DUALITIES, NATURAL AND OTHERWISE

Hilary Priestley

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GAIA 2013

With acknowledgements, in particular, to
Brian Davey and Leonardo Cabrer

From Brian Davey's La Trobe staff profile

Professor Davey's research interests centre on general (= universal) algebra and lattice theory. He is interested in the topological representation of algebras, with an emphasis on natural duality theory, and is particularly interested in applications of duality theory in general and Priestley duality in particular to algebras with an underlying distributive lattice structure.

But how did it all begin?

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25th March, 1971.

Mathematics Department
Monash University,
Clayton, Vic. 3168.

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Mathematical Institute,
OXFORD, ENGLAND.

Dear Dr. Priestly,

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I am greatly interested in this topic and if possible I would like to purchase a copy of your thesis, but if that is not possible copies of these papers would be appreciated.

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Outline

- Setting the scene: natural dualities for quasivarieties and varieties of $\mathfrak{D}$-based algebras
  ($\mathfrak{D}$ the variety of bounded distributive lattices)
  - An example: a hierarchy of prioritised default bilattices
- Natural dualities versus $\mathfrak{D}$-$\mathfrak{P}$-based dualities
- The piggyback strategy: more than just a method for finding economical dualities
  - Natural dualities and $\mathfrak{D}$-$\mathfrak{P}$-based dualities related
  - Examples
  - Coproducts in $\mathfrak{D}$-based varieties

Setting the scene

- Let $\mathfrak{D}$ denote the variety of bounded distributive lattices.
- Take a finitely generated quasivariety $\mathcal{A} = \mathbb{ISP}(\mathfrak{M})$, where $\mathfrak{M}$ is a finite set of finite $\mathfrak{D}$-based algebras. (Jónsson's Lemma ensures that every finitely generated variety of lattice-based algebras can be expressed this way.)
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Priestley duality supplies a dual equivalence between

$$\mathfrak{D} = \mathbb{ISP}(\mathbf{2}) \qquad \text{and} \qquad \mathcal{P} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}}) \quad (\equiv \text{Priestley spaces})$$

set up by hom-functors $\mathsf{H} = \mathfrak{D}(-, \mathbf{2})$ and $\mathsf{K} = \mathcal{P}(-, \underset{\sim}{\mathbf{2}})$. Here

$$\mathbf{2} = \langle \{0,1\}; \wedge, \vee, 0, 1 \rangle \quad \text{and} \quad \underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle.$$

Here, and henceforth, $\mathcal{T}$ denotes the discrete topology.

Objective

To find an **alter ego** $\underset{\sim}{\mathfrak{M}}$ for $\mathfrak{M}$ so there exists a dual equivalence between

$$\mathcal{A} = \mathbb{ISP}(\mathfrak{M}) \quad \text{and} \quad \mathfrak{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathfrak{M}})$$

set up by hom-functors $\mathsf{D} = \mathcal{A}(-, \mathfrak{M})$ and $\mathsf{E} = \mathfrak{X}(-, \underset{\sim}{\mathfrak{M}})$, with ED and DE given by natural evaluation maps.

Can we do this?: YES!

Case in which $\mathfrak{M}$ contains one algebra, $\mathbf{M}$, is probably familiar.

Quick recap: single-sorted case

Let $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is finite, $\mathfrak{D}$-based. Take $\underset{\sim}{\mathbf{M}} = \langle M; R, H, \mathfrak{T} \rangle$ where

$R = \mathbb{S}(\mathbf{M}^2)$ (all algebraic binary relations on M)

$H =$ some suitable set of (partial or total) algebraic operations, with a bound on the arities.

- The **algebraicity** requirements guarantee the hom- functors D and E are well defined, and are embeddings;
- $\mathbf{M}$ **lattice-based**, hence the specified R yields a duality: $\mathbf{A} \cong \mathrm{ED}(\mathbf{A})$ for all $\mathbf{A}$ [by the **NU Duality Theorem**].
- **Upgrading**: with a suitably chosen set H, the duality is **strong** and hence full: $\mathbf{X} \cong \mathrm{DE}(\mathbf{X})$ for all $\mathbf{X} \in \mathfrak{X}$.

