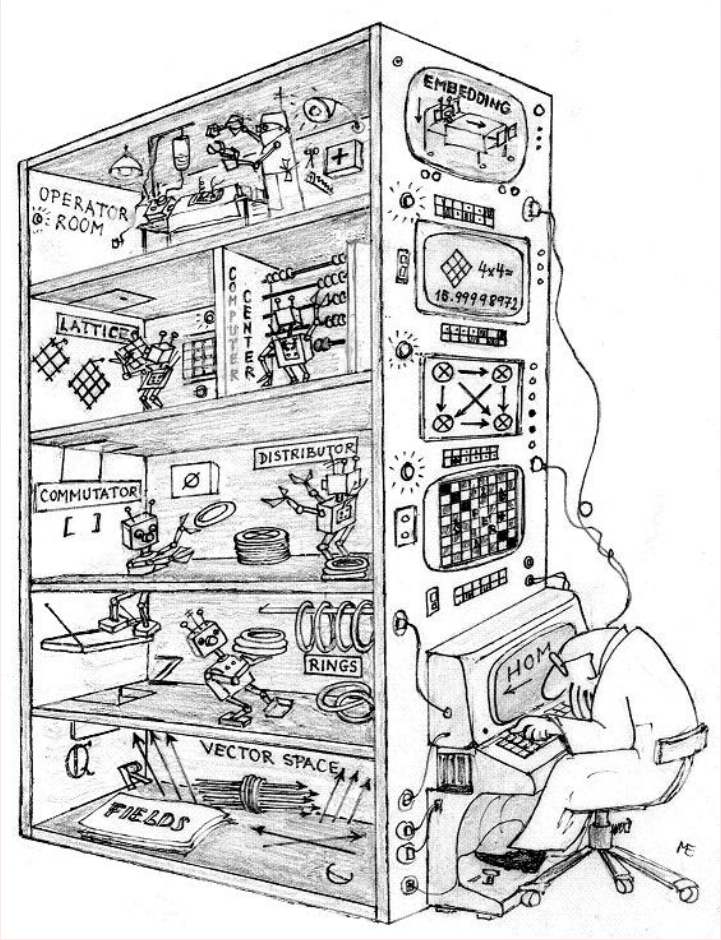
Duality Theory in Algebra, Logik and Computer Science

Workshop I, 13-14 June 2012, Oxford



Dualities for locally hypercompact, stably hypercompact and hyperspectral spaces

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Part I

Variants of compactness and sobriety

Variants of compactness and sobriety

1. Order-theoretical and topological preliminaries
2. Topological concepts of compactness
3. Sobriety and well-filtration

Lower sets, upper sets and feet

Let $P = (X, \leq)$ be a *preordered set* ($\leq$ reflexive and transitive).

- lower set and upper set generated by $Y \subseteq X$, respectively:
  $\uparrow Y = \{x \in X : \exists y \in Y\ (x \geq y)\}$,
  $\downarrow Y = \{x \in X : \exists y \in Y\ (x \leq y)\}$.
- principal ideal and principal filter of $y \in X$, respectively:
  $\downarrow y = \{x \in X : x \leq y\}$,
  $\uparrow y = \{x \in X : x \geq y\}$.
- A foot is a finitely generated upper set (FUS) $\uparrow F$.
- The upper sets form the (upper) Alexandroff topology αP.
- *The lower sets are exactly the closed sets w. r. t. αP.*
- *The feet are exactly the compact open sets w. r. t. αP.*

The specialization order

Let X be a topological space.

- The specialization order of X is given by
$$x \leq y \;\Leftrightarrow\; x \in \overline{\{y\}} \;\Leftrightarrow\; \text{for all open } U\,(x \in U \Rightarrow y \in U).$$
All order-theoretical statements about spaces refer to the specialization order, unless otherwise stated.
- The saturation of a subset $Y \subseteq X$ is the intersection of all neighborhoods of Y; hence it is the upset $\uparrow Y$ generated by Y.
- Specifically, the (neighborhood) core of a point $x \in X$ is the principal filter $\uparrow x$, the intersection of all neighborhoods of x.
- Dually, the point closure of x is the principal ideal $\downarrow x$.

Monotonicity properties

Definition

Let X be a T_0-space.

- A monotone net in X is a map ν from a directed set N into X such that $m \leq n$ implies $\nu(m) \leq \nu(n)$ (with respect to the specialization order).
- X is a monotone convergence space if each monotone net in X has a join to which it converges.
- X is monotone determined if a subset U is open whenever any monotone net converging to a point of U is residually in U. Equivalently, U is open whenever for all directed $D \subseteq X$,

$$\overline{D} \cap U \neq \emptyset \text{ implies } D \cap U \neq \emptyset.$$

The Scott topology

Definition

- A subset U of a poset $P = (X, \leq)$ is Scott open if for all directed subsets D possessing a join, $D \cap U \neq \emptyset \;\Leftrightarrow\; \bigvee D \in U$.
- The Scott topology σP consists of all Scott-open sets.
- The Scott space associated with P is $\Sigma P = (X, \sigma P)$.
- The weak Scott topology $\sigma_2 P$ consists of all upsets U that meet each directed set D with $D^{\uparrow\downarrow} \cap U \neq \emptyset$, where $D^{\uparrow\downarrow} = \bigcap \{\downarrow y : D \subseteq \downarrow y\}$ is the cut closure of D.
- The poset P is up-complete, a dcpo or a domain if all directed subsets of P have joins. In that case, σP coincides with $\sigma_2 P$.

A Min-Max characterization of Scott spaces

Theorem
(ME 2009)

- *The topology of a monotone convergence space is always coarser than the Scott topology.*
- *The topology of a monotone determined space is always finer than the weak Scott topology.*
- *The Scott spaces of domains are exactly the monotone determined monotone convergence spaces.*

The Separation Lemma for locales

Definition

- A locale or frame is a complete lattice L satisfying the infinite distributive law $a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$.
- A locale L enjoys the Separation Lemma or Strong Prime Element Theorem if each element outside a Scott-open filter U in L lies below a prime element outside U.

Theorem

(ME 1986, BB+ME 1993)
The Ultrafilter Principle (**UP**) *resp. Prime Ideal Theorem* (**PIT**) *is equivalent to the Separation Lemma for locales (or quantales).*

Hyper- and supercompactness in spaces

Definition

- A subset C of a space X is hypercompact if $\uparrow C$ is a foot $\uparrow F$.
- A subset C of a space X is supercompact if $\uparrow C$ is a core $\uparrow x$.
- A space is compactly based resp. hypercompactly based resp. supercompactly based if it has a base of compact resp. hypercompact resp. supercompact open sets.
- A space is locally compact resp. locally hypercompact resp. locally supercompact if each point has a neighborhood base of compact resp. hypercompact resp. supercompact sets.

Characterization of hypercompactly based spaces

Definition

A T_0-space is sober if each irreducible closed subset is a point closure.

