DUALITY THEORY AND B L A S T : Selected Themes Part I: Dualities in Various Forms

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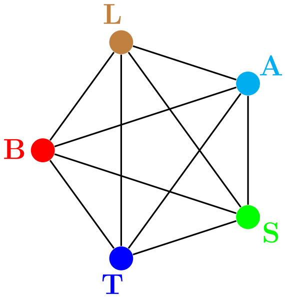
With acknowledgements to very many people



Disclaimers

There's no C in BLAST!

- No Category theory as such in these talks.
- Shall use the language of category theory but little more: no monads, no coalgebras, no finitely presentable algebraic categories, ...
- Perspective on duality theory comes from Algebra.
- Almost all algebras considered will be lattice-based or semilattice-based. (So a big part of the duality story is omitted altogether.)

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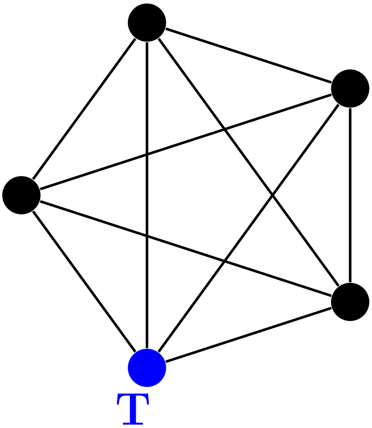
  SO: there's no Elephant in this room!

Outline: Part I

- The framework: Stone duality and Priestley duality as prototype examples
- Dualities for finitely generated lattice-based quasivarieties
- From quasivarieties to varieties: multisorted dualities
- The best of both worlds?
- From algebras to structures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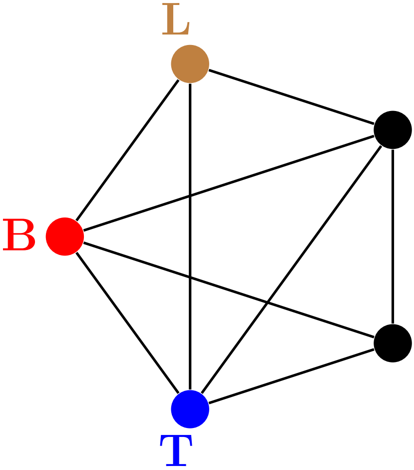
Marshall Stone's legacy

A cardinal principle of modern mathematical research may be stated as a maxim: "One must always topologize."

Marshall Stone, 1938)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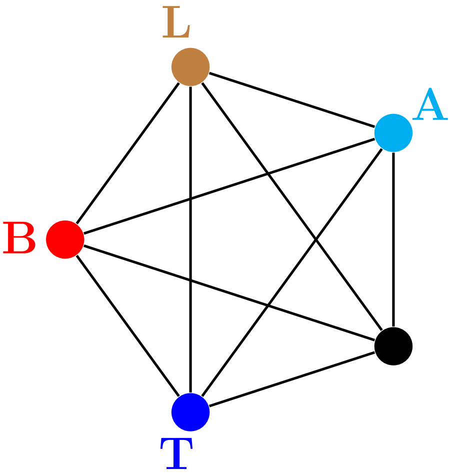
But is topology the whole story? Or the 'right' story?

- Stone's duality for the variety $\mathfrak{B}$ of Boolean algebras uses the dual category of Boolean spaces—purely topological.
- Stone's duality for the variety $\mathfrak{D}$ of (bounded) distributive lattices in terms of spectral spaces again uses a purely topological dual category—the dual objects are T_0-spaces.

Was Stone right 'always to topologize'? YES!

What did his approach conceal? MUCH!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What's special algebraically about $\mathcal{B}$ and $\mathcal{D}$?

- $\mathcal{B} = \mathbb{ISP}(\mathbf{2})$, where $\mathbf{2}$ is the 2-element algebra in $\mathcal{B}$.

- $\mathcal{D} = \mathbb{ISP}(\mathbf{2})$, where $\mathbf{2}$ is the 2-element algebra in $\mathcal{D}$.

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- Boolean algebras: $\boldsymbol{\mathcal{B}} = \mathbb{ISP}(\mathbf{2})$, where $\mathbf{2}$ is the 2-element algebra in $\boldsymbol{\mathcal{B}}$.
  Boolean spaces: $\boldsymbol{\mathcal{S}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_{\mathcal{T}})$, where $\underset{\sim}{\mathbf{2}}_{\mathcal{T}} = \langle\{0,1\};\mathcal{T}\rangle$; here $\mathcal{T}$ denotes the discrete topology.

- Bounded distributive lattices: $\boldsymbol{\mathcal{D}} = \mathbb{ISP}(\mathbf{2})$, where $\mathbf{2}$ is the 2-element algebra in $\boldsymbol{\mathcal{D}}$.

