BEYOND BOOLEAN ALGEBRAS: Lattice-ordered Algebras and Ordered Structures

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[Slightly expanded version of slides for the web]

Boolean algebras

- have been important in many areas,
- have magical, deep theorems,

BUT, as algebras,

- they are very special.

SO extensions of theorems about BA's to related classes of algebras may lie hidden.

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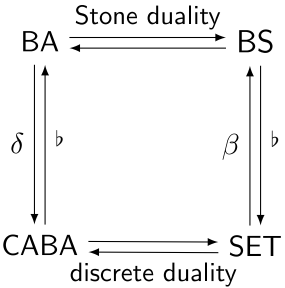
THIS TALK takes some well-known results linked to Stone duality, and seeks to set these in a wider context, using the Boolean algebra case as a launchpad.
AIM: to convey an overall impression of the big picture. [*To appreciate this in general terms, full understanding of the component pieces is not needed, or expected.*]

KEY THEMES

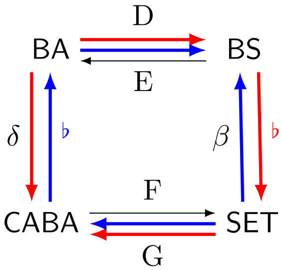
- the role of duality theory in a variety of guises;
- universal mapping properties—a unifying framework for assorted completions and compactifications, of algebras and of structures.

Familiar categories, familiar dualities



BS:	Boolean spaces (*alias* Stone spaces)
CABA:	complete atomic BA's
	(in ZFC) $\cong$ class of powerset algebras
$\flat$:	forgetful functor [*likewise subsequently*]
β:	Stone–Čech compactification functor
δ:	canonical extension functor

Basic Square for BA's, in more detail



The red & blue squares commute:

$$\delta = \mathrm{D} \circ {}^{\flat} \circ \mathrm{G},$$

$$\beta = \mathrm{G} \circ {}^{\flat} \circ \mathrm{D}.$$

D, E and G, F are dual equivalences.

D sends $\mathbf{A} \in$ BA to Ult$(\mathbf{A})$ (ultrafilters, topologised);
E sends $\mathbf{X} \in$ BS to Clop(X) (the clopen subsets of X);

F sends $\mathbf{B} \in$ CABA to its set of atoms;
G is the powerset functor.

$\mathbf{A}^{\delta}$ is the powerset of Ult$(\mathbf{A})$

and $\mathbf{A}$ embeds in $\mathbf{A}^{\delta}$, as the atoms.

Canonical extensions in outline

We give a very brief overview of

- construction,
- characterisation,
- role in logic,
- extending the ideas beyond Boolean algebras to more general ordered structures.

From BA's to Boolean algebras with operators, I

BAO's motivated the introduction of the canonical extension $\mathbf{A}^\delta$ of a Boolean algebra $\mathbf{A}$, as pioneered by Jónsson & Tarski (1951).

Stone duality tells us how to use topology to capture a copy of $\mathbf{A}$ within the powerset algebra $\mathbf{A}^\delta$. J&T took the topology out, by abstractly characterising how $\mathbf{A}$ sits in $\mathbf{A}^\delta$ in infinitary algebraic terms:

1. density property: $\bigvee\bigwedge-$ and $\bigwedge\bigvee-$density, meaning every element of $\mathbf{A}^\delta$ is a join of meets and a meet of joins of elements of (the embedded image of) $\mathbf{A}$ in $\mathbf{A}^\delta$;
2. compactness property. [*thinly concealed topological compactness condition*]

Properties 1. and 2. uniquely characterise $\mathbf{A}^\delta$ among lattice completions of $\mathbf{A}$.

From BA's to Boolean algebras with operators, II

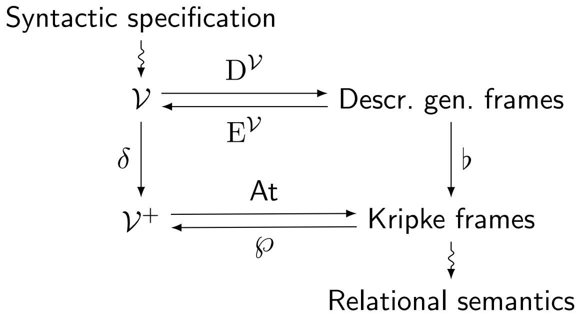
Adding extra operations to Boolean algebras and lifting them to the canonical extensions:

- Canonical extension embeds a BA into a CABA—stronger order-theoretic properties.
- Treat additional operations as an overlay: lift them to operations on the canonical extension with stronger properties.

  Example: an operator is an n-ary operation $f\colon \mathbf{A}^n \to \mathbf{A}$ preserves $\vee$. It lifts to $f^\sigma\colon (\mathbf{A}^\delta)^n \to \mathbf{A}^\delta$ which preserves $\bigvee$.
- (foreshadowing relational semantics for modal logic): we can take the 'trace' of the extended operations on a (Kripke-style) frame sitting inside the canonical extension.