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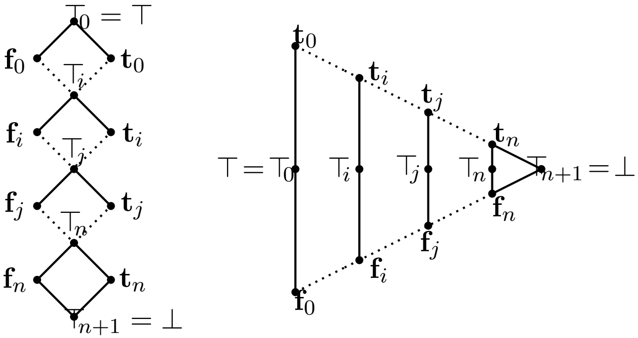
Multisorted case, $|\mathfrak{M}| > 1$ (Davey and Priestley (1984, at LTU)): All the above carries over: Relations and (partial and total) operations are algebraic, but now between members of $\mathfrak{M}$. The structure dual of an algebra $\mathbf{A}$ is based on the disjoint union of hom-sets $\mathcal{A}(\mathbf{A}, \mathbf{M})$, for $\mathbf{M} \in \mathfrak{M}$. Example to follow.

A hierarchy of prioritised default bilattices

(Cabrer, Craig & Priestley, 2013) We set up strong dualities for $\mathbb{ISP}(\mathbf{K}_n)$ and (multisorted) for $\mathbb{HSP}(\mathbf{K}_n)$, where

$$\mathbf{K}_n = (K_n; \wedge_k, \vee_k, \wedge_t, \vee_t, \neg, \top, \bot).$$

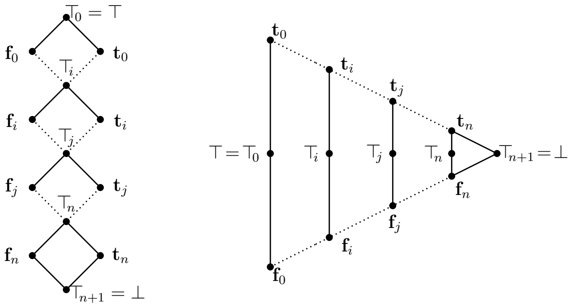


$\mathbf{K}_n$ in knowledge order (left) and truth order (right) with $0 < i < j < n$.

Interpretation for default logic

The elements of K_n represent levels of truth and falsity. Knowledge represented by the truth values at level $m+1$ is regarded as having lower priority priority than from those at level m. Also one thinks of $\mathbf{t}_{m+1}$ as being 'less true' than $\mathbf{t}_m$ and $\mathbf{f}_{m+1}$ as 'less false' than $\mathbf{f}_m$.

Base cases: $\mathbf{K}_0$ and $\mathbf{K}_1$ are Ginsberg's bilattices $\mathcal{FOUR}$ and $\mathcal{SEVEN}$, with $\neg$ added. In $\mathcal{SEVEN}$, $\mathbf{t}_1$ and $\mathbf{f}_1$ may be given the connotation of 'true by default' and 'false by default'.



Algebraic facts

- Every element of $\mathbf{K}_n$ is term-definable; $\mathbf{K}_n$ has no proper subalgebras.
- For $m \leqslant n$ there exists a surjective homomorphism $h_{n,m}\colon K_n \to K_m$.
- For $0 \leqslant m \leqslant n$, there exists $\S_{n,m} \in \mathbb{S}(\mathbf{K}_n^2)$ with elements

  $$\S_{n,m} = \Delta_n \cup \{\, (a,b) \mid a, b \leqslant_k \top_{m+1} \text{ or } a \leqslant_k b \leqslant_k \top_m \,\}.$$

  where $\Delta_n = \{\, (a,a) \mid a \in K_n \,\}$. We have $S_{n,j} \subsetneqq S_{n,i}$ for $0 \leqslant i < j \leqslant n$.
- The subalgebras $\S_{n,m}$ entail every subalgebra of $\mathbf{K}_n^2$ via converses and intersections.
- Each $\mathbf{K}_n$ is subdirectly irreducible.
- $\mathbb{ISP}(\mathbf{K}_n) = \mathbb{HSP}(\mathbf{K}_n)$ iff $n = 0$. For $n \geqslant 1$,

  $$\mathbb{HSP}(\mathbf{K}_n) = \mathbb{ISP}(\mathbf{K}_n, \ldots, \mathbf{K}_0).$$

