Fibred Universal Algebra: Many-Valued Duality Models of Predicate Logics

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Universal Algebra provides foundations of algebraic *propositional* logic. We propose Fibred Universal Algebra or Indexed Universal Algebra as foundations of algebraic *predicate* logic, including both first-order logic and higher-order logic. Duality Theory then provides various models of predicate logic. Among them, Ω-valued models based on dualising objects Ω are striking and useful, relating to many-valued models of set theory (if one category involved is the category of sets); e.g., Boolean-valued, Heyting-valued, and quantum-valued models. Although the framework we develop is of quite universal character, we shall especially focus on substructural logics.

A fibred algebra is defined as a functorial bundle of algebras as follows. Let $\mathcal{V}$ be a variety of ordered algebras in the sense of Universal Algebra; $\mathcal{V}$ may also be taken to be a quasi-variety as in the theory of natural dualities. Let $\mathbf{V}$ denote the category of algebras in $\mathcal{V}$ and their homomorphisms. Then, a fibred algebra (or fibred $\mathcal{V}$-algebra to be precise) is defined as a variety-valued presheaf $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{V}$ for a category $\mathbf{C}$, which may be required to satisfy some conditions like being a CCC (or monoidal closed if $\mathbf{C}$ has a monoidal structure) or being a concrete category in Adámek-Herrlich-Strecker's sense. The former is the case when higher-order logic is concerned. The latter is the case in categorical duality theories as developed by Johnstone and Porst-Tholen. $P(C)$ is called a fibre of P. Such fibred algebras are main objects of study in Fibred Universal Algebra.

Building upon Lawvere's idea of quantifiers as adjoints, we require a fibred algebra $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{V}$ to have adjoints of $P(\pi)$ for all projections π in $\mathbf{C}$ (with so-called Beck-Chevalley conditions); more conditions may be required to represent other logical structures (e.g., equality and comprehension). Such special types of functors provide sound and complete semantics for substructural predicate logics in the sense of axiomatic extensions of Full Lambek Calculus; in these cases, $\mathcal{V}$ is taken to be varieties of residuated algebras. Thus, $\mathcal{V}$ can be the variety of Heyting algebras, and the corresponding soudness and completeness result in this case is well known in categorical logic, and can be extended to substructural predicate logics in general as shown in this work. In such a way, Fibred Universal Algebra provides foundations of algebraic predicate logic.

Duality models are those functors $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{V}$ with the logical conditions that give dual adjunctions between $\mathbf{C}$ and $\mathbf{V}$; e.g., the dual adjunction between spaces and frames provides a model of (topological) geometric logic; dual adjunctions between convex structures and continuous lattices, which we show are essentially the Hofmann-Mislove-Stralka duality, provide models of convex geometric logic; dual adjunctions between sets and Heyting algebras, induced by frames Ω as dualising objects, give rise to sheaf models of intuitionistic logic. The last dual adjunctions can be extended to substructural logics in general, providing models of substructural predicate logics; actually, substructural set theories. We may thus consider that duality for propositional logic is a Lawverian model of predicate logic, and even set theory.