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on

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Abstracts

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Foreword

This booklet contains the abstracts of the talks for the second part of the two-part workshop on Duality Theory in Algebra, Logic and Computer Science to be held in Oxford in the summer of 2012. The first part of the workshop, in June, was very successful and the August workshop promises to be equally stimulating. It has a greater focus on logic and algebra, with less emphasis on topology than the June workshop.

We are grateful to all our participants for the interest they have shown in the workshops and especially to our speakers for their contributions.

Thanks are also due to the Oxford Mathematical Institute for hosting the event. We acknowledge with gratitude the financial support which has made it possible for the workshops to take place: funding from EPSRC, through a Platform Grant awarded to the Mathematical Institute, and the British Logic Colloquium.

Leonardo M. Cabrer
M. Andrew Moshier
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Organisers

Oxford, 14 August 2012

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Applications of Duality on Unification Type Classification

Leonardo Manuel Cabrer
University of Oxford
Joint works with S. Bova and V. Marra

In this work we will show that Priestley dualities for Bounded Distributive Lattices [5] and subvarieties of Pseudocomplemented Lattices [6], natural dualities for Kleene [2] and De Morgan algebras [1], and the duality between finitely presented MV-algebras and rational polyhedra [4], are useful tools to determine the unification type of these classes of algebras. Moreover, using those dualities we will present procedures to determine the type of the unification problems of Bounded Distributive Lattices and some subvarieties of Pseudocomplemented Lattices, Kleene algebras and De Morgan algebras and some classes of MV-algebras.

Given an equational theory E over an algebraic language $\mathcal{L}$, an E-*unification problem* is a finite set of pair of $\mathcal{L}$-terms $U = \{(t_1, s_1), \ldots, (t_n, s_n)\}$. An E-*solution* (E-*unifier*) for U is a substitution σ, defined on the variables of the terms of U, such that $\sigma(t_i)$ is equivalent to $\sigma(s_i)$ modulo E, for each $i \in \{1, \ldots, n\}$.

If σ is an E-unifier for U, we can obtain a family of solutions from σ as follows: let γ be a substitution defined in all the variables of the terms $\{\sigma(t_i), \sigma(s_i) \mid i \in \{1, \ldots, n\}\}$, then clearly $\gamma \circ \sigma$ is also an E-unifier for U. In this case we say that σ is *more general* than $\gamma \circ \sigma$, in symbols $\gamma \circ \sigma \preceq \sigma$. The relation $\preceq$ determines a preorder on the set of E-unifiers of U (denoted by $\mathfrak{U}_{\mathsf{E}}(U)$). This preorder allows us to classify the unification problems depending on its properties.

A μ-*set* for $\mathfrak{U}_{\mathsf{E}}(U)$ is a subset $M \subseteq \mathfrak{U}_{\mathsf{E}}(U)$ such that for all $\sigma_1, \sigma_2 \in M$ if $\sigma_1 \preceq \sigma_2$ then $\sigma_1 = \sigma_2$, and for every $\gamma \in \mathfrak{U}_{\mathsf{E}}(U)$ there exists $\sigma \in M$ such that $\gamma \preceq \sigma$. We say that U has E-type ($type_{\mathsf{E}}(U)$):

- 0: if $\mathfrak{U}_{\mathsf{E}}(U)$ has no μ-sets;
- ∞: if $\mathfrak{U}_{\mathsf{E}}(U)$ has a μ-set of infinite cardinality;
- n: if $\mathfrak{U}_{\mathsf{E}}(U)$ has a finite μ-set of cardinality n.

We say that the equational theory E has type:

- 0: if $\{type_{\mathsf{E}}(U) \mid U \text{ an } \mathsf{E}\text{-unification problem}\} \cap \{0\} \neq \emptyset$;
- ∞: if $\infty \in \{type_{\mathsf{E}}(U) \mid U \text{ an } \mathsf{E}\text{-unification problem}\} \subseteq \{\infty, 1, 2, \ldots\}$;
- ω: if $\{type_{\mathsf{E}}(U) \mid U \text{ an } \mathsf{E}\text{-unification problem}\} \subseteq \{1, 2, \ldots\}$ and there is no n such that $type_{\mathsf{E}}(U) \leq n$ for each E-unification problem U;
- n: if $n \in \{type_{\mathsf{E}}(U) \mid U \text{ an } \mathsf{E}\text{-unification problem}\} \subseteq \{1, \ldots, n\}$.

