety but focusing on quasi-identities rather than equations will allow the consequence relation on the logic side to be captured algebraically.

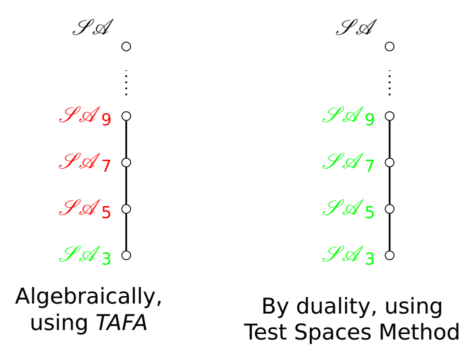
Pro tem, suppose $\mathcal{L}$ is any deductive system. A *rule* $\Gamma \rhd \phi$ for $\mathcal{L}$ consists of a finite set Γ of premises and a conclusion ϕ. It is *admissible* if adding it to $\mathcal{L}$ introduces no new theorems. In **CPC**, admissibility and derivability coincide. In relevant logics the rule $\{\neg p, p \vee q\} \rhd q$ (disjunctive syllogism), for example, is admissible but not derivable. In general, studying admissible rules can yield valuable information about a logic, linking to notions of completeness, and connecting to areas of applied logic such as unification.

Assume $\mathcal{L}$ has algebraic semantics given by a quasivariety $\mathscr{A}$. Then admissibility of a quasi-identity can be defined so that the logical and algebraic notions tally. Moreover, a quasi-identity $\Sigma \Rightarrow \varphi \approx \psi$ is admissible if and only if $\Sigma \Vdash_{\mathbf{F}_{\mathscr{A}}(\omega)} \varphi \approx \psi$; here $\mathbf{F}_{\mathscr{A}}(\omega)$ is the free algebra on countably many generators. It can be deduced that the logical problem of testing $\mathcal{L}$-rules for admissibility translates into that of validating quasi-identities on a suitable family of free algebras.

For Sugihara algebras, $\mathbf{F}_{\mathscr{SA}}(k) = \mathbf{F}_{\mathscr{SA}_{2n+1}}(k)$ so long as $2n+1 \geqslant 2k$. This confines attention to finitely generated quasivarieties. It can be shown that admissibility of quasi-identities can be tested on the particular free algebras $\mathbf{F}_{\mathscr{SA}_{2n+1}}(n+1)$. Here $n+1$ arises because this is the number of elements needed to generate $\mathbf{Z}_{2n+1}$. Metcalfe & Röthlisberger (2013) devised a general algorithmic method which, applied to Sugihara algebras, finds an algebra $\mathbf{A}_n$ of minimum size to substitute for $\mathbf{F}_{\mathscr{SA}_{2n+1}}(n+1)$. Potentially this result is powerful. But can it be exploited in practice? An associated computer package *TAFA* succeeded for $n = 1$, so finding $\mathbf{A}_1$, but failed for $n \geqslant 2$. (In general *TAFA* is feasible only when free algebras have a few million elements at most.)

A new technique is needed to overcome the complexity issues of the admissible rules problem for R-mingle. It is well known that Stone duality for Boolean algebras and Priestley duality for distributive lattices have a "logarithmic" feature. The functor taking a finite object to its dual structure acts like a logarithm, in particular as regards cardinalities. So, for quasivarieties of algebras which have an underlying distributive lattice structure, might there exist dualities which enable the Metcalfe–Röthlisberger algorithms to be recast in an equivalent dual form? The requirements are stringent and an arbitrary dual equivalence of categories would not meet them, so that a Kripke-style semantics based on enriched Priestley duality would not do the job. Fortunately, the theory of *strong dualities*, for which Stone and Priestley duality provide prototypes, is just what is wanted. Crucially, such dualities give direct access to the structures dual to free algebras. Cabrer & Priestley drew on an extensive literature to set up a strong duality for each quasivariety $\mathscr{SA}_{2n+1}$. The dualities were derived by the piggyback method, which exploits Priestley duality for the lattice reducts. Simple cases were treated with computer assistance, capitalising on logariihmicity. Many advances in duality theory in the past 30 years began this way.

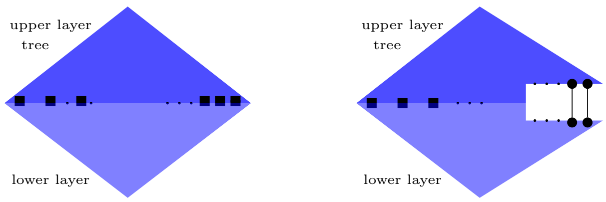
The *Test Spaces Method* of Cabrer, Freisberg, Metcalfe & Priestley (2019) translates the Metcalfe–Röthlisberger algorithms into dual form. This enabled Cabrer & Priestley to describe explicitly the admissibility algebra $\mathbf{A}_n$ for $\mathscr{SA}_{2n+1}$ for arbitrary n. For $\mathscr{SA}_5$ ($n = 2$), it is the 16-element algebra trailed at the start.