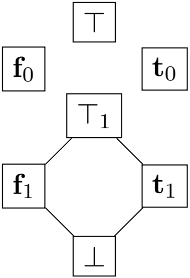
Algebraic binary relations

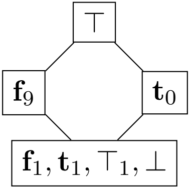
$n = 0$: $\mathbb{ISP}(\mathbf{K}_0)$ is the variety $\mathfrak{DB}$ of **distributive bilattices**. $S_{0,0}$ is the **knowledge order**, $\leqslant_k$.

De Morgan algebras, $\mathcal{M}$, and $\mathfrak{DB}$ compared. $\mathcal{M} = \mathbb{ISP}(\mathbf{M})$ where $\mathbf{M}$ has elements $\mathbf{t}.\mathbf{f}, \top, \bot$, with $\wedge_t, \vee_t, \neg$ as in $\mathbf{K}_0$. and constants $\mathbf{t}$ and $\mathbf{f}$. Here $]\mathbb{S}(\mathbf{M}^2)] = 55$ whereas $]\mathbb{S}(\mathbf{K}_0^2)] = 4$. In both cases, alter ego for an optimal duality includes $\leqslant_k$; for De Morgan we need also $\mathbf{M}$'s non-identity automorphism.

$n = 1$: Algebraic relations $S_{1,0}$ and $S_{1,1}$ on K_1, depicted as quasi-orders.

$S_{1,1}$





$S_{1,0}$

Duality theorems

Theorem

The structure $\underset{\sim}{\mathbf{K_n}} = \langle K_n; S_{n,n}, \ldots, S_{n,0}, \mathcal{T}\rangle$ *yields a strong, and optimal, duality on* $\mathbb{ISP}(\mathbf{K}_n)$.

For $n = 0$ *the dual category* $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{K}_0})$ *is* $\mathfrak{P}$.

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Theorem

Write $\mathbb{HSP}(\mathbf{K}_n)$ *as* $\mathbb{ISP}(\mathfrak{M})$, *where* $\mathfrak{M} = \{\mathbf{K}_0 \dots, \mathbf{K}_n\}$. *Then the alter ego*

$$\underset{\sim}{\mathfrak{M}} = \langle K_0 \dot\cup \dots \dot\cup K_n; \{S_{m,m}\}_{0 \leqslant m \leqslant n}, \{h_{i,i-1}\}_{1 \leqslant i \leqslant n}, \mathcal{T}\rangle,$$

yields a strong, and optimal, duality on $\mathbb{HSP}(\mathbf{K}_n)$.

The dual categories for $\mathbb{ISP}(\mathbf{K}_n)$ and $\mathbb{HSP}(\mathbf{K}_n)$ can be described for general n—comment on this later.

$\mathcal{D}$-based varieties: some old favourites

- De Morgan algebras;
- Kleene algebras;
- pseudocomplemented distributive lattices $\mathcal{B}_\omega$ and finitely generated subvarieties $\mathcal{B}_n$ ($\mathcal{B}_1 =$ Stone algebras);
- Heyting algebras, and Gödel algebras, $\mathcal{G}_n$—generated by an n-element Heyting chain;
- n-valued Łukasiewicz–Moisil algebras;

etc, etc.

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Assume $\mathcal{A}$ is a variety of $\mathcal{D}$-based algebras.