  Priestley spaces: $\boldsymbol{\mathcal{P}}_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_{\mathcal{T}})$, where $\underset{\sim}{\mathbf{2}}_{\mathcal{T}} = \langle\{0,1\};\leqslant,\mathcal{T}\rangle$, where $\leqslant$ is the usual order on $\{0,1\}$ and $\mathcal{T}$ is the discrete topology.

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Priestley spaces are of the form $\langle X;\leqslant,\mathcal{T}\rangle$ where

- $\langle X;\mathcal{T}\rangle$ is compact Hausdorff;
- given $x \nleqslant y$ in X, there exists a $\mathcal{T}$-clopen up-set U with $x \in U$ and $y \notin U$.

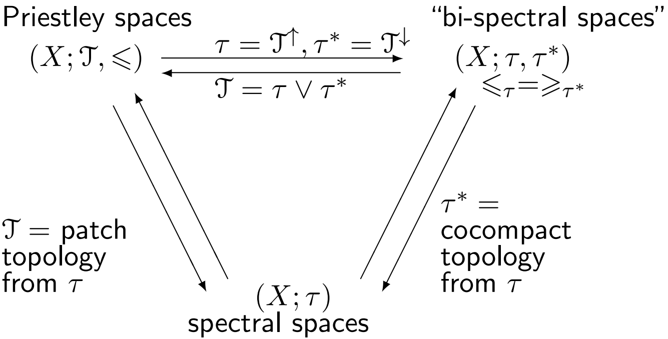
NOTE: this is stronger than saying that $\langle X;\mathcal{T}\rangle$ is a Boolean space and $\leqslant$ is closed in $X \times X$.
Morphisms: $\mathcal{T}$-continuous, $\leqslant$-preserving maps.

Spectral spaces $\cong$ Priestley spaces, as categories

Spectral space: $(X;\tau)$ such that

- compact
- base of compact-opens
- compact-opens closed under finite intersections
- sober [irreducible closed sets are point closures]

Morphisms: f s.t. f^{-1} takes compact-opens to compact-opens



$\mathcal{T}^\uparrow = \mathcal{T}$-open up-sets, $\mathcal{T}^\downarrow = \mathcal{T}$-open down-sets,

$x \leqslant_\tau y$ iff $x \in \mathrm{cl}_\tau\{y\}$ [specialisation order]

Priestley duality in full categorical dress

We have a dual equivalence between

$$\mathcal{D} = \mathbb{ISP}(\mathbf{2}) \qquad \text{and} \qquad \mathcal{P}_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_{\mathcal{T}}) \quad (\equiv \text{Priestley spaces})$$

set up by hom-functors $\mathsf{H} = \mathcal{D}(-, \mathbf{2})$ and $\mathsf{K} = \mathcal{P}_{\mathcal{T}}(-, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$:

$$\mathsf{H}\colon \mathcal{D} \to \mathcal{P}_{\mathcal{T}}, \qquad \begin{cases} \mathsf{H}(\mathbf{A}) = \mathcal{D}(\mathbf{A}, \mathbf{2}) \\ \mathsf{H}(f) = - \circ f \end{cases}$$

$$\mathsf{K}\colon \mathcal{P}_{\mathcal{T}} \to \mathcal{D}, \qquad \begin{cases} \mathsf{K}(\mathbf{X}) = \mathcal{P}_{\mathcal{T}}(\mathbf{X}, \underset{\sim}{\mathbf{2}}_{\mathcal{T}}) \\ \mathsf{K}(\phi) = - \circ \phi \end{cases}$$

Here $\mathbf{2} = \langle \{0,1\}; \wedge, \vee, 0, 1 \rangle$ and $\underset{\sim}{\mathbf{2}}_{\mathcal{T}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$. and $\mathcal{T}$ denotes the discrete topology.

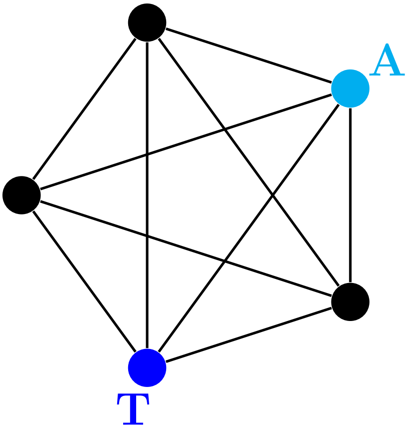
Specifically we have a dual adjunction $(\mathsf{H}, \mathsf{K}, e, \varepsilon)$ where the unit and counit maps are given by evaluations and are isomorphisms.

Other examples?