Via canonical extensions to semantics for logics

Example Modal logics are modelled algebraically by BAO's and relationally by Kripke frames. Outline:



For a variety $\mathcal{V}$ of BAOs, operations lift from $\mathbf{A} \in \mathcal{V}$ to $\mathbf{A}^\delta \in \mathcal{V}^+$. Identities lift to $\mathcal{V}^+$ if negation not involved, but not necessarily if $\neg$ appears.

Putting topology back again

A cardinal principle of modern mathematical research may be stated as a maxim:

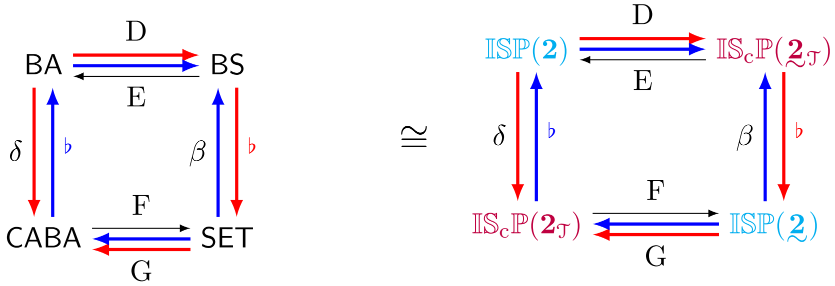
"One must always topologize."

Marshall Stone (1938)

Let $\mathbf{A} \in$ BA. Assume the Boolean Prime Ideal Theorem. This gives BA $= \mathbb{ISP}(\mathbf{2})$. Note that Stone duality is not involved in the construction of $\mathbf{A}^\delta$ from the ultrafilters of $\mathbf{A}$, *alias* the homomorphisms into $\mathbf{2}$. It is used to get the characteristic properties of the canonical extension $\mathbf{A}^\delta$ viewed as a completion of $\mathbf{A}$.

CABA's other persona: CABA is categorically-isomorphic to the category of Boolean-topological BA's, that is, $\mathsf{BA}_{\mathcal{T}} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\mathbf{2}_{\mathcal{T}})$, where $_{\mathcal{T}}$ indicates imposition of the discrete topology, $\mathcal{T}$, and $_c$ that we take closed subalgebras of powers.

Basic Square for BA's revisited



All four categories in right-hand square have generators with universe $\{0,1\}$. $\underset{\sim}{\mathbf{2}}$ denotes the set $\{0,1\}$.

Two of the categories have no topology.
Two are topological prevarieties.

Algebraic semantics for PropCalc—and beyond

[*In spirit, if not in precise detail*] George Boole introduced the idea of algebraic semantics for classical PropCalc, based on BA's.
But PropCalc is just for starters.
Why not extend Boole's fundamental idea of algebraic semantics:

1. add-ons
   - modalities just one possibility of many

and/or

2. relaxations
   - drop, or modify, classical (Boolean) negation;
   - drop, or relax conditions on, implication—eg consider IPC (Intuitionistic Propositional Calculus) or Łukasiewicz logic.

But often keep classical conjunction and disjunction.

The semantic models then need to be more general than BA's and based on eg distributive lattices, Heyting algebras, lattices, or even (for substructural logics) posets, and with or without additional operations.

But are DLs just BAs' 'weak sisters'?

'The theory of distributive lattices is richer than that of Boolean algebras; nevertheless it has had an abnormal development.
First, Stone's 1936 representation for distributive lattices closely imitated his representation for Boolean algebras and turned out to be too contrived. (I have yet to find a person who can state the entire theorem from memory.)
Second, a strange prejudice circulated among mathematicians that distributive lattices are just Boolean algebras' 'weak sisters.'

Gian-Carlo Rota (1973)

Aside: Marshall Stone's legacy: the down-side

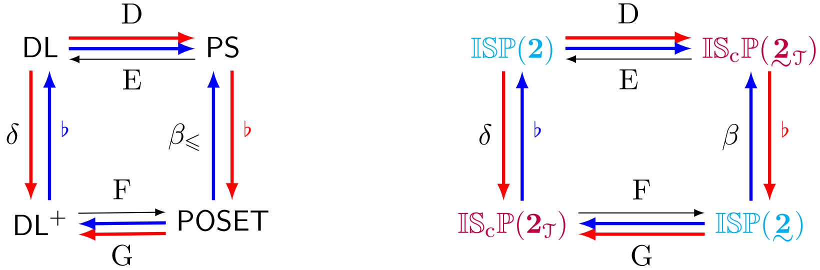
- Stone's purely topological dual structures concealed the importance of relational structures in duality theory.
- The crucially important notion of order was concealed.
- The T_0-space representation theorem for DLs via spectral spaces was way ahead of its time. Importance of T_0-spaces emerged in early development of domain theory in 1970s.

