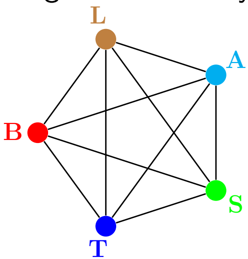
DUALITY THEORY AND B L A S T : Selected Themes Part II: Canonical Extensions, Profinite Completion and Natural Dualities

Hilary Priestley

Mathematical Institute, University of Oxford

With acknowledgements to very many people

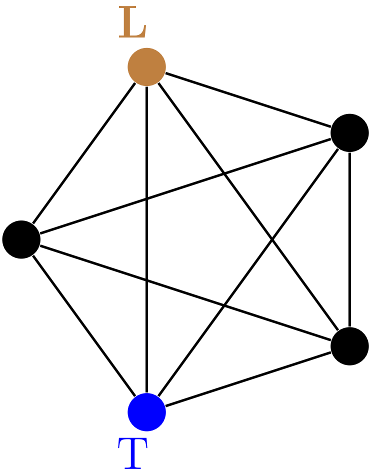


Outline: Part II

with principal sources

- Canonical extensions of unital semilattices and bounded lattices: a fast-track approach
  (Gouveia and Priestley, 2012)
- Canonical extensions of bounded lattices and Choice Principles
  (Erné, 2012)

- The natural extension construction: Bohr compactifications of discrete structures
  (Davey, Gouveia, Haviar and Priestley, 2011→)
- Natural dualities via Ind- and Pro-completion? Some brief comments


L
T

The historical development of the theory of canonical extensions

Motivation came from study of ordered algebras. But shall focus here on the underlying ordered structures.

$\mathfrak{B}$	Boolean algebras	Jónsson and Tarski	1951
$\mathfrak{D}$	Bounded distributive lattices	Gehrke and Jónsson	1994
	Bounded lattices	Gehrke and Harding	2001
$\mathcal{P}$	Posets	Dunn, Gehrke and Palmigiano	2005

But what about (unital) semilattices?

The historical development of the theory of canonical extensions

†	$\mathcal{B}$	Boolean algebras	Jónsson and Tarski	1951
†	$\mathcal{D}$	Bounded distributive lattices	Gehrke and Jónsson	1994
		Bounded lattices	Gehrke and Harding	2001
†	$\mathcal{S}_\wedge$	$\wedge$-semilattices with 1	Gouveia and Priestley	2012
	$\mathcal{S}_\vee$	$\vee$-semilattices with 0		
	$\mathcal{P}$	Posets	Dunn, Gehrke and Palmigiano	2005

† : finitely generated varieties having natural dualities

Canonical extensions of semilattices and lattices: a fast-track approach

(Gouveia and Priestley, 2012, with acknowledgements to Cabrer and to Jipsen & Moshier)

Let $\mathbf{S} \in \mathcal{S}_\wedge$—meet semilattices with 1. Then let

$\mathrm{Filt}(S) =$	filters of $\mathbf{S}$	non-empty up-sets closed under $\wedge$,
$\mathrm{Idl}(\mathbf{S}) =$	ideals of $\mathbf{S}$	directed down-sets

We have order-reversing principal filter embeddings of $\mathbf{S}$ into $\mathrm{Filt}(\mathbf{S})$ denoted by $\uparrow$ and of $\mathrm{Filt}(\mathbf{S})$ into $\mathrm{Filt}^2(\mathbf{S}) = \mathrm{Filt}(\mathrm{Filt}(^2(\mathbf{S}))$, denoted by $\Uparrow$.

We can embed $\mathbf{S}$ in $\mathrm{Filt}^2(\mathbf{S})$ (right way up) via $e\colon a \mapsto \Uparrow(\uparrow a)$.

Note $\mathrm{Filt}^2(\mathbf{S})$ is an algebraic closure system: complete lattice in which meet is given by intersection and directed join by union. ($\mathrm{Filt}^2(\mathbf{S})$ concretely models the free join completion of the free meet completion of $\mathbf{S}$.)