Theorem

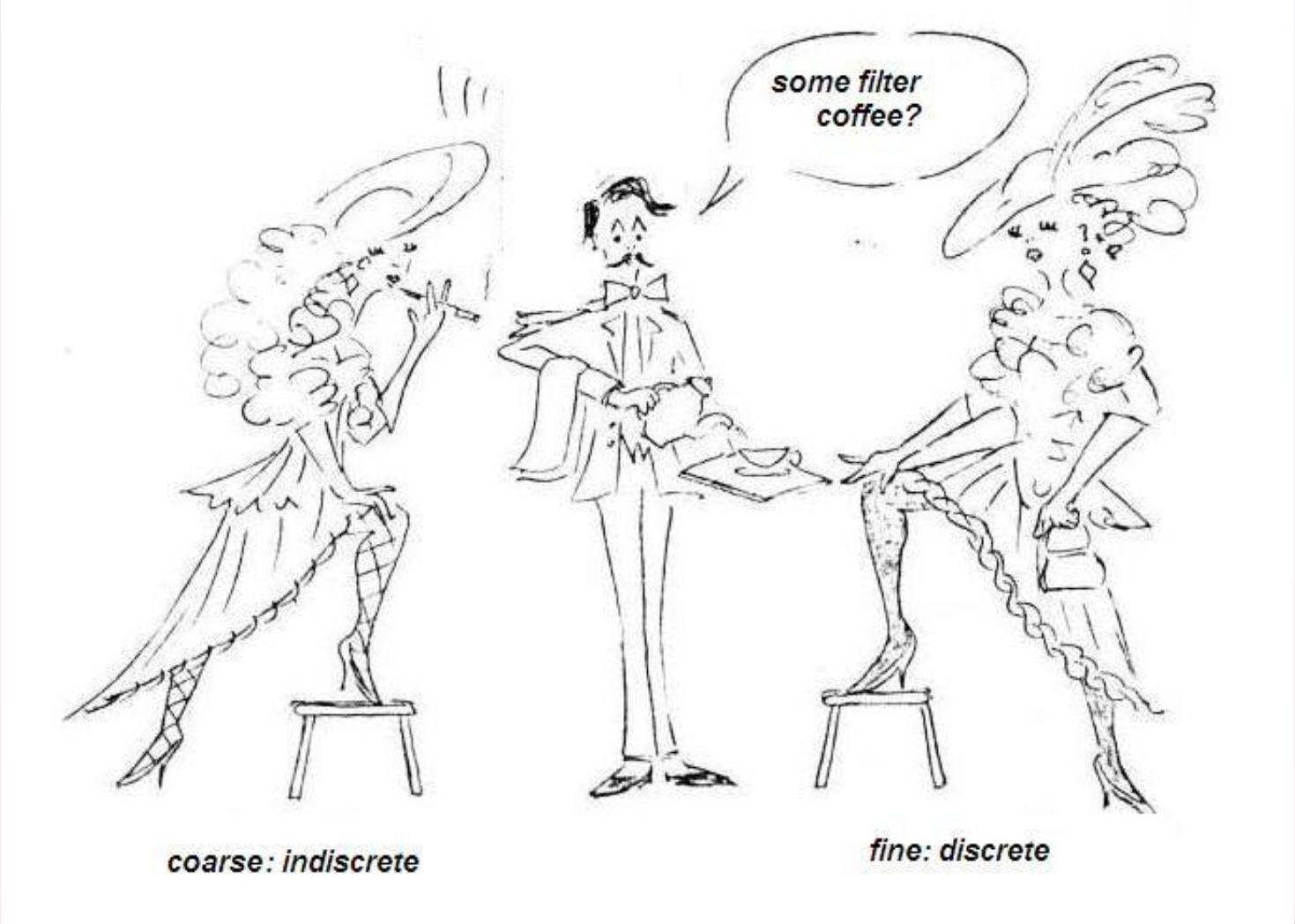
(ME 2009)

(1) *Compact open subsets of monotone determined spaces are hypercompact.*

(2) *The hypercompactly based spaces are exactly the compactly based monotone determined spaces.*

(3) *The hypercompactly based sober spaces are exactly the compactly based Scott spaces of domains.*

Nets and filters



δ-sobriety and well-filtration

Definition

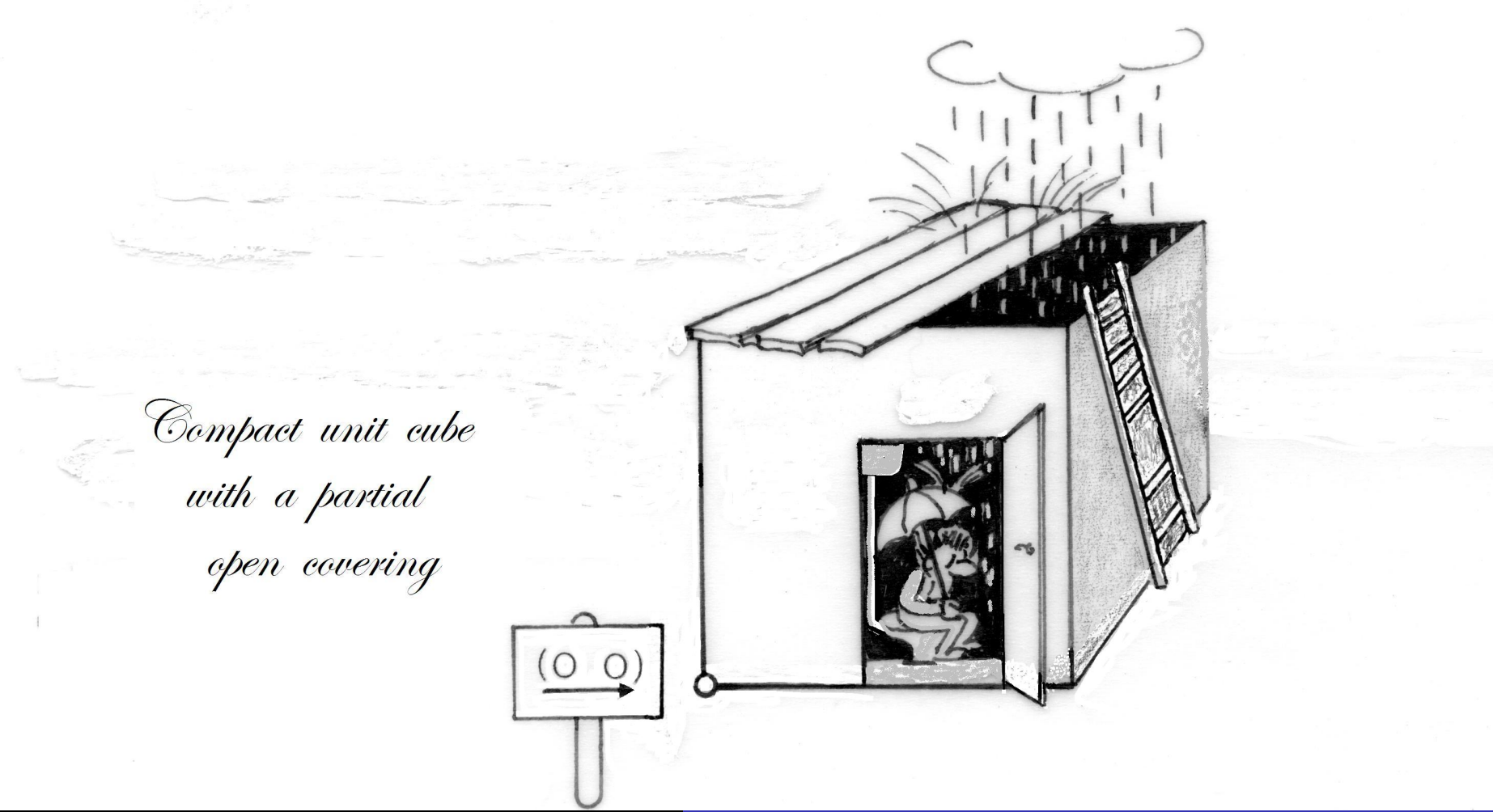
The Lawson dual δP of a poset P consists of all Scott-open filters. A T_0-space X is

- δ-sober if each Scott-open filter of open sets (i.e. each $\mathcal{V} \in \delta\,\mathcal{O}X$) contains all open neighborhoods of its intersection.
- well-filtered (resp. $\mathcal{H}$-well-filtered) if for any filter base $\mathcal{B}$ of compact (resp. hypercompact) saturated sets, each open neighborhood of the intersection $\bigcap \mathcal{B}$ contains a member of $\mathcal{B}$.