1. Take the class $\mathsf{U}(\mathcal{A})$ (the $\mathcal{D}$-reducts).
2. Seek to equip $\mathcal{Z} := \mathsf{HU}(\mathcal{A})$ with additional (relational or functional) structure so, for each $\mathbf{A} \in \mathcal{A}$, $\mathsf{KHU}(\mathbf{A})$ becomes an algebra in $\mathcal{A}$ isomorphic to $\mathbf{A}$;
3. Identify a suitable class of morphisms, making $\mathcal{Z}$ into a category.

If this gives a dual equivalence between $\mathcal{A}$ and $\mathcal{Z}$, we say we have a $\mathcal{D}$-$\mathcal{P}$-based duality. Literature is full of examples!

Bringing things together

We have:

1. Natural duality theory: for any finitely generated $\mathfrak{D}$-based variety we can call on the NU (Strong) Duality Theorem (single-sorted or multisorted).
2. $\mathfrak{D}$-$\mathfrak{P}$-based dualities
3. From, and for, logic: algebraic and relational (Kripke-style) semantics for non-classical propositional logics.

Both (1) and (2) provide valuable tools for studying algebraic properties of $\mathfrak{D}$-based varieties.

Normally, For a given variety, a discrete duality (as in (3)) differs from a $\mathfrak{D}$-$\mathfrak{P}$-based just through the absence or presence of topology. Canonical extensions provide a systematic approach to (3).

Rivals?

	Pro	Con
natural duality	a strong duality can always be found	duality may be complicated (may need entailment to simplify)
	$\mathbf{M}$ governs how the duality works	restriction to finitely generated classes (usually)
	good categorical properties, notably w.r.t. free algebras: $\mathrm{D}(F\mathcal{A})) = \underset{\sim}{\mathbf{M}}^S$	concrete representation is via functions, not sets, if $\lvert M\rvert > 2$
$\mathfrak{D}$-$\mathfrak{P}$-based duality	close relationship to Kripke-style semantics	
	concrete representation via sets	products seldom cartesian; free algebras hard to find

or collaborators?

We'd like to be able to toggle between a natural duality and a $\mathcal{D}$-$\mathcal{P}$-based one.

To get the best of both worlds,

LET'S HITCH A

PIGGYBACK RIDE!

The piggyback strategy: outline

For the case of $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is finite and $\mathcal{D}$-based.

Original objective

- To choose a subset of $\mathbb{S}(\mathbf{M}^2)$ yielding an **economical** duality, so bypassing entailment arguments.

Davey & Werner' (1982, at LTU) Simple piggybacking achieved this for **certain** quasivarieties: **De Morgan algebras**: 55 relations reduced to 2. But no good, e.g., for **Kleene algebras**.

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Further objectives: to use piggybacking

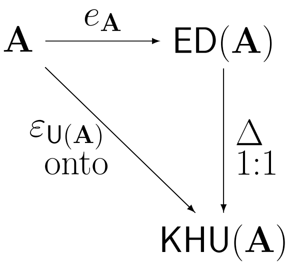
1. given $\mathbf{A} \in \mathcal{A}$, to relate $\mathsf{HU}(\mathbf{A})$ to $\mathsf{D}(\mathbf{A})$ and
2. given (1), to use it to equip $\mathsf{HU}(\mathbf{A})$ with additional structure so as to set up a $\mathcal{D}$-$\mathcal{P}$-based duality for $\mathcal{A}$.

[Concentrate here on (1), and on duality rather than strong duality.]

Davey & Werner's key idea

Find a 1:1 map Δ such that diagram commutes. Then $\varepsilon_{\mathsf{U}(\mathbf{A})}$ surjective $\implies$ $e_{\mathbf{A}}$ surjective.

Given a $\mathfrak{X}$-morphism $\phi\colon \mathsf{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$, we seek a $\mathcal{P}$-morphism $\Delta(\phi)\colon \mathsf{HU}(\mathbf{A}) \to \underset{\sim}{\mathbf{2}}$. IF $\phi \mapsto \Delta(\phi)$ is 1:1 then $e_{\mathbf{A}}$ is surjectve [elementary!].