Properties of the completion $(e, \mathrm{Filt}^2(\mathbf{S}))$

Operations in $\mathrm{Filt}^2(\mathbf{S})$:

$$\bigsqcap e(F) = \Uparrow F \qquad \text{if } F \in \mathrm{Filt}(\mathbf{S}),$$
$$\bigsqcup e(J) = \bigcup e(J) \qquad \text{if } J \text{ is directed.}$$

$\big($ $\bigsqcap$, $\bigsqcup$ used for directed join and down-directed meet.$\big)$

Properties of the completion $(e, \mathrm{Filt}^2(\mathbf{S}))$

Operations in $\mathrm{Filt}^2(\mathbf{S})$:

$$\begin{aligned} \textstyle\bigsqcap e(F) &= \Uparrow F && \text{if } F \in \mathrm{Filt}(\mathbf{S}), \\ \textstyle\bigsqcup e(J) &= \bigcup e(J) && \text{if } J \text{ is directed.} \end{aligned}$$

($\bigsqcap$, $\bigsqcup$ used for directed join and down-directed meet.)

Theorem

(**2/3-canonicity property of** $\mathrm{Filt}^2(\mathbf{S})$) *The completion* $(e, \mathrm{Filt}^2(\mathbf{S}))$ *of* $\mathbf{S}$ *is*

- *compact: for* $F \in \mathrm{Filt}(\mathbf{S})$ *and* $J \in \mathrm{Idl}(\mathbf{S})$,

$$\textstyle\bigsqcap e(F) \leqslant \bigsqcup e(J) \implies F \cap J \neq \emptyset;$$

- $\bigsqcup\bigsqcap$*-dense.*

For a canonical extension we need also $\bigwedge\bigvee$-density. Let's enforce this by restricting to a subset of $\mathrm{Filt}^2(\mathbf{S})$.

The canonical extension of a unital semilattice

Theorem

Let $\mathbf{S} \in \mathcal{S}_\wedge$. *Let*

$$\mathbf{S}^\delta = \{\, \mathcal{F} \in \mathrm{Filt}(\mathrm{Filt}(\mathbf{S})) \mid \mathcal{F} \textit{ is a meet of directed joins of elements from } e(S) \,\}.$$

Then

- e *is an* $\mathcal{S}_\wedge$*-embedding of* $\mathbf{S}$ *into* $\mathbf{S}^\delta$;
- *In* $\mathrm{Filt}^2(\mathbf{S})$ *and in* $\mathbf{S}^\delta$, *meets are given by* $\bigcap$ *and directed joins by* $\bigcup$.
- $(\overline{e}, \mathbf{S}^\delta)$ *is a* **canonical extension** *of* $\mathbf{S}$ (*here* $\overline{e}$ *denotes* e *with codomain restricted to* $\mathbf{S}^\delta$).
- $\mathbf{S}^\delta$ *coincides with the Galois-closed sets for the polarity* $(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S}), R)$, *where* $F\ R\ J$ *iff* $F \cap J = \emptyset$.

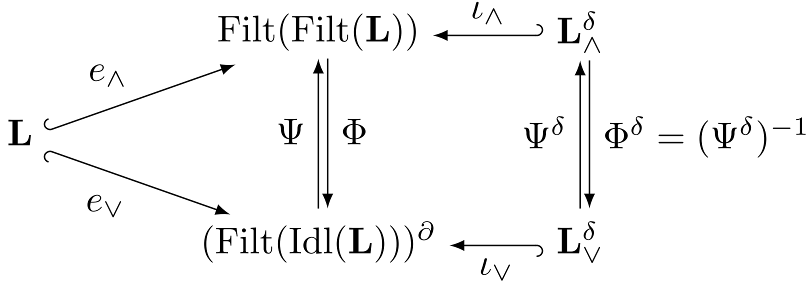
Final statement is a CONSEQUENCE of earlier ones.

Canonical extensions of bounded lattices: relating iterated free completions of semilattice reducts

Let $\mathbf{L}$ be a bounded lattice, with unital semilattice reducts $\mathbf{L}_\wedge$ and $\mathbf{L}_\vee$.

In the diagram, Φ and Ψ define an adjunction. Restriction maps Φ^δ and Ψ^δ set up mutually inverse isomorphisms—they are the maps from the usual polarity $(\mathrm{Filt}(\mathbf{L}), \mathrm{Idl}(\mathbf{L}), R)$.

In general the inclusion maps $\iota_\wedge$ and $\iota_\vee$ are not surjectve.



Back to semilattices: incarnations of $\mathrm{Filt}^2(\mathrm{S})$

Duality strikes again!