Lemma

(1) *X is δ-sober iff its open locale enjoys the Separation Lemma.*
(2) *Every δ-sober space is sober and well-filtered.*
(3) *Every locally compact well-filtered space is δ-sober.*

Tychonoff's Product Theorem



The power of δ-sobriety and Tychonoff's Theorem

Theorem

(ME 2012) *Each of the following statements is equivalent to* **UP**:

- *The Hofmann-Mislove Theorem: sober spaces are δ-sober.*
- *Sober spaces are well-filtered.*
- *Every filter base of compact saturated sets in a sober space has nonempty intersection.*
- *Every filter base of compact saturated sets in a sober space has compact intersection.*
- *Tychonoff's Product Theorem for sober spaces.*
- *Tychonoff's Product Theorem for Hausdorff spaces.*
- *Tychonoff's Product Theorem for two-element discrete spaces.*

Power of sobriety

A sober space and a non-sober neighborhood

Sobrification

The power of Rudin's Lemma

Definition

For any system $\mathcal{Y}$ of sets, each member of the crosscut system

$$\mathcal{Y}^{\#} = \{Z \subseteq \bigcup \mathcal{Y} : \forall\, Y \in \mathcal{Y}\,(Y \cap Z \neq \emptyset)\}$$

is a cutset or transversal of $\mathcal{Y}$.

Theorem

(ME 2012) *Each of the following statements is equivalent to* **UP**:

- *Every system of compact sets whose saturations form a filter base has an irreducible transversal.*
- *Rudin's Lemma: Every system of finite sets generating a filter base of feet has a directed transversal.*
- *Every monotone convergence space is $\mathcal{H}$-well-filtered.*

Part II

Stone-Priestley dualities

Stone-Priestley dualities

4. Order-theoretical concepts of compactness
5. Coherence in spaces and ordered structures
6. Duals of hyper- and superspectral spaces
7. Stone-Priestley dualities

Compactness properties in posets

Definition

Let P be a poset and c an element of P.

- c is compact if $P \setminus \uparrow c$ is Scott closed.
- c is hypercompact if $P \setminus \uparrow c$ is a finitely generated lower set.
- c is supercompact if $P \setminus \uparrow c$ is a principal ideal.
- P is algebraic resp. hyperalgebraic resp. superalgebraic if each element is a directed join of compact resp. hypercompact resp. supercompact elements.

These definitions generalize the corresponding topological ones.

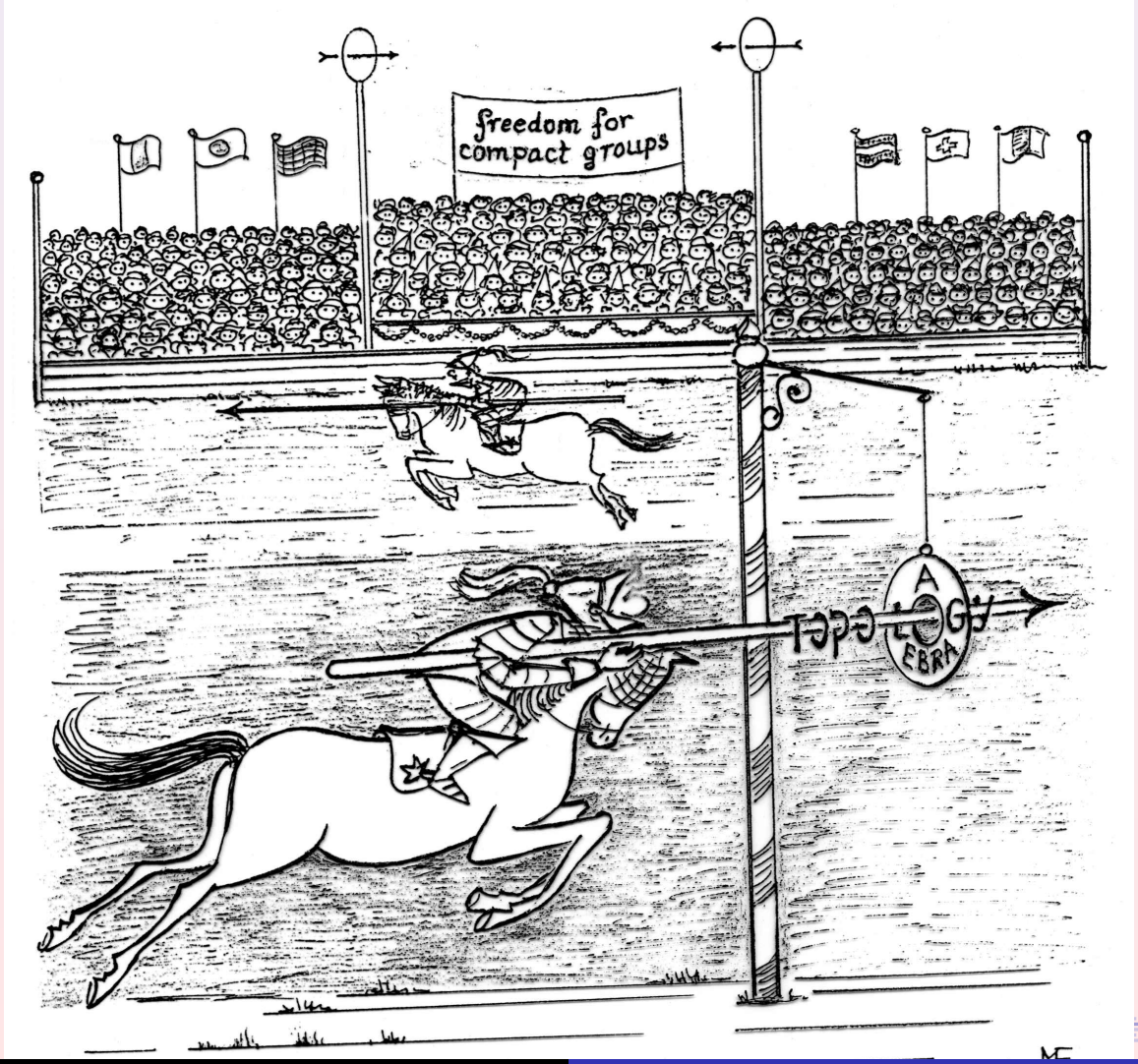
(Quasi-)algebraic and (quasi-)continuous domains

Definition

A domain P is

- algebraic iff each principal filter $\uparrow x$ is the intersection of a filterbase of Scott-open cores.
- quasialgebraic iff each principal filter $\uparrow x$ is the intersection of a filterbase of Scott-open feet.
- continuous iff each principal filter $\uparrow x$ is the intersection of a filterbase of cores having x in their Scott-interior.
- quasialgebraic iff each principal filter $\uparrow x$ is the intersection of a filterbase of feet having x in their Scott-interior.

Topology meets algebra



Topological characterization of domains

Theorem

(1) *The category of algebraic domains is isomorphic to the category of supercompactly based sober spaces and dual to the category of superalgebraic (=completely distributive algebraic) frames* (**ZF**)

(2) *The category of quasialgebraic domains is isomorphic to the category of hypercompactly based sober spaces and dual to the category of hyperalgebraic frames* (**UP**)

(3) *The category of continuous domains is isomorphic to the category of locally supercompact sober spaces and dual to the category of supercontinuous (completely distributive) frames* (**ZF**)

(4) *The category of quasicontinuous domains is isomorphic to the category of locally hypercompact sober spaces and dual to the category of hypercontinuous (dually filter distributive) frames* (**UP**)

Coherence for spaces

Definition

A topological space is

- coherent if finite intersections of compact upsets are compact.
- quasicoherent if finite intersections of cores are compact.
- open coherent if finite intersections of compact open sets are compact.
- A spectral (hyperspectral, superspectral) space is a compactly (hypercompactly, supercompactly) based coherent sober space.

Lemma

For Stone spaces the above three notions of coherence are equivalent.

Coherence for domains and lattices

Definition

- A poset P is quasicoherent if it is quasialgebraic and the Scott space ΣP is coherent.
- A complete lattice is coherent (hypercoherent, supercoherent) if it is algebraic (hyperalgebraic, superalgebraic) and finite meets of compact elements are compact.