Let's see which other dual equivalences follow exactly the same pattern as those between

- Boolean algebras and Boolean spaces
- (Bounded) distributive lattices and Priestley spaces

Emphasise: Priestley spaces are structured topological spaces rather than topological spaces.
So are Boolean spaces, but you don't recognise you have a relational structure rather than a set when the set of relations is empty.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The basic framework: simplest case

Take $\mathbf{M}$ a finite algebra, with underlying set M. Let $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, the quasivariety generated by $\mathbf{M}$.
Special, but already encompasses, besides $\mathfrak{B}$ and $\mathfrak{D}$, a range of classes much studied for their algebraic and logical importance: e.g.

- De Morgan algebras
- Kleene algebras
- Stone algebras
- n-valued Łukasiewicz–Moisil algebras
- distributive bilattices

and significant subclasses of

- Heyting algebras—e.g. Gödel algebras, $\mathcal{G}_n$
- discriminator algebras

Objective

Given $\mathbf{M}$ and $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, we seek an **alter ego** $\underset{\sim}{\mathbf{M}}$ (or **dualising object**) for $\mathbf{M}$ so that there exists a dual equivalence between

$$\mathcal{A} = \mathbb{ISP}(\mathbf{M}) \quad \text{and} \quad \mathcal{X}_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_{\mathcal{T}})$$

set up by a dual adjunction obtained from hom-functors $\mathsf{D} = \mathcal{A}(-, \mathbf{M})$ and $\mathsf{E} = \mathcal{X}_{\mathcal{T}}(-, \underset{\sim}{\mathbf{M}}_{\mathcal{T}})$:

$$\mathsf{D}\colon \mathcal{A} \to \mathcal{X}_{\mathcal{T}}, \qquad \begin{cases} \mathsf{D}(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{M}) \\ \mathsf{D}(f) = - \circ f \end{cases}$$

$$\mathsf{E}\colon \mathcal{X}_{\mathcal{T}} \to \mathcal{A}, \qquad \begin{cases} \mathsf{E}(\mathbf{X}) = \mathcal{X}_{\mathcal{T}}(\mathbf{X}, \underset{\sim}{\mathbf{M}}_{\mathcal{T}}) \\ \mathsf{E}(\phi) = - \circ \phi \end{cases}$$

with ED and DE embeddings, and given by natural evaluation maps.

Hom-sets are structured from the powers in which they sit. Morphisms, being defined by composition, essentially take care of themselves.

But what form should the alter ego take?

The alter ego $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$ will be a discretely topologised structure on M. We shall include in the structure of $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$ relations and sometimes partial (to include total) operations too. Appropriate compatibility between $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$ will be needed.

Given $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$, the generated topological quasivariety $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_{\mathcal{T}})$ is the class of isomorphic copies of topologically closed substructures of powers of $\underset{\sim}{\mathbf{M}}$; the superscript indicates the empty structure is included.

Topology here works the obvious way; relations and partial operations are lifted pointwise to powers and then by restriction to substructures.

Definitions

The topology on M is fixed throughout, so we shall write $\underset{\sim}{\mathbf{M}}$ for the structure we get from $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$ by deleting $\mathcal{T}$.

We say $\underset{\sim}{\mathbf{M}}$

- yields a **pre-duality** if D and E are well-defined functors and ED and DE are embeddings;
- yields a **duality** (or **dualises** $\mathcal{A}$) if in addition $\mathrm{ED}(\mathbf{A}) \cong \mathbf{A}$ for all $\mathbf{A} \in \mathcal{A}$;
- a **full duality** if $\underset{\sim}{\mathbf{M}}$ yields a duality for which $\mathrm{DE}(\mathbf{X}) \cong \mathbf{X}$ for all $\mathbf{X} \in \mathfrak{X}_{\mathcal{T}}$ (then $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$ are dually equivalent).

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Can we get a (full) duality?

- Often NO—a rich theory of dualisability exists, but this would be a sidetrack here.
- Often YES—in particular whenever $\mathbf{M}$ is **lattice-based**. (though this is .certainly not a necessary condition).

Time to reiterate a WARNING: this is not an exercise to which pure category theory can provide a specific answer for specific choices of $\mathbf{M}$.

Good news!

Theorem

(NU Strong Duality Theorem, special case (Davey/Werner; Clark/Davey))

Let $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is a finite lattice-based algebra.

(i) *Let $R = \mathbb{S}(\mathbf{M}^2)$. Then $\underset{\sim}{\mathbf{M}} = \langle M; R \rangle$ yields a duality.*

(ii) *If the duality in (i) is not already full, then a full duality can be obtained by taking R as above and adding to the structure all partial homomorphisms from $\mathbf{M}^k$ to $\mathbf{M}$ for $0 \leqslant k \leqslant n$, where the bound n can be explicitly computed from $\mathbf{M}$.*

If every non-trivial subalgebra of $\mathbf{M}$ is subdirectly irreducible, then $n = 1$ suffices.

If $\mathbf{M}$ has no non-trivial subalgebras, then upgrading need involve only addition of the endomorphisms of $\mathbf{M}$ to $\underset{\sim}{\mathbf{M}}$.