Hofmann–Mislove–Stralka duality for $\mathcal{S}_\wedge$: Let

$$\mathcal{S}_\wedge = \mathbb{ISP}(\mathbf{2}) \qquad \mathbf{2} = \langle \{0,1\}; \wedge, 1 \rangle,$$
$$\mathcal{S}_{\wedge\mathcal{T}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_{\mathcal{T}}) \qquad \underset{\sim}{\mathbf{2}}_{\mathcal{T}} = \langle \{0,1\}; \wedge, 1, \mathcal{T} \rangle.$$

Then $\mathcal{S}_{\wedge\mathcal{T}}$ gives the category of compact 0-dimensional topological semilattices *alias* algebraic lattices with maps preserving $\bigsqcup$ and $\bigwedge$. The hom-functors $\mathsf{D} = \mathcal{S}_\wedge(-, \mathbf{2})$ and $\mathsf{E} = \mathcal{S}_{\wedge\mathcal{T}}(-, \underset{\sim}{\mathbf{2}}_{\mathcal{T}})$ yield a full duality between $\mathcal{S}_\wedge$ and $\mathcal{S}_{\wedge\mathcal{T}}$.

Via characteristic functions we can identify

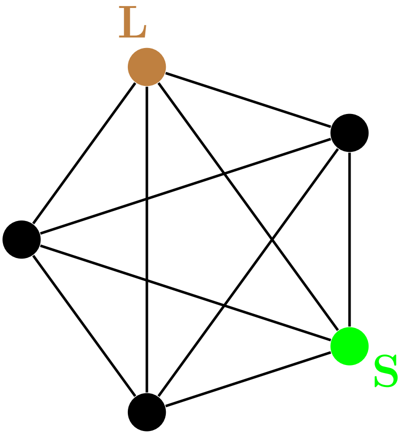
$$\mathrm{Filt}^2(\mathbf{S}) \text{ and } (\mathsf{D}(\mathsf{D}(\mathbf{S})^\flat))^\flat \text{ —denote this by } \widehat{\mathbf{S}}.$$

Here $^\flat$ is the functor which forgets topology.

Canonical extensions, functorially

Easy to show that

- a $\mathcal{S}_\wedge$ morphism $f\colon \mathbf{S} \to \mathfrak{T}$ lifts to $\widehat{f}\colon \widehat{\mathbf{S}} \to \widehat{\mathfrak{T}}$ preserving $\bigsqcup$ and $\bigwedge$;
- by restriction, a $\mathcal{S}_\wedge$ morphism $f\colon \mathbf{S} \to \mathfrak{T}$ lifts to $f^\delta\colon \mathbf{S}^\delta \to \mathfrak{T}^\delta$; this preserves $\bigwedge$ and directed joins of elements from $\overline{e}(S)$;
- a bounded lattice morphism $f\colon \mathbf{L} \to \mathbf{K}$ lifts to a complete lattice homomorphism $f^\delta\colon \mathbf{L}^\delta \to \mathbf{K}^\delta$.
  (Proof by looking at both semilattice reducts and doing a simple diagram-chase.)


L
S

Canonical extensions of distributive lattices, with Choice

If $\mathbf{L} \in \mathfrak{D}$, then in [ZFC] we know, by Priestley duality, that the canonical extension $\mathbf{L}^\delta$ is the up-set lattice of its prime filters (under $\subseteq$) and so is a complete ring of sets. As such, it is

(1) **completely distributive**;

(2) **superalgebraic**: it satisfies one, and hence all, of the equivalent conditions:

  (a) the completely join-prime elements are join-dense;
  (b) $\mathbf{L}^\delta$ is a frame and the completely join-irreducibles are join-dense;
  (c) [splitting pairs] $a \not\leqslant b$ in $\mathbf{L}^\delta$ implies $\exists p, \kappa(p)$ such that $a \not\leqslant p$, $b \not\leqslant \kappa(p)$ and ${\downarrow}p \cup {\uparrow}\kappa(p) = \mathbf{L}^\delta$;

(3) **weakly atomic**.

Canonical extensions of bounded lattices, with and without Choice

For sure, existence and uniqueness of $\mathbf{L}^\delta$ for a bounded lattice $\mathbf{L}$ need only [ZF].

FACTS:

Let $\mathbf{L}$ be a bounded lattice.