Lemma

Not only every coherent, but even every algebraic complete lattice is quasicoherent.

Finite prime decompositions

Definition

Let S be a meet-semilattice.

- A prime ideal of S is a directed proper downset whose complement is a subsemilattice.
- S is an np semilattice if the poset of prime ideals is noetherian (all directed subsets have greatest elements).
- S is an fp semilattice if each element is a finite meet of primes.
- S is an fpi semilattice if each principal ideal is a finite meet of prime ideals.
- A ring or semilattice has property M_f if any element a has only finitely many prime ideals maximal w.r.t. not containing a.

Priestley-Lawson spaces and their duals

Theorem

(ME 2009) *The following statements are equivalent for a space X:*

- *X is a hyperspectral space.*
- *X is a spectral space and carries the Scott topology.*
- *X is the Scott space of a quasicoherent domain.*
- *X is the spectrum of a hypercoherent frame.*
- *X is the upper space of a Priestley-Lawson space.*
- *X is the prime ideal spectrum of a ring with property M_f.*
- *X is the prime ideal spectrum of a distributive lattice with M_f.*
- *X is the prime filter spectrum of an fpi lattice.*

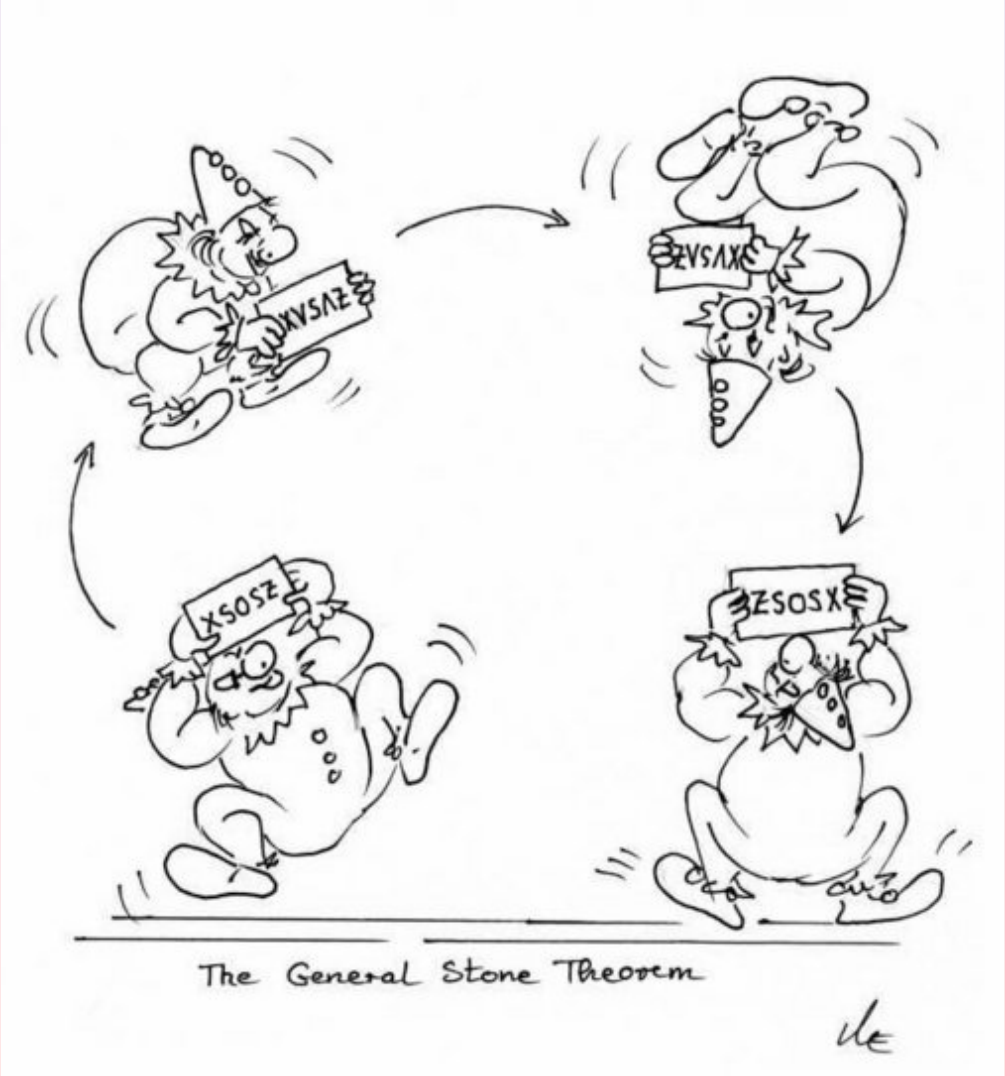
Duality for quasicoherent domains and hyperspectral spaces

Theorem
(ME 2009)
The category of quasicoherent domains is isomorphic to the category of hyperspectral spaces and to the category of Priestley spaces with the Lawson topology.

These categories are dual to the category of hypercoherent frames (via the open set functor) and to the category of fpi lattices (via Priestley duality).

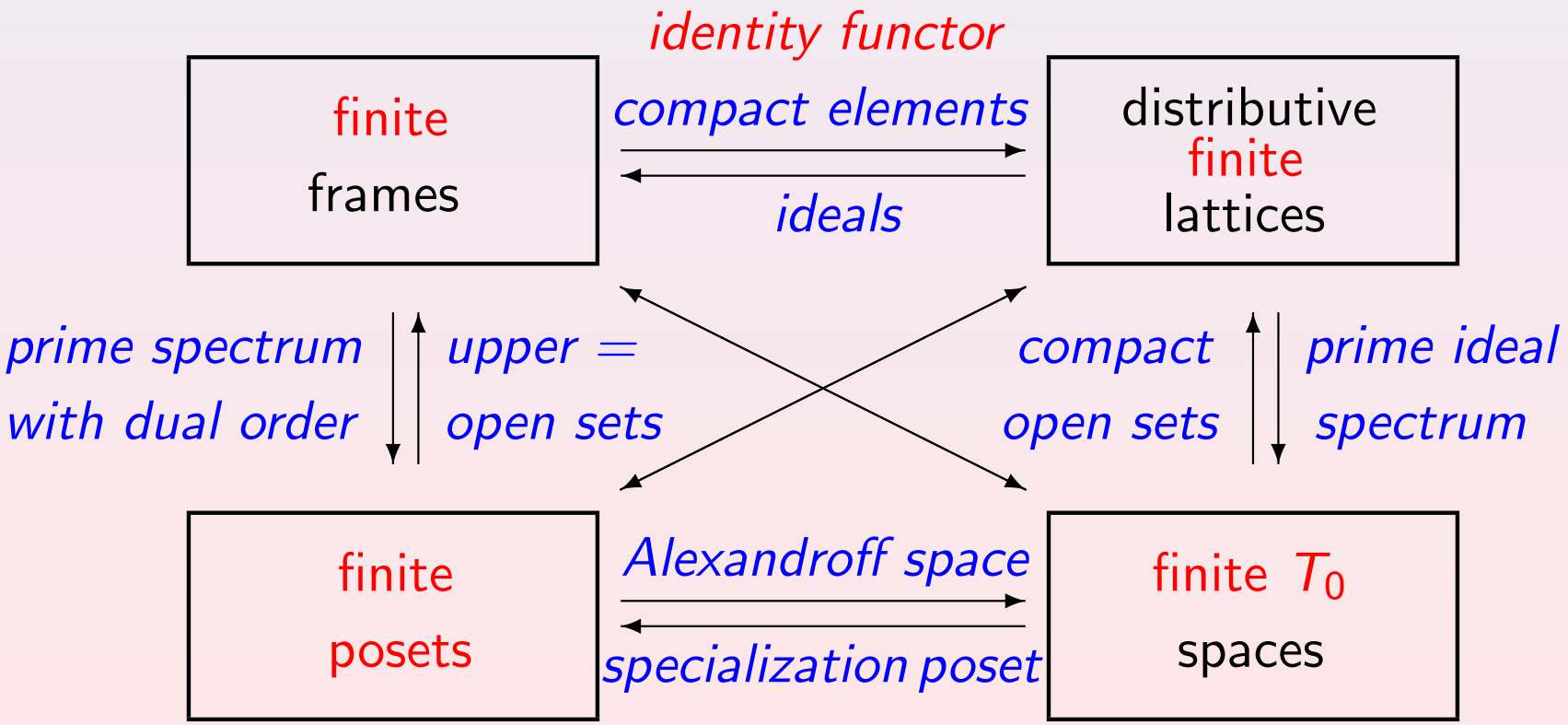
A similar isomorphism and duality holds for the categories of quasicoherent algebraic domains and superspectral spaces, with supercoherent frames and fp lattices as duals.