Moreover key connections were not seen:

- between Birkhoff's representation of finite DLs and Stone's T_0-space one,
- between the representation theory for lattice-ordered algebras and Kripke's relational semantics for associated logics.

The two dualities, Stone's and Priestley's, give isomorphic (not just equivalent) dual categories. Intended application should drive choice. Spectral spaces have enjoyed a renaissance with domain theorists' interest in stably compact spaces (which are to spectral spaces what compact-ordered spaces are to Priestley spaces).

Distributive lattices are not BA's weak sisters



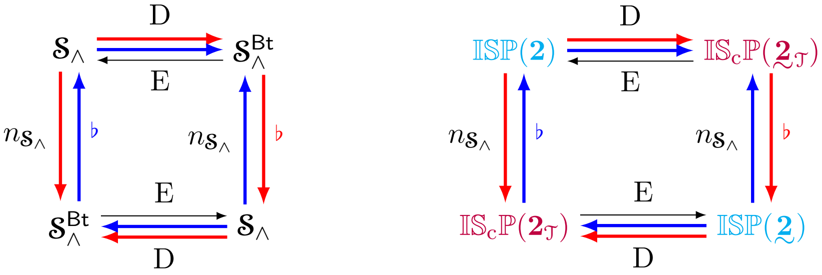
On RHS, generators are the 2-element objects in their categories.

PS:	Priestley spaces (compact, totally order-disconnected)
$\mathsf{DL}^{\mathsf{Bt}}$:	Boolean-topological DL's $= \mathbb{IS}_c\mathbb{P}(\mathbf{2}_{\mathcal{T}})$
	$\cong$ (in (ZFC)) to class DL^+ of all up-set lattices
POSET:	posets
$\beta_{\leqslant}$:	Nachbin order-compactification functor
δ:	canonical extension functor *[information shortly]*

D, E set up Priestley duality. F, G set up Banaschewski duality.

Just as nice as the Boolean algebras Basic Square!

Unital meet semilattices: another classic square



$\mathcal{S}_\wedge$:	meet semilattices with 1
$\mathbf{2} = \underset{\sim}{\mathbf{2}}$:	2-element object in $\mathcal{S}_\wedge$
$\mathcal{S}_\wedge^{\mathsf{Bt}}$:	Boolean-topological meet semilattices with 1
$n_{\mathcal{S}_\wedge}$:	$\mathrm{D} \circ {}^\flat \circ \mathrm{D}$

D and E give a dual equivalence: Hofmann–Mislove–Stralka duality (1976).

Note the symmetry: red and blue squares are the same.

Canonical extensions beyond BA's and BAO's

The breakthrough didn't come for more than forty years:–

Gehrke & Jónsson (1995): canonical extensions for DLO's. Canonical extension construction via Priestley duality for distributive lattices. Handling of operator-lifting relied on Scott topology.
By 2004: G&J had developed a general (topologically-driven) treatment of lifting of arbitrary operations.
Why stop there?

BOOLEAN ALGEBRAS	Jónsson & Tarski	1951
Bounded distributive lattices	Gehrke & Jónsson	1995
Bounded lattices	Gehrke & Harding	2001
Meet semilattices with 1	Gouveia & Priestley	2014
Posets	Gehrke *et al.*	2005$\to$

Red indicates a duality-based Basic Square is available, but for $\mathcal{S}_\wedge$ this does not give the canonical extension. [*Comment later.*]

Canonical extensions of unital semilattices: a fast-track order-theoretic approach (in ZF)

Let $\mathbf{S} \in \mathcal{S}_\wedge$—meet semilattices with 1. Embed $\mathbf{S}$ in $\widehat{\mathbf{S}} := \mathrm{Filt}(\mathrm{Filt}(\mathbf{S}))$ (the filter lattice of the filter lattice of $\mathbf{S}$) via $e\colon a \mapsto \Uparrow(\uparrow a)$. [$\uparrow$ and $\Uparrow$ denote the operators mapping an element to the principal up-set it generates, in $\mathrm{Filt}(\mathbf{S})$ and $\mathrm{Filt}(\mathrm{Filt}(\mathbf{S}))$.]

$\mathbf{S}^\delta := \{\, \mathcal{F} \in \widehat{\mathbf{S}} \mid \mathcal{F}$ is a meet of directed joins of elements from $e(S)\,\}$ is the canonical extension of $\mathbf{S}$.