- $\mathbf{L} \in \mathfrak{D}$ implies $\mathbf{L}^\delta \in \mathfrak{D}$ (Gehrke & Harding, 2001, implicitly).
- $\mathbf{L} \in \mathfrak{D}$ implies $\mathbf{L}^\delta$ a frame (Gehrke, 2011)

QUESTION: What choice principles are required in order that $\mathbf{L}^\delta$ should have the properties of a complete ring of sets, for $\mathbf{L} \in \mathfrak{D}$? What can be said for general bounded lattices?

ANSWERS: given below (Erné, 2012)

Separating filters and ideals

Given a bounded lattice $\mathbf{L}$ let $\mathcal{F} = \mathrm{Filt}(\mathbf{L})$ and $\mathcal{I} = \mathrm{Idl}(\mathbf{L})$.
Given $F \in \mathcal{F}$ and $I \in \mathcal{I}$ with $F \cap I = \emptyset$, there exist $P \in \mathcal{F}$ and $Q \in \mathcal{I}$ such that $F \subseteq P$, $I \subset Q$ and

$Q = L \setminus P$ — Prime Separation Property

$F \cap Q = \emptyset,\ P \cap I = \emptyset$ and $P \cup Q = L$ — Normal Separation Property.

Lemma

[ZF] $\mathbf{NSP} \implies \mathbf{PSP} \implies \mathbf{L}$ *distributive.*

Theorem

In $[\mathbf{Z}F]$*:*

(i) *A bounded lattice* $\mathbf{L}$ *satisfies* $(\mathbf{PSP})$ *iff* $\mathbf{L}^\delta$ *is superalgebraic.*

(ii) *A bounded lattice* $\mathbf{L}$ *satisfies* $(\mathbf{NSP})$ *iff* $\mathbf{L}^\delta$ *satisfies the choice-free formulation of complete distributivity.*

Equivalents of the Ultrafilter Principle

Theorem

The Ultrafilter Principle is equivalent to each of the following:

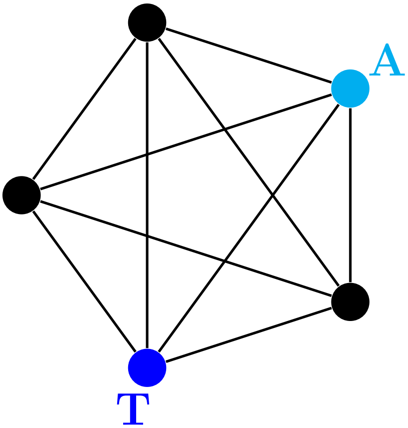
(1) *Any bounded distributive lattice satisfies* $(\mathbf{PSP})$.

(2) *Any bounded distributive lattice satisfies* $(\mathbf{NSP})$.

(3) *Any bounded distributive lattice has a canonical extension with property* $(\star)$.

(4) *Any Boolean lattice has a canonical extension with property* $(\star)$.

Here the property $(\star)$ *may be any of:*
superalgebraic, spatial frame, algebraic, weakly atomic; completely distributive, or versions of this restricted to families of 2-element sets or of finite sets.


A
T

Residually finite (pre)-varieties

Suppose $\mathfrak{M}$ is a set (not necessarily finite) of finite algebras of common type and let $\mathcal{A} = \mathbb{ISP}(\mathfrak{M})$. Except that the term is usually used for varieties, this is **residual finiteness**.
Let

$$\underset{\sim}{\mathfrak{M}}_{\mathcal{T}} = \langle \bigcup \{\, M \in \mathfrak{M} \,\}; R, \mathcal{T} \rangle,$$

where union is disjoint(ified) and R is a set of finitary algebraic relations. Let $\mathcal{X}_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathfrak{M}}_{\mathcal{T}})$. This would be an appropriate set-up for a multisorted natural duality except that now we don't assume $\mathfrak{M}$ is finite.
We have well-defined hom-functors

$$\mathrm{D} \colon \mathcal{A} \to \mathcal{X}_{\mathcal{T}} \quad \text{and} \quad \mathrm{E} \colon \mathcal{X}_{\mathcal{T}} \to \mathcal{A}.$$

Moreover ED and DE are embeddings, and given by (multisorted) evaluation maps.