The General Stone Duality



Stone-Priestley dualities

The two categories in the upper row are equivalent and dual to the isomorphic categories in the lower row



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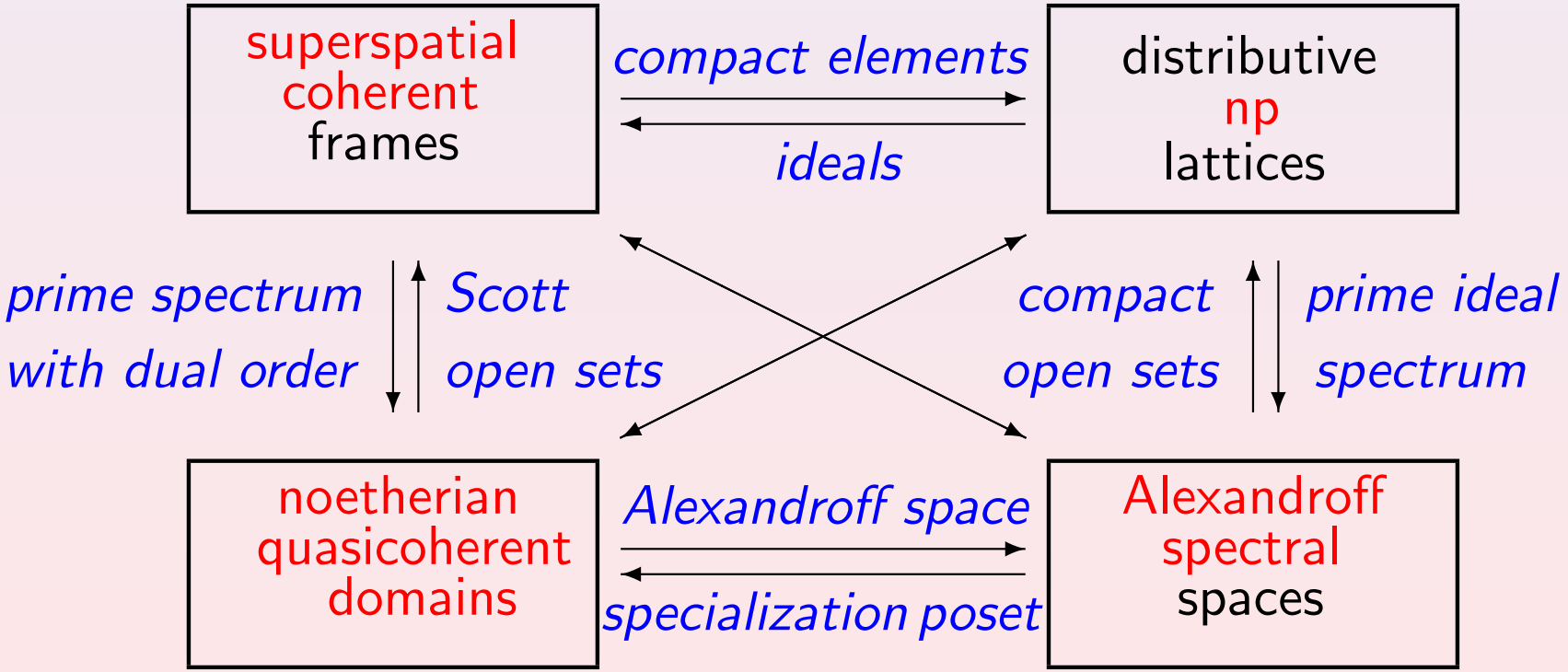

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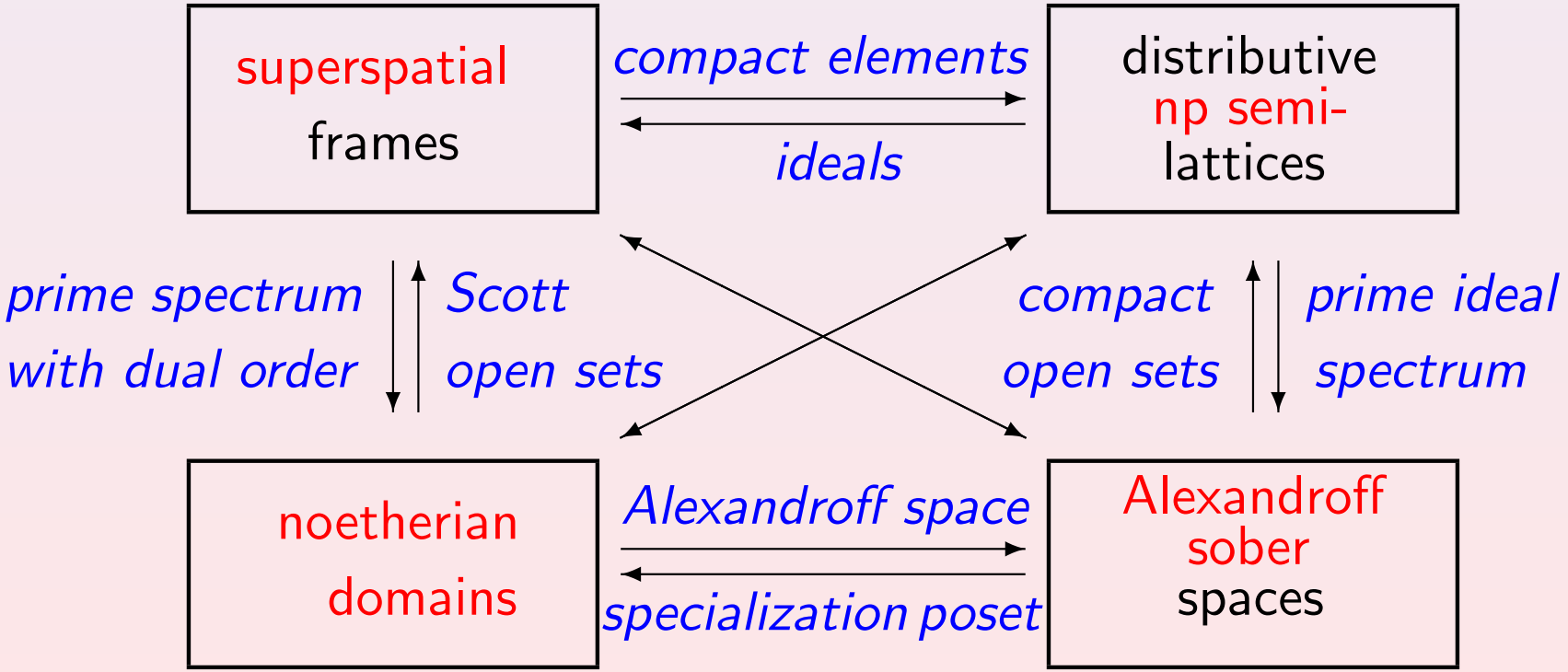