Picking the ingredients apart

- The **compatibility** between $\underset{\sim}{\mathbf{M}}$ and $\mathbf{M}$ achieved by making the structure of $\underset{\sim}{\mathbf{M}}$ 'algebraic' ensures that the hom-functors D and E are well defined, and are embeddings.
- The assumption that $\mathbf{M}$ is **lattice-based** ensures that $\mathbf{M}$ has a 3-ary near-unanimity term (the median). This ensures dualisability and, moreover, that we need at most **binary** algebraic relations.
- The duality will fail to be full if $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$ generates too large a topological quasivariety. Adding extra structure to $\underset{\sim}{\mathbf{M}}$ in the form of suitable partial (taken to include total) operations solves this.
- 'Strong' refers to a condition (with many equivalents, one being $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$ injective in $\mathfrak{X}$) guaranteeing a duality is full. In a strong duality each of the hom-functors D and E interchanges surjections and embeddings—a bonus for applications.
- **Fundamental fact**: $\mathbf{F}\mathcal{A}(S)$, the free algebra on S generators, is such that $\mathrm{D}(\mathbf{F}\mathcal{A}(S)) = \underset{\sim}{\mathbf{M}}_{\mathcal{T}}^{S}$.

Simplifying a duality: entailment

The NU Strong Duality Theorem is powerful, but generally not economical.

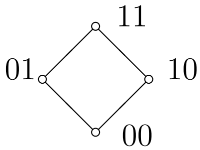
The natural dual space of $\mathbf{A} \in \boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ is $\mathrm{D}(\mathbf{A}) = \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$. Assuming $\underset{\sim}{\mathbf{M}}$ yields a duality, then $\mathbf{A} \cong \mathrm{ED}(\mathbf{A})$, the family of all continuous structure-preserving maps from $\mathrm{D}(\mathbf{A})$ to $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$.
Certainly:

- if a binary relation is preserved, then so is its converse;
- if binary relations r and s are preserved, then so is $r \cap s$;
- trivial relations (finite powers of M and any diagonal subalgebra) are automatically preserved.

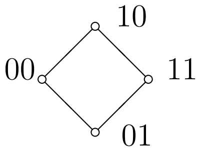
These are instances from a comprehensive list of **entailment constructs**, whereby redundant relations can harmlessly be deleted from an alter ego.

Examples, exploiting entailment

- **Priestley duality** $\mathbb{S}(\mathbf{2}^2)$ contains $\mathbf{2}^2$, diagonal subalgebra $\{(0,0),(1,1)\}$, and the orders $\leqslant$ and $\geqslant$. From above, we need only $\leqslant$.
- **De Morgan algebras:** take $\mathbf{M}$ the 4-element De Morgan algebra, with $\neg$ swapping 00 and 11 and fixing 01 and 10. $\underset{\sim}{\mathbf{M}} = \langle\{00, 01; 10, 11\}; \preccurlyeq, g, \rangle$. where g swaps 01 and 10, and fixes 00 and 11.
  Here $|\mathbb{S}(\mathbf{M}^2)| = 55$.



(lattice reduct of) $\mathbf{M}$



$\langle M; \preccurlyeq\rangle$

Optimising a duality: the Test Algebra Theorem

Suppose we have a dualising alter ego $\underset{\sim}{\mathbf{M}}_{\mathcal{T}} = \langle R; \mathcal{T} \rangle$. Then each $r \in R$ is algebraic, and so is the universe of a subalgebra $\mathbf{r}$ of some $\mathbf{M}^n$: SO $\mathbf{r} \in \boldsymbol{\mathcal{A}}$.

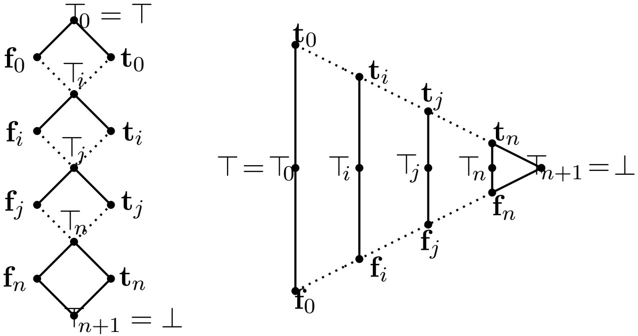
Theorem

The relation r can be discarded from R iff $\mathbf{r} \cong \mathsf{E}'\mathsf{D}'(\mathbf{r})$, where the hom-functors D' and E' are calculated with R replaced by $R' = R \setminus \{r\}$.