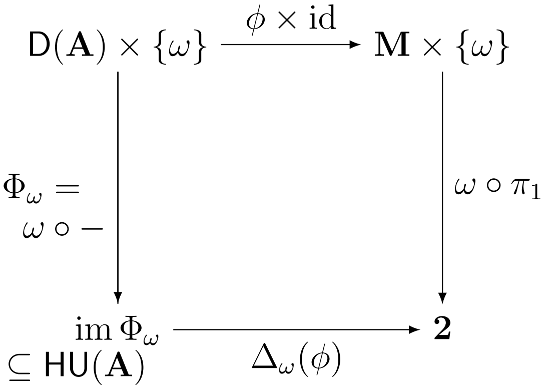


Construction of $\triangle$ for generalised piggybacking

Let $\Omega = \mathsf{HU}(\mathbf{M}) = \mathfrak{D}(\mathsf{U}(\mathbf{M}), \mathbf{2})$ and $\omega \in \Omega$.

Idea: hope that, we can choose $\underset{\sim}{\mathbf{M}}$ so that for each $\mathbf{A}$:

- we get a well-defined map Δ_ω for each $\omega \in \Omega$;
- the union, Δ of the maps Δ_ω is well defined, has domain $\mathsf{U}(\mathbf{A})$ and has the properties required.



Choosing R

For $\omega_i, \omega_2 \in \Omega$, consider the sublattice of $\mathsf{U}(\mathbf{M}^2)$:

$$(\omega_1, \omega_2)^{-1}(\leqslant) := \{\, (a,b) \in \mathbf{M}^2 \mid \omega_1(a) \leqslant \omega_2(b) \,\}.$$

Let R_{ω_1,ω_2} be the set (possibly empty) of **algebraic relations** maximal w.r.t. being contained in $(\omega_1, \omega_2)^{-1}(\leqslant)$. Let

$$R = \bigcup \{\, R_{\omega_1,\omega_2} \mid \omega_1, \omega_2 \in \Omega \,\}.$$

Theorem

Let $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, with $\mathbf{M}$ $\mathfrak{D}$-based. Then R yields a duality on $\mathcal{A}$.

Key ingredient in proof: the maps Φ_ω, for $\omega \in \Omega$, jointly map onto $\mathsf{HU}(\mathbf{A})$ for any $\mathbf{A} \in \mathcal{A}$. This relies on special properties of Priestley duality.

Fix $\mathbf{A} \in \mathcal{A}$, The dual space $\mathsf{D}(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{M})$ is viewed as a closed substructure of $\underset{\sim}{\mathbf{M}}^A$ and carries relations $r^{\mathsf{D}(\mathbf{A})}$, for $r \in R$, obtained by pointwise lifting.

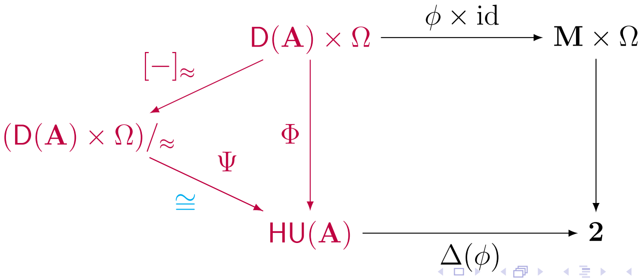
Relating $\mathsf{D}(\mathbf{A})$ and $\mathsf{HU}(\mathbf{A})$

Define $\preccurlyeq$ on $\mathsf{D}(\mathbf{A}) \times \Omega$ by

$$(x, \omega_1) \preccurlyeq (y, \omega_2) \iff (x, y) \in r^{\mathsf{D}(\mathbf{A})} \text{ for some } r \in R_{\omega_1, \omega_2}.$$