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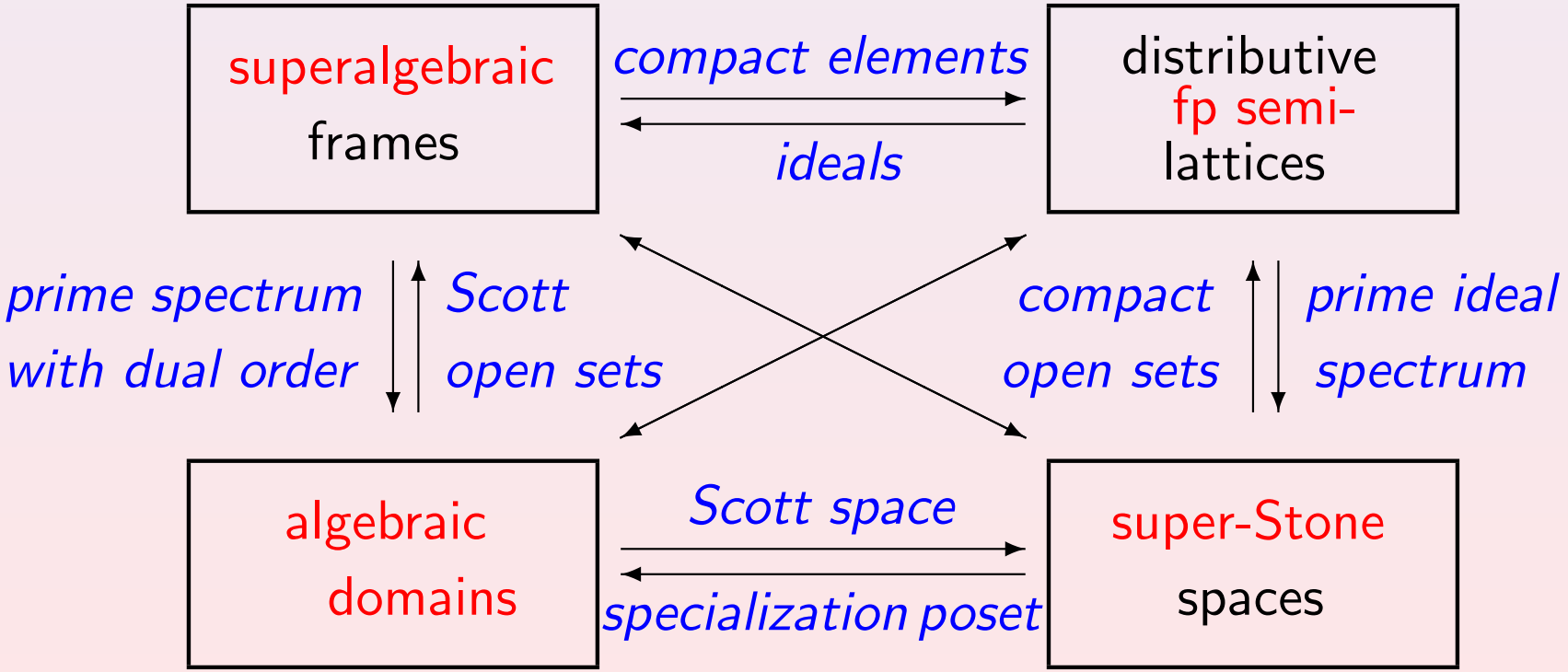

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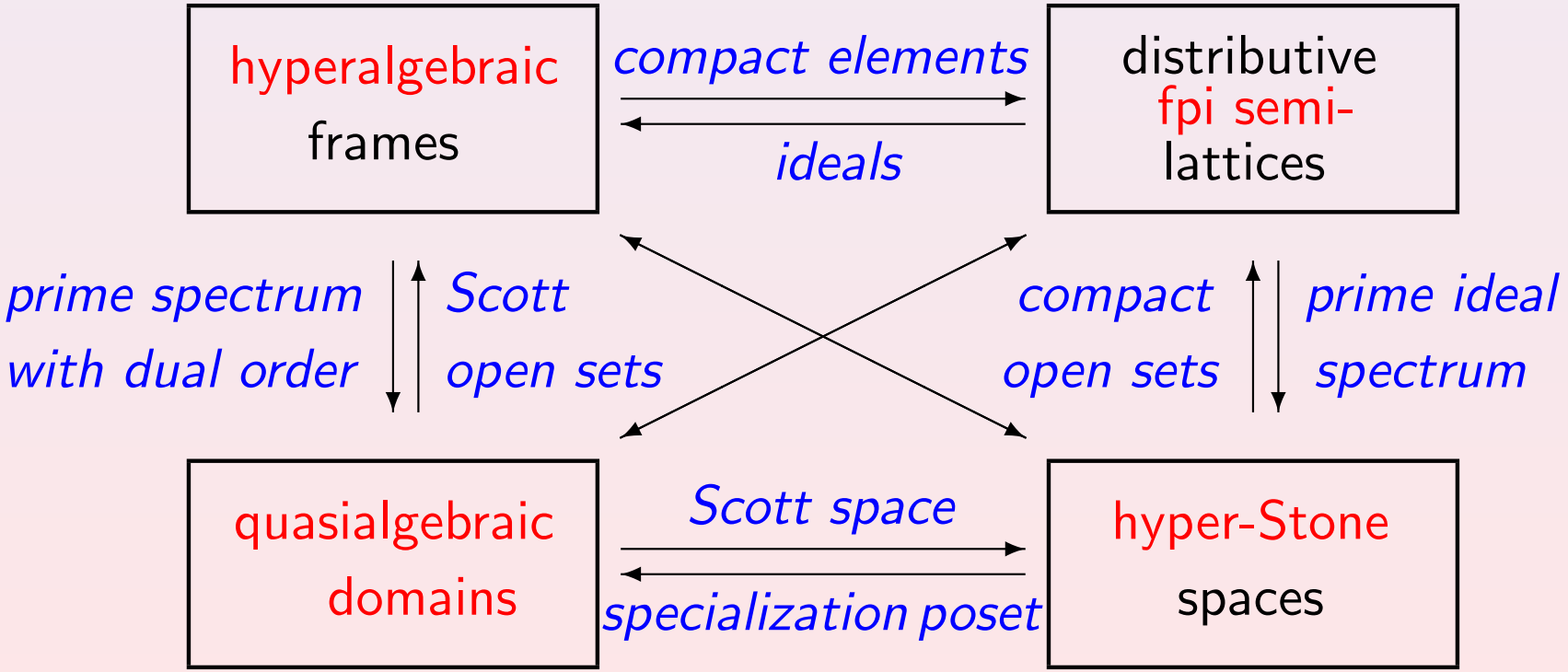