Example: a hierarchy of prioritised default bilattices

(Cabrer, Craig & Priestley, 2013) We set up strong dualities for $\mathbb{ISP}(\mathbf{K}_n)$ and (later, multisorted) for $\mathbb{HSP}(\mathbf{K}_n)$, where

$$\mathbf{K}_n = (K_n; \wedge_k, \vee_k, \wedge_t, \vee_t, \neg, \top, \bot).$$



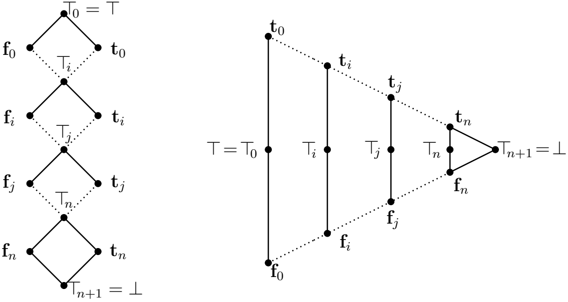
$\mathbf{K}_n$ in knowledge order $\leqslant_k$ (left) and truth order $\leqslant_t$ (right) with $0 < i < j < n$.

For $n > 0$, neither set of lattice operations is monotonic w.r.t. the other.

Interpretation for default logic

The elements of K_n represent levels of truth and falsity. Knowledge represented by the truth values at level $m+1$ is regarded as having lower priority than from those at level m. Also one thinks of $\mathbf{t}_{m+1}$ as being 'less true' than $\mathbf{t}_m$ and $\mathbf{f}_{m+1}$ as 'less false' than $\mathbf{f}_m$.

Base cases: $\mathbf{K}_0$ and $\mathbf{K}_1$ are Ginsberg's bilattices $\mathcal{FOUR}$ and $\mathcal{SEVEN}$, with $\neg$ added. In $\mathcal{SEVEN}$, $\mathbf{t}_1$ and $\mathbf{f}_1$ may be given the connotation of 'true by default' and 'false by default'.



Algebraic facts

- Every element of $\mathbf{K}_n$ is term-definable; $\mathbf{K}_n$ has no proper subalgebras.
- For $m \leqslant n$ there exists a surjective homomorphism $h_{n,m} \colon K_n \to K_m$.
- For $0 \leqslant m \leqslant n$, there exists $\mathcal{S}_{n,m} \in \mathbb{S}(\mathbf{K}_n^2)$ with elements

  $$\mathcal{S}_{n,m} = \Delta_n \cup \{ (a,b) \mid a, b \leqslant_k \top_{m+1} \text{ or } a \leqslant_k b \leqslant_k \top_m \}.$$

  where $\Delta_n = \{ (a,a) \mid a \in K_n \}$. We have $S_{n,j} \subsetneq S_{n,i}$ for $0 \leqslant i < j \leqslant n$.
- The subalgebras $\mathcal{S}_{n,m}$ entail every subalgebra of $\mathbf{K}_n^2$ via converses and intersections.
- Each $\mathbf{K}_n$ is subdirectly irreducible.
- $\mathbb{ISP}(\mathbf{K}_n) = \mathbb{HSP}(\mathbf{K}_n)$ iff $n = 0$. For $n \geqslant 1$,

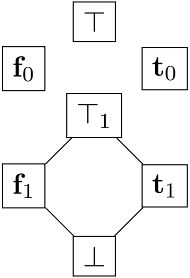
  $$\mathbb{HSP}(\mathbf{K}_n) = \mathbb{ISP}(\mathbf{K}_n, \dots, \mathbf{K}_0).$$

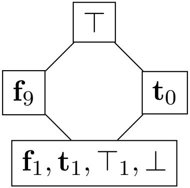
The algebraic binary relations: illustrations

$n = 0$: $\mathbb{ISP}(\mathbf{K}_0)$ is the variety $\mathcal{DB}$ of **distributive bilattices**. $S_{0,0}$ is the **knowledge order**, $\leqslant_k$.

$n = 1$: Our binary algebraic relations are $S_{1,0}$ and $S_{1,1}$ on K_1; these can be depicted as quasi-orders.

$S_{1,1}$





$S_{1,0}$

Theorem

Duality theorem for $\mathbb{ISP}(\mathbf{K}_n)$: *The structure $\underset{\sim}{\mathbf{K_n}} = \langle K_n; S_{n,n}, \ldots, S_{n,0}, \mathcal{T}\rangle$ yields a strong, and optimal, duality on $\mathbb{ISP}(\mathbf{K}_n)$.*

For $n = 0$, the dual category $\mathbb{IS}_c\mathbb{P}^+(\mathbf{K}_0)$ is $\mathfrak{P}_\mathcal{T}$.

From $\mathbb{ISP}$ to $\mathbb{HSP}$

In general $\mathbb{HSP}(\mathbf{M}) \neq \mathbb{ISP}(\mathbf{M})$.