Theorem

Let $\approx$ be the equivalence relation $\preccurlyeq \cap \succcurlyeq$, Then $\approx$ is $\ker \Phi$, where $\Phi \colon (x, \omega) \mapsto \omega \circ x$. Moreover, the map $\Psi \colon (\mathsf{D}(\mathbf{A}) \times \Omega)/{\approx} \to \mathsf{HU}(\mathbf{A})$ given by $\Psi([(x, \omega)]) = \omega \circ x$ is well defined and a Priestley space isomorphism. *The map $\Psi \colon (\mathsf{D}(\mathbf{A}) \times \Omega)/{\approx} \to \mathsf{HU}(\mathbf{A})$ given by $\Psi([(x, \omega)]) = \omega \circ x$ is well defined and a Priestley space isomorphism.*

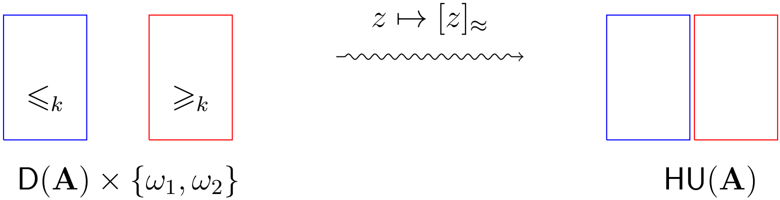


Example 1: prioritised default bilattices revisited

Product representation —and beyond

$\mathbb{HSP}(\mathbf{K}_0)$ is the much-studied variety $\underline{\mathbf{D}}$ of distributive bilattices. Such bilattices are **interlaced**: each pair of lattice operations is monotonic with to the other lattice order.

Bilattice theorists' favourite theorem: Every interlaced bilattice with negation decomposes as a product $\mathbf{L} \odot \mathbf{L}$, where $\mathbf{L}$ is a lattice, and the truth operations are those of $\mathbf{L} \times \mathbf{L}$, the knowledge operations are those of $\mathbf{L} \times \mathbf{L}^{\partial}$ and $\neg(a.b) = (b, a)$.
In the (bounded) distributive case, the product representation and our dualities are tied together.



Beyond K_0: duality to the rescue!

For $n > 0$, the bilattice $\mathbf{K}_n$ is NOT interlaced. Product representation FAILS and bilattice theory provides no structure theorem.
smallskip But in the natural duality for $\mathbb{HSP}(\mathbf{K}_n)$, we have we have an explicit description of the dual category $\mathfrak{X} = \mathbb{IS}_c\mathbb{P}^+(\mathfrak{M})$, where

$$\mathfrak{M} = \langle K_0 \,\dot\cup \ldots \dot\cup\, K_n; \{S_{m,m}\}_{0 \leqslant m \leqslant n}, \{h_{i,i-1}\}_{1 \leqslant i \leqslant n}, \mathcal{T}\rangle,$$

Loosely, the objects in $\mathfrak{X}$ are stratified: each of its $(n+1)$ layers is a Priestley space, and successive layers linked by the maps $\mathsf{HU}(h_{i,i-1})$.
This gives rise to a **generalisation of the product representation**, whereby each $\mathbf{A} \in \mathbb{HSP}(\mathbf{K}_n)$ can be realised as sitting inside a product of distributive bilattices $(\mathbf{L}_0 \odot \mathbf{L}_0) \times \cdots \times (\mathbf{L}_n \odot \mathbf{L}_n)$.

Example 2: Gödel algebras, $\mathcal{G}_n = \mathbb{ISP}(\mathsf{n})$

Just a glimpse, to illustrate a point.

The quasivariety generated by the n-element Heyting chain $\boldsymbol{n}$ Is endodualisable (Davey (1976)); optimised version by Davey & Talukder (2005). Historically, it has been a mantra: 'Use endomorphisms as far as possible; avoid partial operations except where unavoidable.'

BUT $\mathcal{G}_n$ also has a simple piggyback duality using graphs of partial endomorphisms (Davey & Werner).