Theorem

- *e is an $\mathcal{S}_\wedge$-embedding of $\mathbf{S}$ into $\mathbf{S}^\delta$;*
- *meets in $\widehat{\mathbf{S}}$ and in $\mathbf{S}^\delta$ are given by $\bigcap$ and directed joins by $\bigcup$;*
- *each $\mathcal{F} \in \widehat{\mathbf{S}}$ is a directed join of filtered meets from $e(S)$;*
- *$\mathbf{S}$ embeds in the complete lattice $\mathbf{S}^\delta$) via e.*

[In fact $\mathbf{S}^\delta$ coincides with the Galois-closed sets for the polarity $(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S}), R)$, where $F\ R\ J$ iff $F \cap J = \emptyset$.]

Summing up the canonical extensions philosophy

Nowadays (Gehrke in particular): Given a class of ordered algebras

- Canonical extension exists for underlying ordered structures.
- Work with the abstract characterisation, suppressing the mode of construction.
- Treat additional structure, and homomorphisms, as an overlay.
- Uniform approach, not case-by-case for each class.
- For many logics, get complete relational semantics by going via canonical extensions;
- Canonical extensions should come before, not from, topological dualities, when available.

BUT

- Topological-relational dualities obtained as a byproduct of canonical extensions are 'hand-me-downs': they often do not work smoothly as an algebraic tool.

[*Return to this at the end*].

Towards a general framework

What's the general pattern behind the various Basic Squares? In particular, is there some general process encompassing the various downward vertical arrows? We've encountered, at least in passing

- canonical extension, as an order-theoretic completion.
- with CABA identified with Boolean-topological BA's, the canonical extension can be seen as a compactification of algebras.
- Stone–Čech compactification and Nachbin order-compactification: compactifications of relational structures.

The best known of these is the Stone–Čech compactification. This is a special compactification, which can be characterised by a universal mapping property.

We seek a compactification functor on suitable classes of structures and which has a suitable universal mapping property.

Structures and topological structures

In the various squares so far, we met finitely generated prevarieties of both algebras (in fact varieties) and of purely relational structures, and associated finitely generated topological prevarieties. Why not both together?

(total) structure	$(A; G^A, R^A)$	$G^A \subseteq$ finitary operations $R^A \subseteq$ finitary relations
algebra		$R^A = \emptyset$
topological structure	$(A; G^A, R^A, \mathcal{T}^A)$	$\mathcal{T}$ a compact topology operations continuous, relations closed

For a structure $\mathbf{M}$ and associated topological structure $\mathbf{M}_{\mathcal{T}}$ let

$$\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M}), \qquad \boldsymbol{\mathcal{A}}_{\mathcal{T}} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\mathbf{M}_{\mathcal{T}}).$$

Fix a structure $\mathbf{M}$ and a compatible compact Hausdorff topology $\mathcal{T}$.

Natural extension functor on a class of structures

Let $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$ ($\mathbf{M}$ as above) and let $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. Then the evaluation map

$$e_{\mathbf{A}} \colon \mathbf{A} \to \mathbf{Z} := \mathbf{M}_{\mathcal{T}}^{S}, \quad S := \text{underlying set of } X_{\mathbf{A}} := \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}),$$
$$\text{where } e_{\mathbf{A}}(a)(x) := x(a) \quad (a \in A,\ x \in X_{\mathbf{A}}),$$

is an embedding of structures.

$$\boldsymbol{\mathcal{A}} \;\underset{\flat}{\overset{n_{\boldsymbol{\mathcal{A}}}}{\rightleftarrows}}\; \boldsymbol{\mathcal{A}}_{\mathcal{T}}$$

The natural extension, $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$, of $\mathbf{A}$, is $\overline{e_{\mathbf{A}}(\mathbf{A})}$, the closure of $e_{\mathbf{A}}(\mathbf{A})$ in $\mathbf{Z}$.

- $n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ (given structure coordinatewise, and induced product topology) belongs to $\boldsymbol{\mathcal{A}}_{\mathcal{T}}$;
- $n_{\boldsymbol{\mathcal{A}}}$ can be defined on morphisms to yield a functor.

$\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{A}}_{\mathcal{T}}$ and $n_{\boldsymbol{\mathcal{A}}}$ are built from $\mathbf{M}$ and $\mathbf{M}_{\mathcal{T}}$ alone.

No duality in sight!

What IS this natural extension gadget?

Theorem (Brute force description, finite case)

Let $\mathbf{M} = (M; G, R)$ *be a finite structure, let* $\mathbf{A} \in \mathcal{A} := \mathbb{ISP}(\mathbf{M})$*, and let* $z \colon \mathcal{A}(\mathbf{A}, \mathbf{M}) \to M$ *be a map. Then TFAE are equivalent:*

(i) $z \in n_{\mathcal{A}}(\mathbf{A})$.