Finding admissibility algebras: comparisons
Red labels: failure. Green labels: success

Sugihara algebras are interesting in their own right. The piggyback dualities for the quasivarieties $\mathscr{SA}_{2n+1}$ encode a method for translating from the duals of free algebras $\mathbf{F}_{\mathscr{SA}_{2n+1}}(s)$, for any s, to a pictorial dual representation of their lattice reducts, based on the Birkhoff/Priestley duality between finite distributive lattices and finite pointed ordered sets.

Here pleasant surprises appear: the focus switches to the combinatorics of ordered sets.



For $\mathbf{F}_{\mathscr{SA}}(s)$

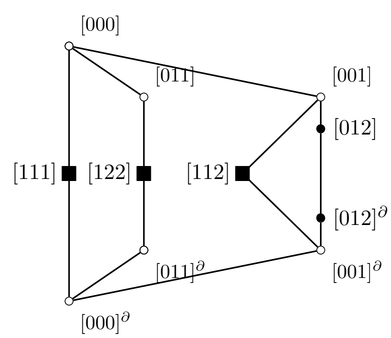
For the free algebra $\mathbf{F}_{\mathscr{SA}_{2n+1}}(n+1)$ related to admissibility algebra $\mathbf{A}_n$

Stylised pictures of mirrored trees

Squares indicate glued pairs of points, circles those min/max points which are not glued

The Birkhoff/Priestley dual of $\mathbf{F}_{\mathscr{SA}}(s) = \mathbf{F}_{\mathscr{SA}_{2s+1}}(s)$ is a 'mirrored tree'. This is formed from a downward-growing finite tree and its upward-growing reflection (that is, the order dual), glued together by identifying minimal points in the upper layer to corresponding maximal points in the lower layer, as shown. For $\mathbf{F}_{\mathscr{SA}_{2s-1}}(s)$, which arose with admissibility algebras, the picture is similar, but with less glueing between layers.

Attention then focuses on describing an upper layer tree in detail. It turns out that this can always be built from an easily specified subtree, the *skeletal tree*. The mirroring construction applied to this gives a skeleton for the entire mirrored tree.
The skeletal tree carries a canonical labelling. Exploiting this, the entire upper layer tree can be obtained recursively, with multiple copies of subtrees of the skeletal tree.



Mirrored skeletal tree for $\mathbf{F}_{\mathscr{SA}_5}(3)$

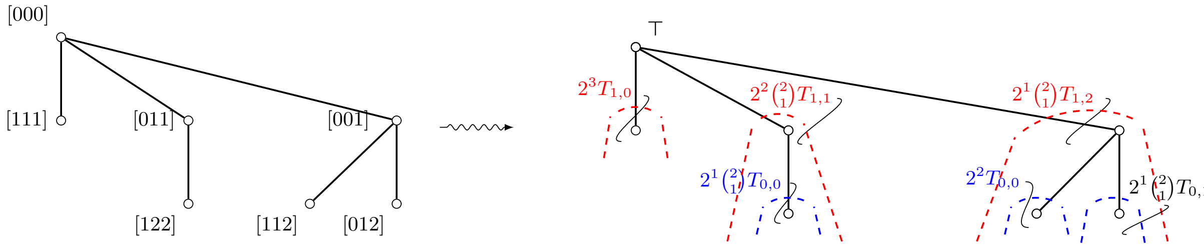
[Solid symbols denote the points which determine how to glue the layers.]

Specifically, the upper layer tree, $T_{n,s}$, is determined by letting $T_{k,t} = \mathbf{1}$ (a single element) if $k = 0$ and/or $t = 0$ and otherwise

$$T_{k,t} = \dot{\bigcup}_{r=1}^{t} \left(2^r \tbinom{t}{r} T_{k-1,t-r}\right) \oplus \top;$$

cP denotes the disjoint union of c copies of a poset P and $\top$ is a newly-added top element.

Below, this process is illustrated pictorially for $n = 2$ and $s = 3$, the situation in which the 16-element admissibility algebra $\mathscr{SA}_2$ could not be obtained algebraically from $\mathbf{F}_{\mathscr{SA}_5}(3)$ by computer. The size of this free algebra is the large number given in the trailer.



Skeletal tree for $F_{\mathscr{SA}_5}(3)$

The upper layer tree $T_{2,3}$ recursively

Revealing secrets

What can then be deduced about free algebras? Any finitely generated free algebra in some $\mathscr{SA}_{2n+1}$ or in $\mathscr{SA}$ has an associated mirrored tree and hence its underlying lattice can be accessed using elementary properties of Birkhoff/Priestley duality. For $k \geqslant 0$ let $L_{k,0} = \mathbf{1}$, the 1-element lattice, and for $k \geqslant 1$ and $t \geqslant 1$ let

$$L_{k,t} = \prod_{r=1}^{t} \Big(L_{k-1,t-r}{}^{\sharp} \Big)^{2^r \binom{t}{r}}.$$

Here $^\sharp$ denotes the addition of bottom and top elements. To take an example, $\mathbf{F}_{\mathscr{SA}}(s) = L_{2s,s}$. The powers which appear in the recursion indicate why, and more significantly how, free algebras get so enormous. These algebras, and the associated tree-based structures, invite further investigation, both structurally and combinatorially. Work is on-going!

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