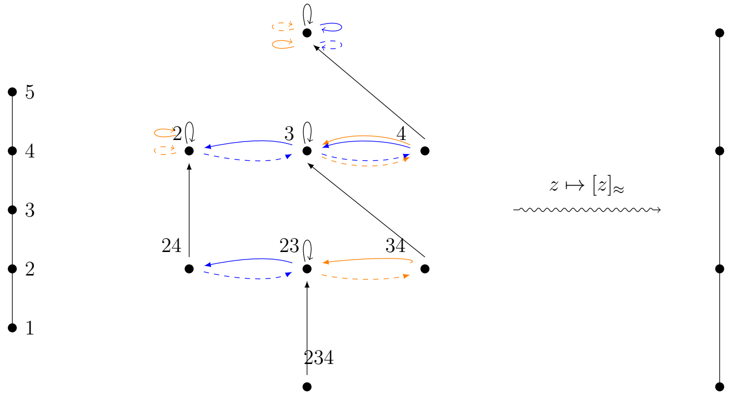
We shall deviate from our general strategy above: we dualise via simple piggybacking, with one endomorphism and two selected piggyback relations, to give a clear picture. This makes the translation from natural to $\mathfrak{D}$-$\mathcal{P}$-based duality transparent. For an algebra $\mathbf{A} \in \mathcal{G}_n$, we can visualise D$)\mathbf{A})$ as having at most $(n-1)$ layers: collapsing to HU$(\mathbf{A})$ occurs within layers, and a single endomorphism gives the ordering between layers.

Application (Cabrer & Priestley, work in progress) : description of coproducts of finite algebras in $\mathcal{G}_n$.

Illustration: from D$(\mathbf{5})$ to HU$(\mathbf{5})$

Elements of D$(\mathbf{5})$ are uniquely determined by, and so labelled by, their ranges.

These relations act on D$(\mathbf{5})$ as shown: endomorphism (black); piggyback relations (blue and orange, with converses shown dashed). The transitive closure of the union of these relations gives the quasi-order $\approx$.



$\mathbf{5})$ D$(\mathbf{5}) = \mathcal{G}_5(\mathbf{5}, \mathbf{5})$ HU$(\mathbf{5})$

Coproducts via duality

Cabrer and Priestley, 2012)

Assume $\mathcal{A} = \mathbb{ISP}(\mathfrak{M})$ is a finitely generated $\mathfrak{D}$-based quasivariety. Take $\Omega \subseteq \dot{\bigcup}_{\mathbf{M}\in\mathfrak{M}} \mathsf{HU}(\mathbf{M})$ to satisfy

$(\mathrm{Sep})_{\mathfrak{M},\Omega}$: for all $\mathbf{M} \in \mathfrak{M}$, given $a, b \in \mathbf{M}$ with $a \neq b$, there exists $\mathbf{M}_{a,a} \in \mathfrak{M}$, $u \in \mathcal{A}(\mathbf{M}, \mathbf{M}_{a,b})$ & $\omega \in \Omega \cap \mathsf{HU}(\mathbf{M}_{a,b})$ such that $\omega(u(a)) \neq \omega(u(b))$.

- $\Omega = \dot{\bigcup} \mathsf{HU}(\mathbf{M})$ always works —as we used earlier for piggybacking when $\mathfrak{M} = \{\mathbf{M}\}$.
- Simple piggybacking needs $\mathrm{Sep}_{\mathbf{M},\omega}$: 'one $\mathbf{M}$, one ω' case.

Given $\mathfrak{K} \subseteq \mathcal{A}$, there is a $\mathfrak{D}$-homomorphism

$$\chi_{\mathfrak{K}}\colon \coprod \mathsf{U}(\mathfrak{K}) \to \mathsf{U}(\coprod \mathfrak{K}),$$

We can work with the dual map $\mathsf{H}(\chi_{\mathfrak{K}})$ to get iff conditions

(E) for $\chi_{\mathfrak{K}}$ to be **injective** (for any $\mathfrak{K}$);

(S) $\chi_{\mathfrak{K}}$ to be **surjective** (for any $\mathfrak{K}$).