(ii) z *is locally an evaluation, that is, for* $Y \subseteq \mathcal{A}(\mathbf{A}, \mathbf{M})$*,* Y *finite,* $\exists a \in A$ *such that* $\forall y \in Y \, (z(y) = y(a))$*;*

(iii) z *preserves every finitary relation on* M *that forms a substructure of the appropriate power of* $\mathbf{M}$*.*

When $G = R = \emptyset$, as happens when $\mathbf{M} = \mathbf{2} \in \mathsf{BA}$, we get

$$n_{\mathcal{A}}(\mathbf{A}) = \text{ all maps from } \mathcal{A}(\mathbf{A}, \mathbf{M}) \text{ into } M.$$

SO $n_{\mathsf{BA}}(\mathbf{A})$ and $\mathbf{A}^{\delta}$ are the same, as topological BA's.

The natural extension as a reflection

[As before] Let $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is a structure and $\mathbf{M}_{\mathcal{T}}$ a compact Hausdorff topological structure.

Theorem

The natural extension functor $n_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}_{\mathcal{T}}$ is a reflection of $\mathcal{A}$ into the (non-full) subcategory $\mathcal{A}_{\mathcal{T}}$. Specifically [universal mapping property]*:*

> *for each $\mathbf{A} \in \mathcal{A}$, each $\mathbf{B} \in \mathcal{A}_{\mathcal{T}}$ and every $g \in \mathcal{A}(\mathbf{A}, \mathbf{B}^{\flat})$,*
> *$\exists\ !\ h \colon \mathcal{A}_{\mathcal{T}}(n_{\mathcal{A}}(\mathbf{A}), \mathbf{B})$ with $h \circ e_{\mathbf{A}} = g$.*

Consequently, $n_{\mathcal{A}}$ is left adjoint to the forgetful functor $^{\flat} \colon \mathcal{A}_{\mathcal{T}} \to \mathcal{A}$.

Extension: can replace the single structure $\mathbf{M}$ by a set $\mathcal{M}$ of structures of common type. Call $\mathbb{ISP}(\mathcal{M})$ 'residually finite' if elements in $\mathcal{M}$ are all finite.

Profinite completions (only available for algebras)

Another compactification construction that might fit into our scheme?

[Thinking more algebraically than categorically]

Assume that $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ where $\mathcal{M}$ is a set of finite algebras. [If $\mathcal{A}$ is a variety, this says $\mathcal{A}$ is residually finite.] Given $\mathbf{A} \in \mathcal{A}$ there exists a Boolean-topological algebra $\mathrm{Pro}_{\mathcal{A}}(\mathbf{A})$, its profinite completion. It can be built concretely as the limit in $\mathcal{A}_{\mathcal{T}}$ of an inverse system whose members have $\mathcal{A}$-reducts which are finite quotients of $\mathbf{A}$; moreover $\mathbf{A}$ embeds in $\mathrm{Pro}_{\mathcal{A}}(\mathbf{A})^{\flat}$.

The profinite completion has, and is characterised by, a universal mapping property, different in form from that for the natural extension. But we have

Theorem

For $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ *as above,* $\mathrm{Pro}_{\mathcal{A}}(\mathbf{A}) = n_{\mathcal{A}}(\mathbf{A})$ *for all* $\mathbf{A} \in \mathcal{A}$*.*

Profinite completions and canonical extensions

Theorem

If $\boldsymbol{\mathcal{A}}$ *is* BA, DL, or any finitely generated variety of lattice-based algebras of finite type,

$$\mathbf{A}^{\delta} = \mathrm{Pro}_{\boldsymbol{\mathcal{A}}}(\mathbf{A})^{\flat} \quad \text{for all } \mathbf{A} \in \boldsymbol{\mathcal{A}}.$$

A parting of the ways:
Not true in general for $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{S}}_{\wedge}$ that canonical extension coincides with profinite completion. No big surprise.

Canonical extension depends only on underlying order. Profinite completion $\mathrm{Pro}_{\boldsymbol{\mathcal{A}}}$ can vary with $\boldsymbol{\mathcal{A}}$. N&S conditions available for $\mathrm{Pro}_{\mathsf{DL}}(\mathbf{A})$ to coincide with profinite completion of either or both of $\mathbf{A}$'s semilattice reducts. Uses sophisticated 1980's results from domain theory due to Mislove, Lawson involving locally finite meet breadth.

Bohr compactification of (discretely topologised) algebras

- A famous construction in context of abelian groups (almost periodic functions, ...).
- Extended to certain other classes, eg semilattices (Holm (1964), Hart & Kunen (1999), with fragmentary treatment of many examples).
- Two flavours:
  b (compact Hausdorff setting);
  b_0 (Boolean-topological setting, when applicable).
- Characterised by a universal mapping property.
- Existence: as a 'maximal compactification'—not concrete.
- (Sometimes) concrete description via a duality.

Why restrict to algebras?