The natural extension functor, for a class $\mathbb{ISP}(\mathfrak{M})$ of algebras

Let $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathfrak{M})$ (as above) and let $\boldsymbol{\mathcal{A}}_{\mathfrak{T}} = \mathbb{IS}_{c}\mathbb{P}^{+}(\underset{\sim}{\mathfrak{M}}_{\mathfrak{T}})$. Then there exists a covariant functor $n_{\boldsymbol{\mathcal{A}}} \colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}_{\mathfrak{T}}$ with the following properties:

- $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ is a Boolean-topological algebra whose algebra reduct belongs to $\boldsymbol{\mathcal{A}}$;
- $n_{\boldsymbol{\mathcal{A}}}$ is a reflector into a (non-full) subcategory of $\boldsymbol{\mathcal{A}}_{\mathfrak{T}}$ and is left-edjoint to the forgetful functor $^{\flat}$ from $\boldsymbol{\mathcal{A}}_{\mathfrak{T}}$ to $\boldsymbol{\mathcal{A}}$.

But where does this comes from?
We haven't involved the hom functors D and E yet ...

The natural extension via paired adjunctions

$$
\begin{array}{ccc}
\mathcal{A} & \underset{\mathsf{E}}{\overset{\mathsf{D}}{\rightleftarrows}} & \mathcal{X}_{\mathcal{T}} \\
{\scriptstyle n_{\mathcal{A}}}\downarrow\uparrow{\scriptstyle \flat} & & \downarrow{\scriptstyle \flat} \\
\mathcal{A}_{\mathcal{T}} & \underset{\mathsf{G}}{\overset{\mathsf{F}}{\rightleftarrows}} & \mathcal{X}
\end{array}.
$$

Here the lower adjunction works just the same way as the upper one, except that the topology has been moved from $\underset{\sim}{\mathfrak{M}}$ to $\boldsymbol{\mathfrak{M}}$:
$\mathcal{A}_{\mathcal{T}} = \mathbb{IS}_{c}\mathbb{P}(\boldsymbol{\mathfrak{M}}_{\mathcal{T}})$ and $\mathcal{X} = \mathbb{ISP}(\boldsymbol{\mathfrak{M}})$.

Paired Adjunctions Theorem: TFAE, if $\underset{\sim}{\mathbf{M}}$ is of finite type:

(1) outer square commutes, i.e, $n_{\mathcal{A}}(\mathbf{A}) = \mathsf{G}(\mathsf{D}(\mathbf{A})^{\flat})$, $\forall \mathbf{A} \in \mathcal{A}$.

(2) $n_{\mathcal{A}}(\mathbf{A})$ consists of all multisorted maps
$\alpha\colon \bigcup_{\mathbf{M}\in\mathfrak{M}} \mathcal{A}(\mathbf{A}, \mathbf{M}) \to M$ that preserve the structure of $\underset{\sim}{\mathfrak{M}}$,
for all $\mathbf{A} \in \mathcal{A}$ [no topology!].

And If $\boldsymbol{\mathfrak{M}}$ is finite, (1) and (2) are equivalent to

(3) $\underset{\sim}{\mathfrak{M}}$ yields a multisorted duality between $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$.

Given (3), we can use (2) to describe $n_{\mathcal{A}}(\mathbf{A})$ for $\mathbf{A} \in \mathcal{A}$.

So what is this natural extension gadget?

The **profinite completion** $\mathrm{Pro}_{\mathcal{A}}(\mathbf{A})$ of an algebra $\mathcal{A}$ in a finitely generated variety of the form $\mathcal{A}$ is the projective limit of the finite quotients of $\mathbf{A}$. The residual finiteness assumption implies that each $\mathbf{A} \in \mathcal{A}$ has a **profinite completion** $\mathrm{Pro}_{\mathcal{A}}(\mathbf{A})$. If $\mathcal{A}$ is a residually finite pre-variety, but not a variety, we just restrict to finite quotients which belong to $\mathcal{A}$.
FACT: there is a canonical embedding $\mu_{\mathbf{A}} \colon \mathbf{A} \to \mathrm{Pro}_{\mathcal{A}}(\mathbf{A})$.

[Profinite objects in a category, are, loosely, those which are built from (discretely topologised) finite ones by means of filtered colimits. In context of algebras, profinite objects should be viewed as topological algebras.]

Theorem
Let $\mathcal{A} = \mathbb{ISP}(\mathfrak{M})$ be a residually finite pre-variety. Then $n_{\mathcal{A}}(\mathbf{A})$ and $\mathrm{Pro}_{\mathcal{A}}(\mathbf{A})$ are isomorphic as topological algebras.