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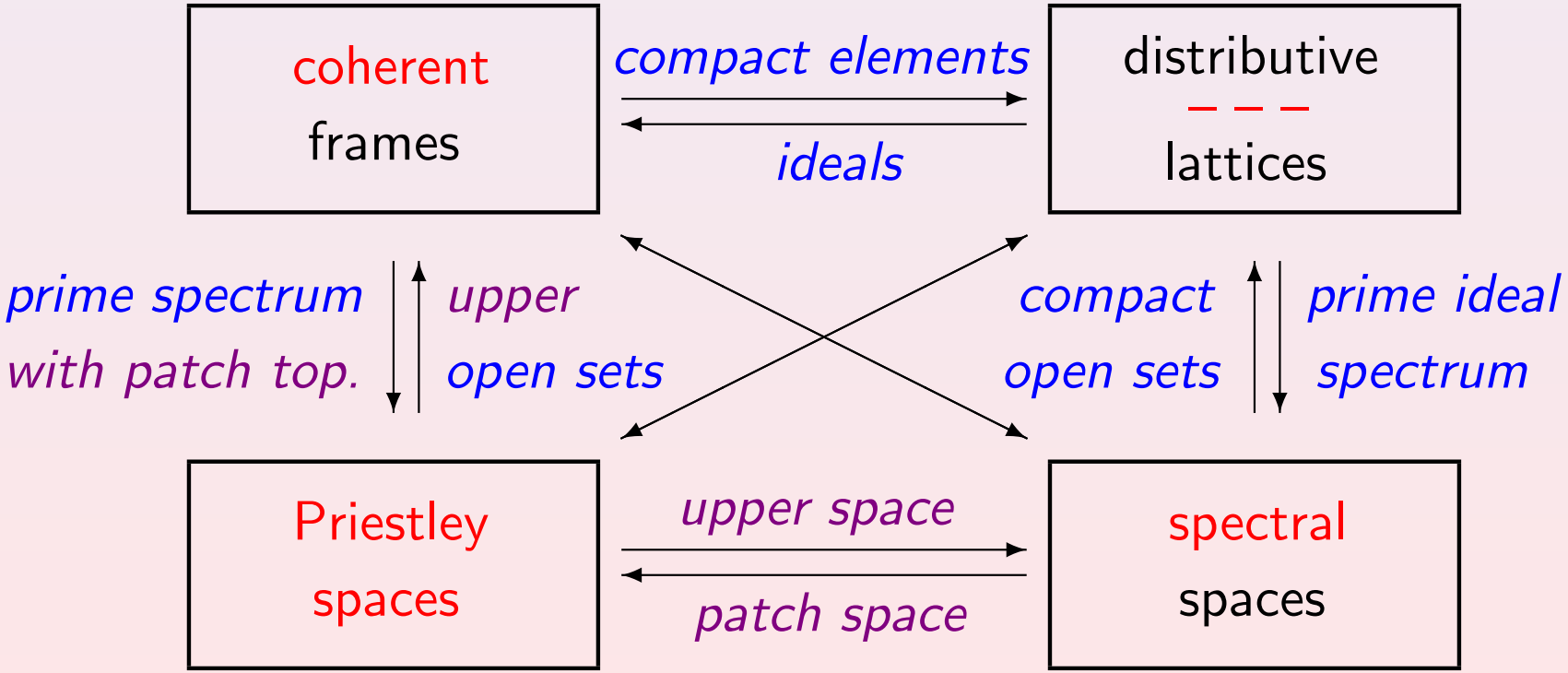

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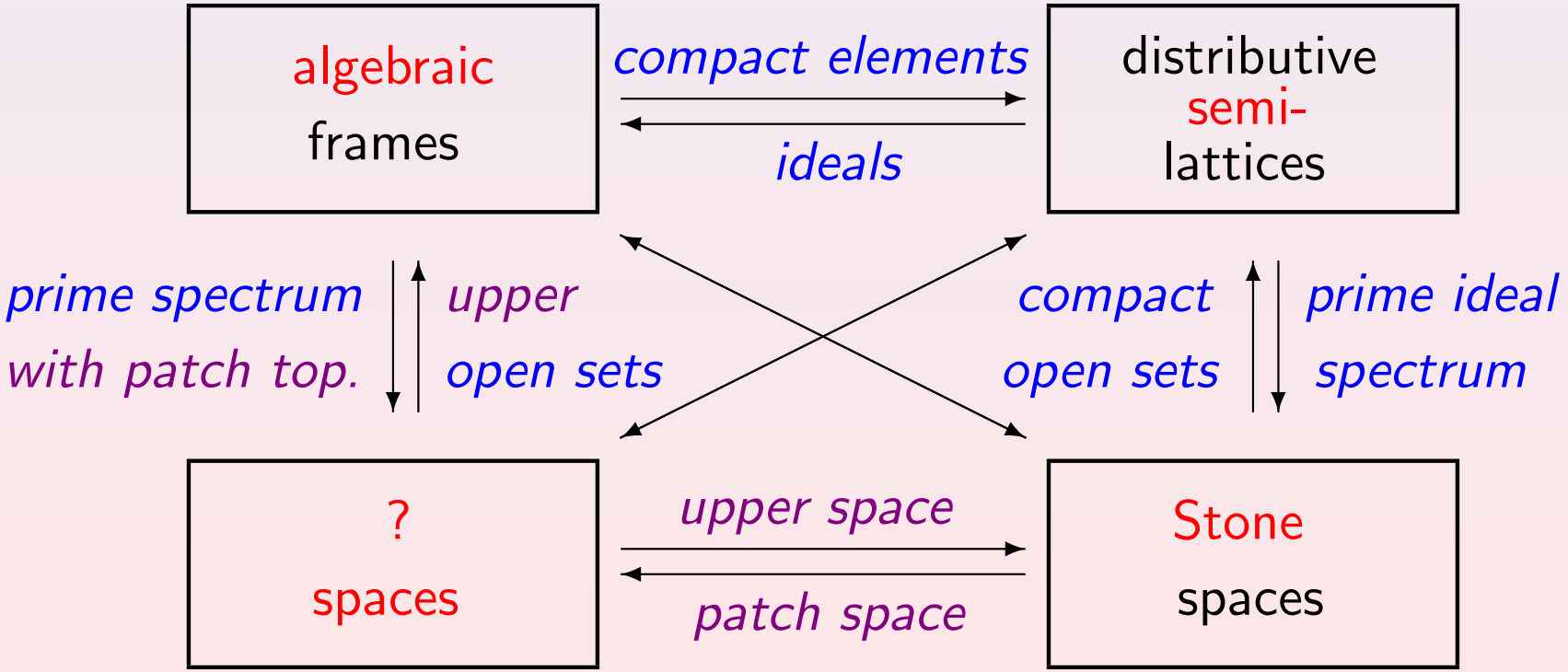

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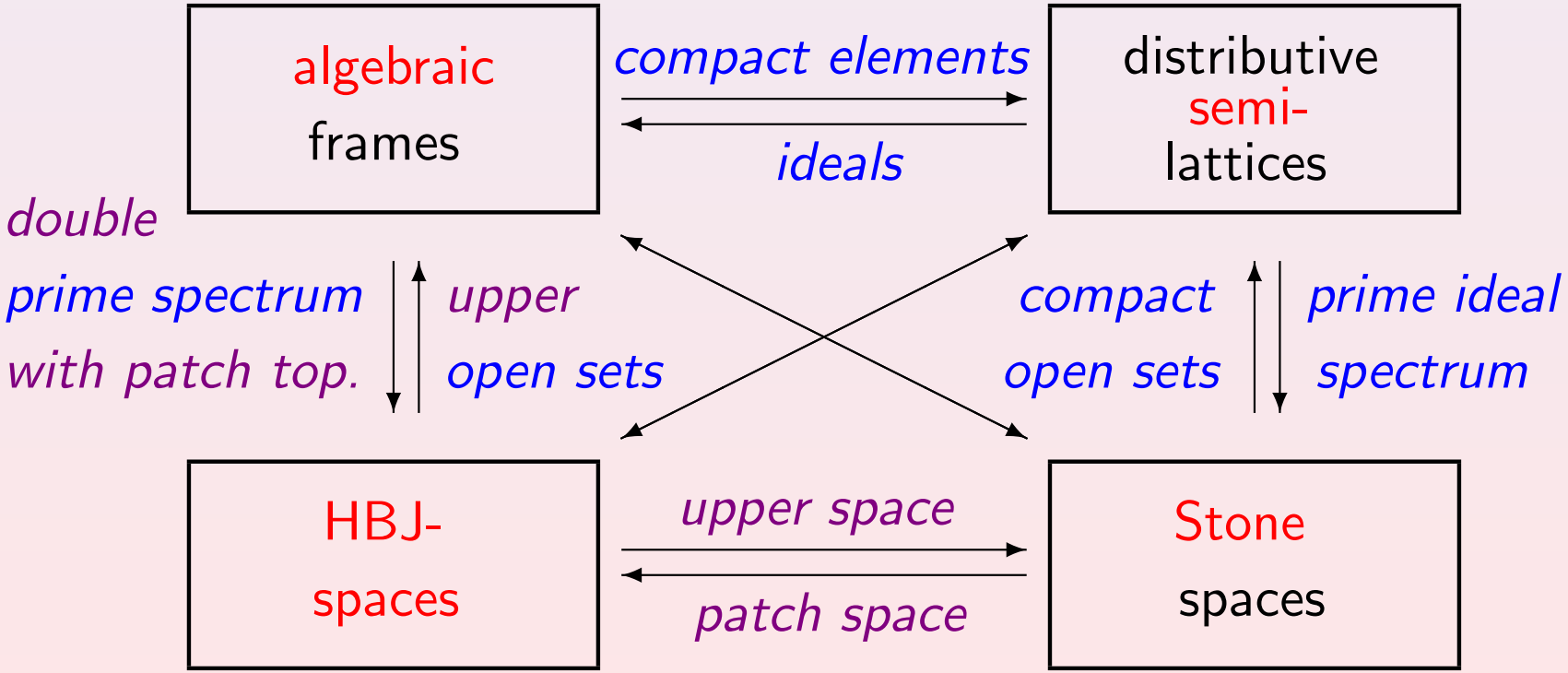