However **Jónsson's Lemma** implies that for any **finitely generated lattice-based variety** $\mathcal{A} = \mathbb{HSP}(\mathbf{M})$ we do have

$$\mathbb{HSP}(\mathbf{M}) = \mathbb{ISP}(\mathfrak{M}) \text{ where } \mathfrak{M} = \mathbb{HS}(\mathbf{M}),$$

so that $\mathfrak{M}$ is a finite set of finite algebras, $\mathbf{M}_0, \ldots, \mathbf{M}_n$.

Multisorted dualities

We look for an alter ego $\mathfrak{M}_{\mathcal{T}} = \langle M_0 \,\dot\cup \ldots \dot\cup\, M_n; R, H, \mathcal{T}\rangle$, where now $\mathcal{T}$ is the disjoint union of the discrete topologies on each M_i. and R and H are algebraic, 'between $\mathbf{M}_i$'s'. SO an algebraic binary relation is a subalgebra of some $\mathbf{M}_i \times \mathbf{M}_j$.

We form powers of $\mathfrak{M}_{\mathcal{T}}$ 'by sorts': $\mathfrak{M}_{\mathcal{T}}^S = M_0^S \,\dot\cup \ldots \dot\cup\, M_n^S$, with R, H and $\mathcal{T}$ lifted in the obvious way.

The generated topological quasivariety $\mathfrak{X}_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}^+(\mathfrak{M}_{\mathcal{T}})$ has objects which are multisorted structures which are isomorphic copies of closed substructures of powers of $\mathfrak{M}_{\mathcal{T}}$. Morphisms in $\mathfrak{X}_{\mathcal{T}}$ are continuous maps preserving the sorts and the structure amongst them.

Given $\mathbf{A} \in \mathcal{A}$, we let $\mathsf{D}(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{M}_0) \,\dot\cup \cdots \dot\cup\, \mathcal{A}(\mathbf{A}, \mathbf{M}_n)$,

Given $\mathbf{X} = \mathbf{X}_0 \,\dot\cup \ldots \dot\cup\, \mathbf{X}_n \in \mathfrak{X}$, we let $\mathsf{E}(\mathbf{X}) = \mathfrak{X}(\mathbf{X}, \mathfrak{M}_{\mathcal{T}})$, viewing it as a subalgebra of $\mathbf{M}_0^{X_0} \times \cdots \times \mathbf{M}_n^{X_n}$.

Everything extends from the single-sorted case to the multisorted one. In particular the NU Strong Duality Theorem. Now $\mathsf{D}(\mathbf{F}\mathcal{A}(S)) = \mathfrak{M}_{\mathcal{T}}^S$.

Duality theorem for $\mathcal{A} = \mathbb{HSP}(\mathbf{K}_n)$

Theorem

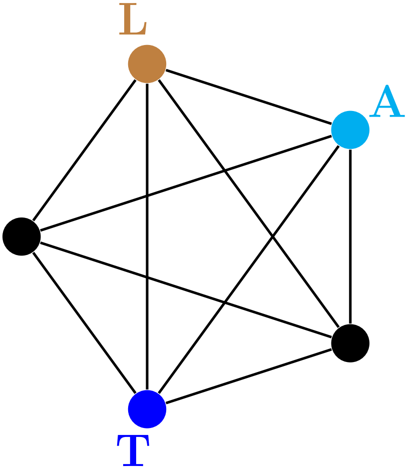
Write $\mathbb{HSP}(\mathbf{K}_n)$ as $\mathbb{ISP}(\mathfrak{M})$, where $\mathfrak{M} = \{\mathbf{K}_0 \dots, \mathbf{K}_n\}$. Then the alter ego

$$\mathfrak{M}_{\mathcal{T}} = \langle K_0 \,\dot\cup \dots \dot\cup\, K_n; \{S_{m,m}\}_{0\leqslant m\leqslant n}, \{h_{i,i-1}\}_{1\leqslant i\leqslant n}, \mathcal{T}\rangle,$$

yields a strong, and optimal, duality on $\mathbb{HSP}(\mathbf{K}_n)$.

The dual category for $\mathbb{HSP}(\mathbf{K}_n)$ can be described for general n.

This duality leads to a structure theorem for members of $\mathbb{HSP}(\mathbf{K}_n)$ which is beyond the reach of traditional bilattice methods.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$\mathcal{D}$-$\mathcal{P}_{\mathcal{T}}$-based dualities

Assume $\mathcal{A}$ is a variety of $\mathcal{D}$-based algebras, not necessarily finitely generated.

1. Take the class $\mathsf{U}(\mathcal{A})$ (the $\mathcal{D}$-reducts).
2. Seek to equip the associated class of Priestley spaces, $\mathcal{Z} := \mathsf{HU}(\mathcal{A})$, with additional (relational or functional) structure so that, for each $\mathbf{A} \in \mathcal{A}$, $\mathsf{KHU}(\mathbf{A})$ becomes an algebra in $\mathcal{A}$ isomorphic to $\mathbf{A}$;
3. Identify a suitable class of morphisms, to make $\mathcal{Z}$ into a category.

