Extending Functions to the Natural Extension

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Introduction. The concept of canonical extension has been defined successively for various classes of algebras. Some kind of general paradigm arise rather easily: the types of the considered algebras have a well definite reduct—the "dominant", or base operations–and the other operations are somewhat secondary and expected to have some compatibility relations with respect to the base operations (in case of lattice-based algebras, the extra operations were first considered to be operators, then isotone functions, and now just arbitrary maps). One can find in litterature mainly two ways to obtain the canonical exten- sion. It sometimes happens that the canonical extension comes all in one, by a process in which all operations (base and extra) receive a single treatment—think of profinite or natural completions. But in other cases, we need a two step process. First construct the canonical extension of the designated reduct, and then extend each extra operation by some density procedure. In this case it may happen that more than one choice is available.

The lattice-based case has been up to now considered at length. But recently Davey, Gouveia, Haviar and Priestley have considered canonical extensions (under the name of natural extensions) of algebras lying in an internally residually finite prevariety $\mathcal{A}$, that is, a class of the form $\mathbb{ISP}(\mathcal{M})$ for some set $\mathcal{M}$ of finite algebras. For facility, we suppose here $\mathcal{A} = \mathbb{ISP}(\underline{P})$ where $\underline{P}$ is a finite algebra. In this case, the canonical or natural extension is obtained (in a single step) as a double dual construction. If $A \in \mathcal{A}$, its "proto-dual" is $A^* = \mathcal{A}(A, \underline{P})$ and A naturally embeds into P^A by the evaluation map $e(a \mapsto e_a(\varphi) = \varphi(a)$ for $\varphi \in A^*)$. Then the natural extension A^η of A is the closure of $e(A)$ within P^{A^*} (with the product topology of P endowed with the discrete topology). This situation will be the designated reduct of the situation of our talk. In other words, *we consider a map $u \colon A \to B$ between algebras of $\mathcal{A}$ and see how to lift u to the level of the natural extensions of A and B.*

If we want to parallel the lattice-based case, we first have to refine the topology on P^{A^*}.

1 The δ-topology

Suppose P and X are topological members of a subcategory $\mathcal{X}$ of Set. We introduce the δ-topology on P^X as follows. By a *partial continuous morphism* $f \colon X \to P$ we mean a continuous morphism $f \colon \operatorname{dom} f \to P$ where $\operatorname{dom} f$ is a closed subobject of X. The δ-topology on $\mathcal{X}(X, P)$ has as open basis the sets of the form

$$O_f = \{\, x \in \mathcal{X}(X, P) \mid f \subseteq x \,\}.$$

For instance, let P be the 2-element discrete space. if X has the cofinite topology, then you get the product topology. If X is the dual space of a Boolean algebra B, then the δ-topology on 2^X coincides with the usual δ-topology (also called σ-topology by Gehrke and Jónsson) on the canonical extension 2^X of B since the latter has for basis the intervals $[F, O]$ where $F \subseteq O$,

F closed and O open in X: we have $[F, O] = O_f$ for the partial continuous f which sends x to 0 if $x \in F$ and to 1 if $x \notin O$.

Though obvious, the following observations are essential:

1) *if x is a continuous morphism $X \to P$, then x is isolated in $\mathcal{X}(X, P)$;*
2) *if P is injective in $\mathcal{X}$, then the continuous morphisms are dense in $\mathcal{X}(X, P)$.*

2 Extensions of maps into finite algebras

In view of the results of the previous section, to make things easy and to avoid technicalities, we shall have the following standing assumptions for the rest of the paper: We work in a pre-variety $\mathcal{A} = \mathbb{ISP}(\underline{P})$ generated by a finite algebra $\underline{P}$. The set P is endowed with a topological structure $\underset{\sim}{P}$ in which the topology is discrete and *which leads to a natural duality on* $\mathcal{A}$. On the dual class $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{P})$, we consider two categories. Firstly the dual category $\mathcal{X}$ of $\mathcal{A}$, whose morphisms respect $\underset{\sim}{P}$ and in particular are continuous. Secondly, if we denote by $\underset{\sim}{P}^b$ the non topological part of $\underset{\sim}{P}$, we also have the category $\mathcal{X}^b$, whose morphisms only respect $\underset{\sim}{P}^b$. In addition to $\underset{\sim}{P}$ leading a natural duality on A, we also assume that $\underset{\sim}{P}$ is injective in $\mathcal{X}$.

With these assumptions, we know that each A is isomorphic with its double dual $\mathcal{X}(A^*, \underset{\sim}{P})$ (with $A^* = \mathcal{A}(A, \underline{P})$), that the natural extension of A is $A^\eta = \mathcal{X}^b(A^*, \underset{\sim}{P}^b)$, and that the points of A are isolated and dense in A^η with respect to the product topology on A^η.

We now have a map $u\colon A \to B$ $(A, B \in \mathcal{A})$ where B is finite, and we want to extend it to A^η. Let $x \in A^\eta$. Then the family of all $u(V \cap A)$, where V runs through the δ-neighborhoods of x, is a lower directed family of non-empty finite sets: it has a least member that we shall denote by $\widetilde{u}(x)$.

Definition. The *natural* or canonical *extension* of $u\colon A \to B$ is the map $\widetilde{u}\colon A^\eta \to \mathcal{P}(B) : x \mapsto \widetilde{u}(x)$. Of course if $|\widetilde{u}(x)| = 1$ for all x, that is, if u is *smooth*, then $\widetilde{u}$ may be thought of as a map $A \to B$. But unfortunately, *if u is not smooth, it is not possible to choose continuously an element $u'(x)$ in each $\widetilde{u}(x)$.*

3 Extensions of maps

We now have a map $u\colon A \to B$ where A and B are arbitrary in $\mathcal{A}$. To reduce this situation to that of the previous section, we fix a finite subset F of the dual $B^* = \mathcal{A}(B, \underline{P})$ of B. Then $u_F\colon A \to B \to P^F$ defined as the composition with u of the projection pr_F along $F\colon B \cong \mathcal{X}(B^*, \underset{\sim}{P}) \subseteq P^{B^*} \to P^F$ is a map from A into the finite algebra P^F, and therefore has an extension $\widetilde{u_F}$. Let $u(x, F) = \{\, y \in P^{B^*} \mid pr_F(y) \in \widetilde{u_F}(x) \,\}$. Then the following can be shown.

Theorem. *For each $x \in A^\eta$,*

$$\widetilde{u}(x) := \bigcap \{\, u(x, F) \mid F \text{ finite} \subseteq B^* \,\}$$

is a non-empty closed subset of B^η.

This leads to the following:

Definition. The *natural extension* of $u\colon A \to B$ is the map $\widetilde{u}\colon A^\eta \to \Gamma(B^\eta)\colon x \mapsto \widetilde{u}(x)$, where $\Gamma(B^\eta)$ is the space of closed subsets of B^η.

Theorem. *If A^η is endowed with the δ-topology and $\Gamma(B^\eta)$ with the co-Scott topology, then $\widetilde{u}$ is a continuous extension of u.*