Examples and comments I

The Paired Adjunctions Theorem gives access to $n_{\mathcal{A}}(\mathbf{A})$, and in a quite explicitl way if we have $\mathfrak{M}$ finite and a set R yielding a natural duality on $\mathcal{A}$. Profinite completions are hard to describe directly.

Lattices:

- Take $\mathcal{A} = \mathfrak{D} = \mathbb{ISP}(\mathbf{2})$. Then $n_{\mathfrak{D}}(\mathbf{L})$, as calculated from the Paired Adjunctions Theorem, is exactly the canonical extension $\mathbf{L}^{\delta}$, as the latter was defined by Gehrke and Jónsson. So we can re-obtain the known result that $\mathbf{L}^{\delta} \cong \mathrm{Pro}_{\mathfrak{D}}(\mathbf{L})^{\flat}$.
- If $\mathcal{A}$ is a finitely generated lattice-based variety of finite type, then $\mathbf{A}^{\delta} \cong \mathrm{Pro}_{[} \mathcal{A}^{\flat}$ (Harding, 2006 + Gouveia 2009). $n_{\mathcal{A}}(\mathbf{A})$ provides another description.

Examples and comments II

Semilattices, and beyond:

Consider $\mathcal{S}_\wedge$. Then HMS duality tells us that $n_{\mathcal{S}_\wedge}(\mathbf{S})^\flat \cong \mathrm{Filt}^2(\mathbf{S})$. As a topological algebra, $n_{\mathcal{S}_\wedge}(\mathbf{S}) \cong \mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S})$ is known as the **Bohr compactification** of $\mathbf{S} \in \mathcal{S}_\wedge$.

Note that HMS duality is special in that the algebra persona and the alter ego persona are the same, apart from the topology. (cf. Pontryagin duality for abelian groups, where same phenomenon occurs—this is a rare instance of a natural duality with an infinite generating algebra, in this case the circle group.)

Examples and comments III

Lattices and their unital semilattice reducts: completions compared

For $\mathbf{S} \in \mathcal{S}_\wedge$, the profinite completion $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S}) = \widehat{\mathbf{S}}$ and the canonical extension $\mathbf{S}^\delta$ in general do NOT coincide—indeed, $\widehat{\mathbf{S}}$ may have strictly larger cardinality than $\mathbf{S}^\delta$.

Now let $\mathbf{L}$ be a bounded distributive lattice. Then the canonical extensions $\mathbf{L}^\delta$, $\mathbf{L}^\delta_\wedge$ and $\mathbf{L}^\delta_\vee$ are all the same. (Canonical extension uniquely determined by the underlying poset.)

The profinite completions $\mathrm{Pro}_{\mathfrak{D}}(\mathbf{L})$, $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{L}_\wedge)$ and $\mathrm{Pro}_{\mathcal{S}_\vee}(\mathbf{L}_\vee)$ all exist, but may differ:—

Theorem

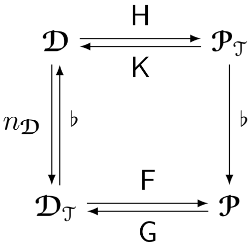
(Gouveia and Priestley, 2012) *For $\mathbf{L} \in \mathfrak{D}$, the three profinite completions coincide if and only if the Priestley dual space* $\mathsf{H}(\mathbf{L})$ *contains no infinite antichain.*

Proof relies on HMS and Priestley dualities and a lot of theory of continuous lattices.

Two dualities in partnership: Priestley duality and Banaschewski duality revisited: paired adjunctions

$\mathcal{D} := \mathbb{ISP}(\mathbf{2})$, $\qquad \mathcal{P} := \mathbb{ISP}(\underset{\sim}{\mathbf{2}})$ (posets),

$\mathcal{P}_\mathcal{T} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}_\mathcal{T})$, $\qquad \mathcal{D}_\mathcal{T} := \mathbb{IS}_c\mathbb{P}(\mathbf{2}_\mathcal{T})$ (Boolean-topological DLs)



Here the top adjunction gives Priestley duality. The bottom one gives the duality between $\mathcal{P}$ and $\mathcal{D}_\mathcal{T}$ (Banaschewski, 1976). Symbol $^\flat$ denotes the functor forgetting topology.

From algebras to structures

The term **Bohr compactification** suggests we are thinking in terms of structures rather than algebras. Indeed we should. As hinted in first talk, taking Banaschewski duality as a potential example, the natural duality framework extends to this wider setting.