If this gives a dual equivalence between $\mathcal{A}$ and $\mathcal{Z}$, we say we have a $\mathcal{D}$-$\mathcal{P}_{\mathcal{T}}$-**based duality**.

Literature is full of examples!

Which way to go?

1. Natural duality theory: for any finitely generated $\mathfrak{D}$-based variety we can call on the NU Strong Duality Theorem (single-sorted or multisorted).
2. $\mathfrak{D}$-$\mathfrak{P}_{\mathfrak{T}}$-based dualities.
3. From, and for, logic: algebraic and relational (Kripke-style) semantics for non-classical propositional logics.

Both (1) and (2) provide valuable tools for studying algebraic properties of $\mathfrak{D}$-based varieties.

Normally, For a given variety, a discrete duality (as in (3)) differs from a $\mathfrak{D}$-$\mathfrak{P}_{\mathfrak{T}}$-based just through the absence or presence of topology. Canonical extensions provide a systematic approach to (3).

Rivals?

	Pro	Con
natural duality	a strong duality can always be found	duality may be complicated (may need entailment to simplify)
	$\mathbf{M}$ governs how the duality works	restriction to finitely generated classes (usually)
	good categorical properties, notably w.r.t. free algebras: $\mathrm{D}(F\mathcal{A})) = \underset{\sim}{\mathbf{M}}_{\mathfrak{T}}^{S}$	concrete representation is via functions, not sets, if $\lvert M \rvert > 2$
$\mathfrak{D}$-$\mathcal{P}_{\mathfrak{T}}$-based duality	close relationship to Kripke-style semantics	
	concrete representation via sets	products seldom cartesian; free algebras hard to find

Dualities in collaboration

Priestley duality per se has excellent properties: in particular

- embeddings/surjections in $\mathcal{D}$ correspond to surjections/embeddings in $\mathcal{P}_{\mathcal{T}}$;
- finite products in $\mathcal{D}$ correspond to finite disjoint unions in $\mathcal{P}_{\mathcal{T}}$;
- coproducts in $\mathcal{D}$ correspond to cartesian products in $\mathcal{P}_{\mathcal{T}}$;
- it is 'logarithmic'—a significant asset computationally.

With a $\mathcal{D}$-$\mathcal{P}_{\mathcal{T}}$-based duality one can exploit to some extent these excellent features of the parent duality. For a suitable class $\mathcal{A} = \mathbb{HSP}(\mathbf{M})$, it can be a great way to get a handle on, e.g.

- congruences, and subdirectly irreducible algebras,
- subalgebras, in particular of $\mathbf{M}^2$.

But a $\mathcal{D}$-$\mathcal{P}_{\mathcal{T}}$-based duality will seldom give easy access to free algebras or more to generally coproducts in $\mathcal{A}$. For such categorical notions we want a categorically natural duality—a natural duality.

SO: to get the best of both worlds we'd like to have BOTH a $\mathcal{D}$-$\mathcal{P}_{\mathcal{T}}$-based duality and a natural one, when available.

From a natural dual space to the associated Priestley dual space

For simplicity, take $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is finite and $\mathfrak{D}$-based. Let $\mathsf{H}\colon \mathfrak{D} \to \mathcal{P}_{\mathcal{T}}$ and $\mathsf{K}\colon \mathcal{P}_{\mathcal{T}} \to \mathfrak{D}$ be the hom-functors setting up Priestley duality. Assume we have a forgetful functor $\mathsf{U}\colon \mathcal{A} \to \mathfrak{D}$, given on objects by a term-reduct.

The key to linking $\mathsf{D}(\mathbf{A})$ and $\mathsf{HU}(\mathbf{A})$ (for any $\mathbf{A} \in \mathcal{A}$) is

$$\Omega = \mathsf{HU}(\mathbf{M}).$$

For $\omega_i, \omega_2 \in \Omega$, consider the following sublattice of $\mathsf{U}(\mathbf{M}^2)$:

$$(\omega_1, \omega_2)^{-1}(\leqslant) := \{\, (a, b) \in \mathbf{M}^2 \mid \omega_1(a) \leqslant \omega_2(b) \,\}.$$

Let R_{ω_1,ω_2} be the set (possibly empty) of **algebraic relations** maximal w.r.t. being contained in $(\omega_1, \omega_2)^{-1}(\leqslant)$.

FACT (part of the Multisorted Piggyback Duality Theorem)

$$R = \bigcup \{\, R_{\omega_1,\omega_2} \mid \omega_1, \omega_2 \in \Omega \,\}.$$

yields a duality on $\mathcal{A}$.