Zero-dimensional version of Bohr compactification

Again let $\mathcal{A}$ be a class of structures, treated as discretely topologised when context demands.

Notation: $\mathcal{A}^{\mathrm{Bt}}$ is the class of Boolean-topological structures which have $\mathcal{A}$-reducts.

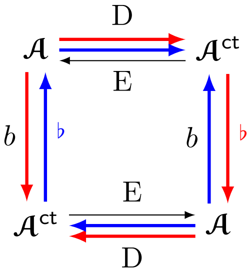
Let $\mathbf{A} \in \mathcal{A}$. Assume $b_0(\mathbf{A}) \in \mathcal{A}^{\mathrm{Bt}}$ is such that $\exists\ \iota_{\mathbf{A}} \colon \mathbf{A} \hookrightarrow b_0(\mathbf{A})$, embedding $\mathbf{A}$ as a dense substructure.

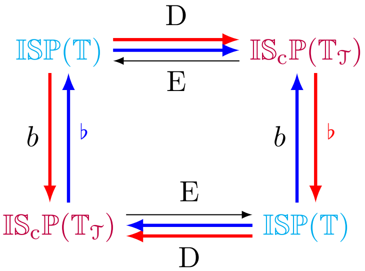
Then $b_0(\mathbf{A})$ is the zero-dimensional Bohr compactification of $\mathbf{A}$ if the following universal mapping property holds:

> *Given* $\mathbf{B} \in \mathcal{A}^{\mathrm{Bt}}$ *&* $g \in \mathcal{A}(\mathbf{A}, \mathbf{B}^\flat)$, $\exists !\ h \in \mathcal{A}^{\mathrm{Bt}}(b_0(\mathbf{A}), \mathbf{B})$ *with* $h \circ \iota_{\mathbf{A}} = g$.

Bohr compactification, b (the compact Hausdorff version), for discrete structures. Allow the topology to be compact Hausdorff rather than Boolean. Replace $\mathcal{A}^{\mathrm{Bt}}$ by $\mathcal{A}^{\mathrm{ct}}$ (compact Hausdorff topological structures with $\mathcal{A}$-reducts).

A classic: abelian groups





$\mathcal{A}$:	abelian groups
$\mathcal{A}^{ct}$:	compact Hausdorff abelian groups
$\mathbb{T}$:	circle group
$\mathbb{T}_{\mathcal{T}}$:	circle group with usual topology
b:	$\mathrm{D} \circ {}^{\flat} \circ \mathrm{D}$

D and E give a dual equivalence: Pontryagin duality.

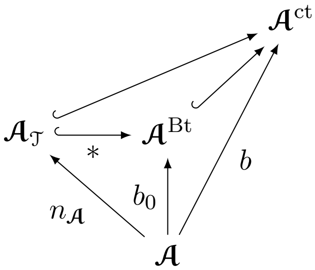
Red and blue squares are the same. This is just like the $\mathcal{S}_{\wedge}$ Basic Square.

Towards reconciliations?

Stocktake: Again consider a structure $\mathbf{M}$ and associated topological structure $\mathbf{M}_{\mathcal{T}}$ and

$$\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M}), \qquad \boldsymbol{\mathcal{A}}_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}(\mathbf{M}_{\mathcal{T}}).$$

We have now seen three constructions, $n_{\boldsymbol{\mathcal{A}}}$, b_0 and b having universal mapping properties which look much the same except that their codomain categories are different.



$*$ —only when $(M;\mathcal{T}) \in$ BS.

Trivially,

- $b(\mathbf{A}) = b_0(\mathbf{A})$ if $b(\mathbf{A}) \in \boldsymbol{\mathcal{A}}^{\mathrm{Bt}}$;
- $b_0(\mathbf{A}) = n_{\boldsymbol{\mathcal{A}}}(\mathbf{A})$ if $h_0 \in \boldsymbol{\mathcal{A}}_{\mathcal{T}}$.

Converse may not be true?

Why seek reconciliations? Concrete descriptions of Bohr compactifications rarely available. Natural extension less elusive.

Coincidence of b_0 and $n_{\mathcal{A}}$

Two ways we can get coincidence:

strongly $\mathbb{IS}_c\mathbb{P}(\mathbf{M}_{\mathcal{T}}) =: \mathbf{A}_{\mathcal{T}} = \mathcal{A}^{\mathrm{Bt}}$ (same universal property for both);

weakly For $\mathbf{A} \in \mathcal{A}$ we have $n_{\mathcal{A}}(\mathbf{A}) = b_0(\mathbf{A})$ iff $b_0(\mathbf{A}) \in \mathcal{A}_{\mathcal{T}}$.