Coproduct embedding and surjectivity theorems

Theorem

The following are equivalent to (E)*:*

(1) *there exists* $\mathbf{M} \in \mathcal{A}_{\mathrm{fin}}$ *and* $\omega \in \mathsf{HU}(\mathbf{M})$ *such that* $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ *and* $(\mathrm{Sep})_{\mathbf{M},\omega}$ *holds;*

(2) *as* (1) *but with, additionally,* $\mathbf{M}$ *subdirectly irreducible.*

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Theorem

The following are equivalent to (S)*:*

(1) *for every* n*-ary* $\mathcal{A}$*-term* t *(*$n \geqslant 1$*)* $\exists$ *unary* $\mathcal{A}$*-terms* $t_1, \ldots, t_n$ *and an* n*-ary* $\mathfrak{D}$*-term* s *such that for every* $\mathbf{A} \in \mathcal{A}$ *and every* $a_1, \ldots, a_n \in \mathbf{A}$

$$t^{\mathbf{A}}(a_1, \ldots, a_n) = s^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1), \ldots, t_n^{\mathbf{A}}(a_n));$$

(2) $|R_{\omega_1,\omega_2}| \leqslant 1$ *for every* $\omega_1, \omega_2 \in \Omega$*;*

(3) *for* $\mathbf{A} \in \mathcal{A}$ *and every* $\mathfrak{D}$*-sublattice* $\mathbf{L}$ *of* $\mathsf{U}(\mathbf{A})$*, then either* L *contains no* $\mathcal{A}$*-subalgebra of* $\{\, \mathbf{B} \in \mathbb{S}(\mathbf{A}) \mid B \subseteq L \,\}$ *is empty has a top element.*

A medley of examples

$\mathrm{E}^{\checkmark}\mathrm{S}^{\checkmark}$	Boolean Algebras De Morgan algebras n-valued pre-Łukasiewicz–Moisil algebras Stone algebras
$\mathrm{E}^{\checkmark}\mathrm{S}^{\times}$	Gödel algebra varieties $\mathfrak{G}_n$ $(n \geqslant 3)$ Varieties $\mathfrak{B}_n$ $(2 \leqslant n < \omega)$ Q-lattice varieties $\mathfrak{D}_{p0}$ and $\mathfrak{D}_{q1}$ $(p \geqslant 1,\ q \geqslant 0)$
$\mathrm{E}^{\times}\mathrm{S}^{\checkmark}$	Kleene algebras MV-algebra varieties $\not\supseteq \mathbf{Ł}_{p \cdot q}$ (whenever $p \neq q$, primes) n-valued Łukasiewicz–Moisil algebras $(n \geqslant 2)$
$\mathrm{E}^{\times}\mathrm{S}^{\times}$	Non-singly generated varieties of Heyting algebras MV-algebra varieties $\supseteq \mathbf{Ł}_{pq}$ (for some $p \neq q$, primes) Q-lattice varieties $\mathfrak{D}_{pq}$ $(q \geqslant 2)$

Coproducts and piggybacking

Theorem

Let $\mathcal{A}$ be a finitely generated $\mathcal{D}$-based quasivariety.

(i) *$\mathcal{A}$ admits free products iff it admits a single-sorted duality.*

(ii) U *satisfies* (E) *iff $\mathcal{A}$ admits a simple piggyback duality.*

(iii) U *satisfies* (S) *iff $\mathcal{A}$ admits a piggyback duality (single-sorted or multisorted), such that $|R_{\omega_1,\omega_2}| \leqslant 1$ for all $\omega_1, \omega_2 \in \mathsf{HU}(\mathbf{M})$.*

(iv) U *preserves coproducts (hat is, (E) and (S) hold) iff $\mathcal{A}$ has a simple piggyback duality that is a $\mathcal{D}$-$\mathcal{P}$-based duality.*

Familiar classics: reconciliations

- (iv) holds for De Morgan algebras and for Stone algebras.
- Can handle coproducts of Kleene algebras via a reflector from De Morgan algebras. This strategy generalises.

How to sum up?

Brian Davey has gone a very long way in convincing algebraists that topology and algebra make good bed-fellows.

In this he has followed in the footsteps of Marshall Stone (1938):

A cardinal principle of modern mathematical research may be stated as a maxim: "One must always topologize."

Poster of Stone 10/24/2006 05:54 PM

Marshall Stone

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