In [3] Ghilardi, translates the traditional equational unification of a theory E to the algebraic unification of the equational class of algebras $\mathcal{V}$ that E determines. In this translations unification problems become finitely presented algebras, and unifiers are homomorphisms from that algebras into projective algebras in the class $\mathcal{V}$. It is here where dualities play a central role, since different kinds of dualities have been used in the literature to describe free, projective and finitely presented algebras of equational classes of algebras.

In this talk we will present a list of cases where the use of dualities have been central to provide algorithms to calculate the unification type of certain E-unification problems.

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Concrete Coalgebraic Modal Logic

Liang-Ting Chen
University of Birmingham
Joint work with Achim Jung

As introduced by Larry Moss, coalgebraic modal logic generalises modal logic and different approaches have been proposed so far, e.g. cover modality, predicate liftings, abstract logic to name but a few. Some of them have been studied in different categories rather than the category of sets, and the general framework based on a dual adjunction is now almost standard. However, a general study based on *concrete dualities* [1] has not yet been exploited. In this talk, I will introduce our ongoing project on concrete coalgebraic modal logic. Firstly I will formulate few basic notions concretely and show connections in different approaches, for examples objects of modalities and (generalised) predicate liftings.

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The Construction of Free Algebras via a Functor on Partial Algebras

Dion Coumans
Radboud University of Nijmegen
Joint work with Sam van Gool

In this talk we introduce a new setting, based on partial algebras, for studying constructions of finitely generated free algebras. We give a method to describe, for certain varieties V, the finitely generated free V-algebras as the colimit of a chain of finite partial algebras that is obtained by repeated application of a functor. We give sufficient conditions on V for our method to apply and use duality theory to show that our method applies in particular to certain classes of modal algebras. Finally we discuss some on-going research on relating the construction of finitely generated free Heyting algebras, as described by N. Bezhanishvilli and M. Gehrke, to the universal model in intuitionistic logic.

The Structure of Skew Distributive Lattices

Karin Cvetko Vah
University of Ljubljana

In the talk I will present the standard decomposition theorems for skew lattices. The emphasize will be given to the description of the structure of skew distributive lattices. We shall see that any skew distributive lattice is a distributive lattice of rectangular bands. Further information is given by the so called coset structure. We shall discuss the role of the cosets in the recently obtained duality for skew distributive lattices, which will be presented at this workshop by Sam Van Gool, and is a joined work with Andrej Bauer, Mai Gehrke, Sam Van Gool and Ganna Kudryavtseva.

Generalised Gelfand Spectra for Noncommutative Operator Algebras and Multi-Valued Logic for Quantum Systems

Andreas Döring
University of Oxford

Gelfand duality provides an enormously useful bridge between commutative C^*-algebras algebras and (locally) compact Hausdorff spaces. Yet, in quantum theory noncommutative C^*-algebras play a key role. For these, a suitable notion of spectrum is largely lacking. I will show that with each unital C^*-algebras one can associate a presheaf topos with a distiguished object, the spectral presheaf, that generalises the Gelfand spectrum of the commutative case. This assignment is shown to be functorial, providing a step towards noncommutative Gelfand duality. In the case of von Neumann algebras, the spectral presheaf can also be seen as a generalised Stone spectrum. I will discuss some topologies on the spectral presheaf, in particular the daseinisation topology, and how they relate to propositions about the quantum system at hand. This leads to a new, multi-valued and intuitionistic form of logic for quantum systems.

Stone Duality for First-order Logic: A Nominal Approach to Logic and Topology

Murdoch James Gabbay
Heriot-Watt University

We give a finite and purely equational axiomatisation of first-order classical logic in nominal algebra, and using an accompanying semantics in nominal sets we prove a duality theorem for a suitable generalisation of Stone spaces. The technical details are non-trivial and include new ideas including sigma- and amgis-algebras. There is also the motivation that it is possible at all, and in a relatively straightforward manner, to use nominal logics and semantics to extend duality to logic and topology in the presence of terms, variables, and bindings. This has never been done before in quite this way, and I even hope it will exemplify a new family of duality proofs.

Spectral-like and Priestley-style Duality for Distributive Hilbert Algebras with Infimum

María Esteban Garcia
University of Barcelona
Joint work with S. A. Celani and R. Jansana

Recently, they have been studied [5, 6, 4, 1] Spectral-like and Priestley-style dualities for Distributive meet semilattices and Hilbert Algebras (the algebraic counterpart of the implicative fragment of intuitionistic logic). By *Spectral-like* we mean dualities based on compactly based sober spaces. By *Priestley-style* we mean dualities based on Priestley spaces.