A clutch of off-the-peg theorems

Let $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ with M finite. Strong coincidence $\equiv$ the condition that $\mathcal{A}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\mathbf{M}_{\mathcal{T}})$ is standard, *viz.* $\mathcal{A}_{\mathcal{T}}$ consists of the Boolean-topological models of the quasi-equations defining $\mathcal{A}$. When $\mathbf{M}$ is a finite algebra, standardness has been extensively studied, principally when $\mathcal{A}$ is a variety. Standardness occurs/fails for quite deep reasons. Main general theorems combine universal-algebraic features of $\mathcal{A}$ with syntactic properties.

Some examples in which $\mathcal{A}_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}(\mathbf{M}_{\mathcal{T}})$ is standard:

- $\mathbf{M}$ a BA or DL;
- [when $\mathbb{ISP}(\mathbf{M}) = \mathbb{HSP}(\mathbf{M})$] $\mathbf{M}$ a group, semigroup, ring, lattice or unary algebra.

New roles for two classic 'bad' examples from domain theory

1. Posets: $\exists$ a Boolean space with a closed order relation which is not a Priestley space (Stralka (1978))—ie $\exists P \in \mathfrak{P}$ (POSET) such that $n_{\mathfrak{P}}(P) \neq b_0(P)$.

   The DL Basic Square asserted correctly that $n_{\mathfrak{P}}$ agrees with the Nachbin order compactification functor $\beta_{\leqslant}$. Fact: $\beta_{\leqslant}(P) \in$ PS for any poset P so we get $b = b_0$.

2. Unital meet semilattices: $\exists$ $\mathbf{A} \in \mathfrak{S}_\wedge$ such $b_0(\mathbf{A}) \neq b(\mathbf{A})$. (Derivation from Fundamental Theorem on Compact 0-dimensional Semilattices (which characterises $\mathfrak{S}_\wedge^{\mathsf{Bt}}$) and a famous example due to Lawson (1983).)

Stocktaking: what have we revealed so far?

For suitable classes $\mathcal{A}$ of structures (not just algebras):

- A new kid on the block, $n_{\mathcal{A}}$, among compactifications of structures.
  Coincides with profinite completion when this is available.
- In our various Basic Square examples, the left-hand and right-hand down arrows are instances of the same (natural extension) construction. We don't have one construction for algebras and a different one for relational structures.
- Natural extension (*alias canonical extension*) for BA's and Stone–Čech compactification occur in partnership, through Stone duality.

- Wider framework for Bohr-type compactifications.
- New insight into b_0 when $\mathcal{A}_{\mathcal{T}}$ is a standard topological prevariety.
- Reconciliations and examples of non-coincidences.

Unfinished business

- A more refined concrete description of $n_{\mathcal{A}}(\mathbf{A})$ would be nice.
- New Basic Square examples?
- Generalisations and further reconciliation questions.

Duality back on stage, briefly

[As before] Let $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is a structure and $\underset{\sim}{\mathbf{M}}_{\mathcal{T}} = (M; G', R', \mathcal{T})$ an alter ego: a compact Hausdorff topological structure which is compatible with $\mathbf{M}$ in the sense that operation in G' is a $\mathcal{A}$-homomorphism and each relation in R' is a substructure of appropriate power of $\mathbf{M}$.

Examples: See the DL Basic Square. Note the topology-swapping.

- $M = \{0, 1\}$, $\mathbf{M} = \mathbf{2} = (M; \vee, \wedge, 0, 1)$ in DL; $\underset{\sim}{\mathbf{M}}_{\mathcal{T}} = (M; \leqslant, \mathcal{T})$ in PS (for Priestley duality).
- $M = \{0, 1\}$, $\mathbf{M} = (M; \leqslant)$ in POSET, $\underset{\sim}{\mathbf{M}}_{\mathcal{T}} = (M; \vee, \wedge, 0, 1, \mathcal{T})$ (for Banaschewski duality).

Let $\mathfrak{X}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{M}}_{\mathcal{T}})$. Then $\exists$ hom-functors $\mathrm{D} = \mathcal{A}(-, \mathbf{M})$ and $\mathrm{E} = \mathfrak{X}_{\mathcal{T}}(-, \underset{\sim}{\mathbf{M}}_{\mathcal{T}})$ giving a dual adjunction between $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$ with each evaluation $e_{\mathbf{A}}$ an embedding. If $\mathbf{A} \cong \mathrm{ED}(\mathbf{A}) = e_{\mathbf{A}}(\mathbf{A})$ for each $\mathbf{A} \in \mathcal{A}$, then (definition) we have a (natural) duality. D and E not necessarily set up a dual equivalence.

Dualisability

Not all algebras or structures $\mathbf{M}$ have an alter ego $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$ which yields a duality.