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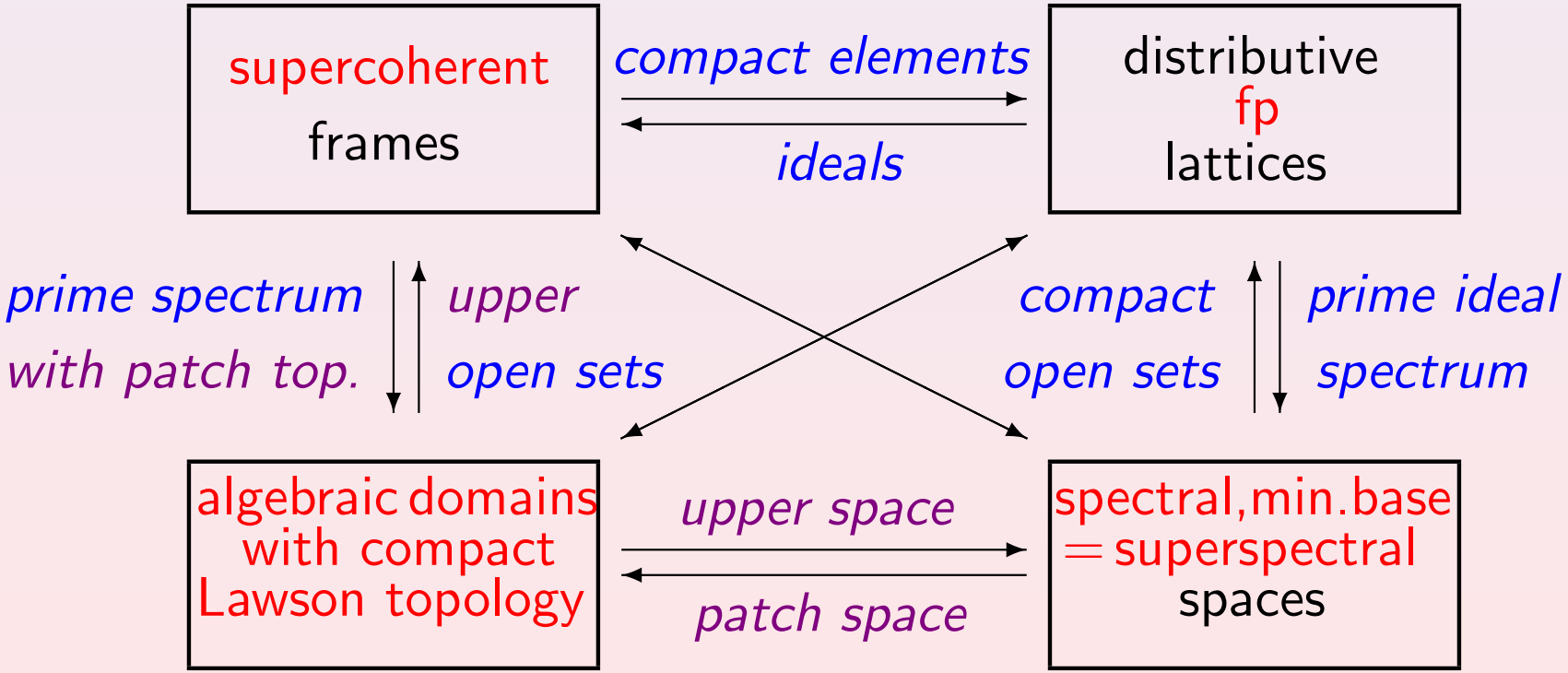

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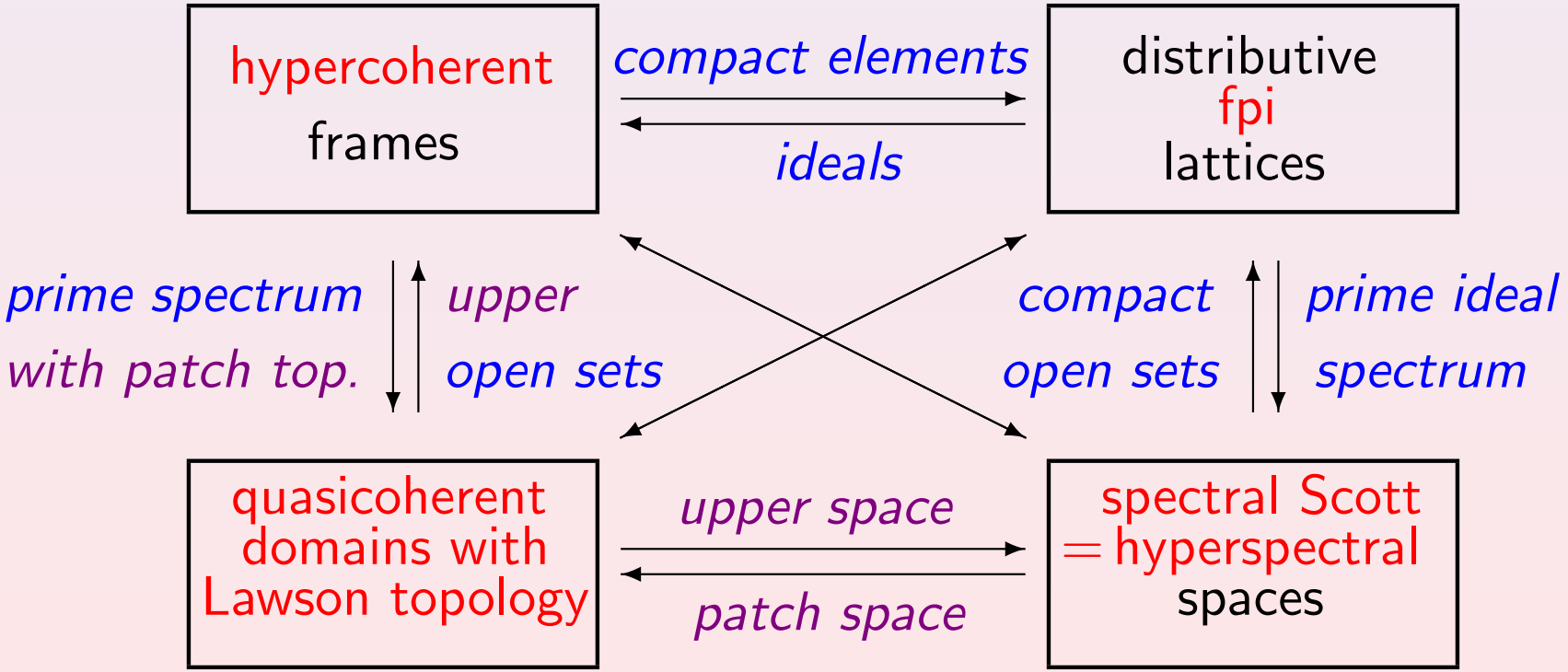

Open end: Mile-Stones of duality