From $\mathsf{D}(\mathbf{A})$ to $\mathsf{HU}(\mathbf{A})$, continued

Fix $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, Remember that the dual space $\mathsf{D}(\mathbf{A}) = \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ is viewed as a closed substructure of $\underset{\sim}{\mathbf{M}}^{A}$ and carries relations $r^{\mathsf{D}(\mathbf{A})}$, for $r \in R$, obtained by pointwise lifting.
Define $\preccurlyeq$ on $\mathsf{D}(\mathbf{A}) \times \Omega$ by

$$(x, \omega_1) \preccurlyeq (y, \omega_2) \iff (x, y) \in r^{\mathsf{D}(\mathbf{A})} \text{ for some } r \in R_{\omega_1, \omega_2}.$$

Theorem

(Cabrer & Priestley, 2012) Let $\approx$ be the equivalence relation $\preccurlyeq \cap \succcurlyeq$, Then the map $\Psi \colon (\mathsf{D}(\mathbf{A}) \times \Omega)/{\approx} \to \mathsf{HU}(\mathbf{A})$ given by

$$[(x, \omega)]_{\approx} \longmapsto \omega \circ x \quad (x \in \mathsf{D}(\mathbf{A}),\ \omega \in \mathsf{HU}(\mathbf{M}))$$

is well defined and a Priestley space isomorphism.

SO: $\mathbf{Y}_{\mathbf{A}} = (\mathsf{D}(\mathbf{A}) \times \Omega)/{\approx}$ 'is' the Priestley dual of $\mathsf{U}(\mathbf{A})$.

Remarks

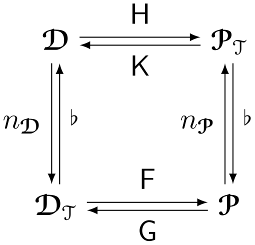
- When any additional operations in $\mathbf{A}$ are determined by the underlying lattice order, then $\mathbf{A}$ is uniquely determined by $\mathsf{HU}(\mathbf{A})$. This happens, e.g., whenever $\mathbf{M}$ is a Heyting algebra or is pseudocomplemented.
- In general, we fully recapture $\mathbf{A}$ only once we equip $\mathbf{Y}_{\mathbf{A}}$ with extra structure to model additional operations. Work in progress as to how to do this in general; special cases easy to handle—whatever, what happens with $\mathbf{M}$ fully determines process for general $\mathbf{A}$.
- In a few cases, there exists $\omega \in \Omega$ such that $\mathsf{D}(\mathbf{A}) \times \{\omega\} \cong \mathsf{HU}(\mathbf{A})$ and, at the level of Priestley space reducts, the natural duality 'is' a $\mathfrak{D}$-$\mathcal{P}_{\mathfrak{T}}$-based duality. This happens, e.g., for De Morgan algebras and Stone algebras.
- The translation process is the key to understanding how coproducts work in finitely generated $\mathfrak{D}$-based quasivarieties (Cabrer and Priestley, 2012) . If you didn't hear Leo Cabrer's talk on this at TACL, too bad!

Two dualities in partnership: Priestley duality and Banaschewski duality

$\mathcal{D} := \mathbb{ISP}(\mathbf{2})$, $\mathcal{P} := \mathbb{IS}^0\mathbb{P}^+(\underset{\sim}{\mathbf{2}})$ (posets),

$\mathcal{P}_{\mathcal{T}} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\underset{\sim}{\mathbf{2}}_{\mathcal{T}})$, $\mathcal{D}_{\mathcal{T}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\mathbf{2}_{\mathcal{T}})$ (Boolean-topological DLs)

Technical note: $\mathbb{P}$ *allows empty indexed products, yielding the total 1-element structure;* $\mathbb{P}^+$ *doesn't. Operator* $\mathbb{S}$ *excludes the empty structure while* $\mathbb{S}^0$ *includes it, when there are no nullary operations.*



Here the top adjunction gives Priestley duality. The bottom one gives the duality between $\mathcal{P}$ and $\mathcal{D}_{\mathcal{T}}$ (Banaschewski, 1976). Symbol $^\flat$ denotes the functor forgetting topology.

Unanswered questions

- Does the duality between $\mathcal{P}$ and $\mathcal{D}_{\mathcal{T}}$ fit into a theory of natural dualities, for structures, rather than for algebras?
  ANSWER: YES. We can formulate a notion of compatibility between two structures $\mathbf{M}_1$ and $\underset{\sim}{\mathbf{M}}_2$ on the same finite set M (each may include relations and partial (including total) operations). But dualisability questions are non-trivial in general. The duality between $\mathcal{P}$ and $\mathcal{D}_{\mathcal{T}}$ fits into this generalised framework.
- **Buy one, get one free?**: When In general, does one duality (such as that between $\mathcal{D}$ and $\mathcal{P}_{\mathcal{T}}$), have a partner duality obtained by swapping the topology from one category to the other?
  ANSWER: Yes, sometimes, but not always.
- In the example, what are the vertical arrows in the square diagram doing?
  ANSWER: It might have something to do with **canonical extensions** . . ..

Stories for another day!