Examples of finite algebras $\mathbf{M}$ for which a dualising alter ego $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$ exists/does not exist:

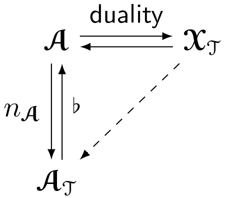
- ✓ all lattice-based algebras.
- ✓/× groups: classification known (but hard).
- ✓/× semilattices & semigroups (incomplete classification).
- ✓/× commutative rings (N&S condition for dualisability known).

For $\mathbf{M}$ which is a structure (which is not an algebra) and/or M is infinite: dualisable examples are scarcer and theory is less well developed.

Natural extension description refined

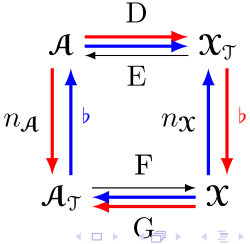
Theorem ($n_{\mathcal{A}}$ element-wise)

Let $\mathbf{M} = (M; G, R)$ *be a finite structure,* $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ *and* $\underset{\sim}{\mathbf{M}}_{\mathcal{T}} = (M; G', R', \mathcal{T})$ *a dualising alter ego. Then* $n_{\mathcal{A}}(\mathbf{A})$ *consists of the maps from* $\mathcal{A}(\mathbf{A}, \mathbf{M})^\flat$ *to* $\underset{\sim}{\mathbf{M}}$ *which preserve* G' *and* R'*.*



Theorem (the full works, by topology-swapping)

Let $(M; G, R)$ *and* $(M; G', R')$ *be finite structures such that each, with topology, acts as an alter ego for the prevariety generated by the other and one pair of hom-functors is a dual equivalence. Then the other is too and the red & blue squares commute.*



Rival dualities: very, very brief comments

Assume $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is a finite algebra with a distributive lattice reduct. Then one may seek for $\mathcal{A}$

- [operations and topology handled together] a natural duality based on an alter ego $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$. A dualising alter ego does always exist, but may be complicated.

- [Via Priestley duality: topology first, then additional operations] a duality based on Priestley spaces, obtained by applying Priestley duality to the DL-reducts and then capturing additional structure relationally.

- [Via canonical extensions: additional operations first, then topology] In many cases, can find a duality based on topological frames, that is, by topologising the canonical extensions of the algebras, equipped with lifted operations.

Rival dualities for DL-based algebras: reconciliation?

- Sufficient conditions now known under which there is a tight connection between the dual space $\mathrm{D}(\mathbf{A})$ of an algebra $\mathbf{A}$ under a natural duality and the Priestley dual space of $\mathbf{A}$'s DL-reduct. A best-of-both-worlds scenario.
- [Loosely] Translation from natural dual space to PS-based dual space may be seen as a process whereby multiple copies of $\mathrm{D}(\mathbf{A})$ are stuck together in a structured way.

- [Via canonical extensions] Get a discrete duality (Kripke-style) first, then introduce topological frames. Usually, but not always, the Kripke semantics and the topological components separate, with (first-order) conditions being as in the finite case and the (second-order) topological conditions being just the necessary ones. This happens, eg, with the duality between Heyting algebras and Esakia spaces.

Summing up, I

The natural extension functor $n_{\mathcal{A}}$ provides a framework which links constructions important in logic (canonical extensions), in algebra and topology (Bohr compactifications), in algebra and category theory (profinite completions), and in duality theory.

- Natural extension functor $n_{\mathcal{A}}$ exists for any class $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ where $\mathcal{M}$ a set of (1) finite structures or (2) infinite structures which support a compatible compact Hausdorff topology).
- $n_{\mathcal{A}}$ is a reflection; left adjoint to $^\flat \colon \mathcal{A}_{\mathcal{T}} \to \mathcal{A}$.
- In case (1):
  - Each $n_{\mathcal{A}}(\mathbf{A})$ can be concretely described; refined description available when $\mathcal{A}$ has a natural duality.
  - For algebras, $n_{\mathcal{A}}$ agrees with profinite completion.
- Case (2) covers in particular $\mathcal{A} =$ abelian groups (via Pontryagin duality).
- New examples embraced, in case (1) and in case (2).

Summing up, II

Notion of Bohr compactification (both b and b_0) extends from algebras to structures, so a range of different universal constructions are united.

- $n_{\mathbf{A}}$ may, but need not, equal b_0 or b, locally or globally.

  Such coincidences, when they occur, throw light on (inherently abstract) Bohr compactifications.
- Theory of standard topological prevarieties elucidates when coincidences do occur and provides many examples.

Canonical extension is inherently an order-theoretic completion. Big symmetric difference between scope of this and the above constructions; it coincides with these only in VERY special cases.

References and acknowledgements

Talk draws on the work of very many people, far too numerous for their important contributions to be acknowledged individually.

For further details, in particular on the newer developments, see the list of recent papers at

https://people.maths.ox.ac.uk/hap/

and the references therein.

In conclusion

Well done

GEORGE BOOLE

and

THANK YOU!