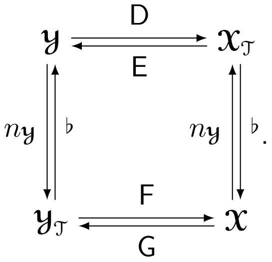
Compatibility: $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}$ are **compatible structures** on the same finite set M (operations, relations and partial operations allowed) if the structure of $\underset{\sim}{\mathbf{M}}$ is preserved by the operations and partial operations of $\mathbf{M}$ and the relations are substructures.

- This notion is symmetric.
- No presumption that $\mathbf{M}$ is "algebraic" and $\underset{\sim}{\mathbf{M}}$ "relational".
- We can extend this to the multisorted setting.

[A little care is needed over the inclusion, or not, in the discrete or topologised generated classes. Not a big issue, and we slur over it here.]

A unifying framework: the natural extension functor for structures

With a little care, the natural extension functor works, as before, but now for classes $\mathfrak{X}$ and $\mathfrak{Y}$ which are generated by a pair of mutually compatible structures. We get a Paired Adjunctions Theorem based on a commuting diagram:



CAVEAT: The identification of the natural extension $n_{\mathfrak{Y}}(\mathbf{Y})$ with a profinite completion is NOT available when $\mathfrak{Y}$ is a class of structures with relations as well as operations in the type.

Natural extension: examples

The natural extension functor for classes of structures $\mathbb{ISP}(\mathfrak{M})$ should be seen as providing a common umbrella for assorted results seen variously as belonging to algebra or to topology:

- Profinite completions in the context of a residually finite variety, with an explicit description if the variety has a natural duality.
- Stone-Čech compactification of a discrete space.
- ordered Stone-Čech compactification of a poset, or of a quasi-ordered set.
- Hybrid algebraic/relational examples can be found
- ....

A dual equivalence on the cheap: Hofmann–Mislove–Stralka duality for semilattices

$$\mathcal{S} = \mathbb{ISP}(\mathbf{2}) \qquad \wedge, 1 - \text{semilattices}$$

$$\mathcal{Z} = (\mathcal{S}_\wedge)_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}(\mathbf{2}_{\mathcal{T}}) \qquad \text{compact } 0\text{-dimensional semilattices}$$

(Here we have two categories rather than four.)

On discretely topologised, objects the topology does no work, so

$$\mathcal{Z}_{\mathsf{fin}} \text{ ``is'' } \mathcal{S}_{\mathsf{fin}}.$$

With this identification the evaluation maps are just identities. SO we have a dual equivalence at the level of finite objects.
Easy:

$\mathcal{S}$ is built from $\mathcal{S}_{\mathsf{fin}}$ by taking directed (cofiltered) limits,

$\mathcal{Z}$ is built from $\mathcal{Z}_{\mathsf{fin}}$ by taking projective limits (filtered colimits).

and the limits/colimits are preserved by the functors.

Are Ind- and Pro-completions the whole story?

Take a finitely generated quasivariety $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$.

(1) Assume we can find an alter ego $\underset{\sim}{\mathbf{M}}$ (of finite type) and topological quasivariety $\mathfrak{X}_{\mathfrak{T}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ such that there exists a full duality between $\mathcal{A}_{\text{fin}}$ and $(\mathfrak{X}_{\mathfrak{T}})_{\text{fin}}$.

(2) **IF** $\underset{\sim}{\mathbf{M}}$ is a **total** structure (i.e. contains no partial operations) then we get a full duality between $\mathcal{A}$ and $\mathfrak{X}_{\mathfrak{T}}$, by lifting via Ind- and Pro-completions.

BUT

(3) There exist examples in which we cannot achieve (1) without including partial operations in $\underset{\sim}{\mathbf{M}}$ and when we do so, $\mathfrak{X}_{\mathfrak{T}}$ is not the Pro-completion of $(\mathfrak{X}_{\mathfrak{T}})_{\text{fin}}$.

CATCH 22!

Moral: There's more to natural duality theory than abstract category theory can address.

The inheritance from Marshall Stone?

Marshall Stone

Pathways, Patios, Walls and More.

We invite you to come in and browse...for *ideas*, inspiration, and the products and **know-how** to put it all together.

About Us | Uses & Applications | Products | Literature | Do-It-Yourself | Contact Us | News & Events

Rocky Mount, VA and Kernersville, NC