Distributive Hilbert Algebras with infimum, or $DH^{\wedge}$-algebras, are included in the variety that is the algebraic counterpart of the implicative fragment of intuitionistic logic, augmented with a conjunction that is not necessarily the residuum of the implication, as it is in the intuitionistic case.

In this talk on our two forthcoming papers [7, 8], we introduce two special kinds of spaces, based on compactly based sober spaces and Priestley spaces respectively, that were inspired by the dualities previously mentioned. We prove that the categories of these spaces, together with certain kinds of relations, are dually equivalent to the category of $DH^{\wedge}$-algebras and semi-homomorphisms. We restrict this result to give two dualities for the category of $DH^{\wedge}$-algebras and homomorphisms. This dualities generalizes the ones presented in [3, 2] for implicative semi-lattices. We show how each one of these categories can be construct from the other. Moreover, we use the Spectral-like duality to give a dual characterization of the main classes of filters for these algebras, namely (irreducible) meet filters, (irreducible) implicative filters and absorbent filters.

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Sheaf Representations of MV-algebras via Priestley Duality

Mai Gehrke
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Joint work with S. van Gool and V. Marra

MV-algebras, introduced by Chang in 1958 as semantics for the infinite-valued Lukasiewicz logic, have been studied extensively for their own sake and relative to their connections to various other areas of mathematics. Representation theory has played an important role in the theory of MV-algebras while duality theory has not been used except in some special subclasses such as locally finite or finitely presented MV-algebras. From the point of view of duality theory, the full class of MV-algebras provides an interesting case study because a simple split between topological and discrete components of duality is obstructed for a different reason than the one known from modal logic. This was the topic of several joint papers with Hilary Priestley in which we developed extended dualities, based on Priestley duality, for a type of varieties of distributive lattice ordered algebras including the variety of MV-algebras. However, in this talk I will mainly focus on more recent work joint with Sam van Gool and Vincenzo Marra. We study the lattice duals of arbitrary MV-algebras. Thus none of the special dualities developed by MV-algebraists apply. We show that the two representation theorems for arbitrary MV-algebras as algebras of global sections of sheaves of MV-algebras with special properties, obtained, respectively, by Filipoiu and Georgescu in 1995 and by Dubuc and Poveda in 2010, may be seen as stemming from decompositions of the lattice duals. Further, we use our analysis of the dual spaces of MV-algebras to establish a broad MV-algebraic generalisation of Kaplansky's classical 1947 result stating that a compact Hausdorff space is uniquely determined up to homeomorphism by its lattice of real-valued continuous functions. In this work, the theory of canonical extensions plays an important role in the analysis of the interaction between the lattice dual spaces and the multiplication of MV-algebras.

A Non-commutative Priestley Duality

Sam van Gool
Radboud University Nijmegen
Joint work with A. Bauer, K. Cvetko-Vah, M. Gehrke and G. Kudryavtseva

In this talk on our paper [2], we describe a new duality for skew distributive lattices. *Skew lattices* [4] consist of two idempotent semigroup operations on the same underlying set, which are related by certain absorption identities, but are not necessarily commutative. Skew (distributive) lattices form a natural non-commutative generalization of (distributive) lattices. For more details on the algebraic properties of skew lattices, see, e.g., Karin Cvetko-Vah's lecture in this conference.

It is known [5] that so-called *left-handed skew distributive lattices* can be embedded in partial function algebras, whose operations are given by "*restriction*" and "*override*". We strengthen this result by proving that a category of left-handed skew distributive lattices with zero is in a dual equivalence with a category of *sheaves, or étalé spaces, over local Priestley spaces.* In particular, every skew distributive lattice is isomorphic to an *algebra of local sections* over the compact open downsets of a local Priestley space.

Although our duality for left-handed skew distributive lattices is not a natural duality in the sense of Clark, Davey, Priestley et al., it does make use of the idea that the *points* of the dual space of a given algebra should correspond to certain *quotients* of that algebra. In the case of left-handed skew distributive lattices, a point of the dual space can be regarded as an element of quotient of the left-handed skew distributive lattice which is 'primitive'. Here, the primitive left-handed skew distributive lattices form an easy to understand, but infinite, subclass of left-handed skew distributive lattices, which generates the entire variety.

Our results generalize both Priestley duality [6] and the recent development of Stone duality for skew Boolean algebras [1, 3].

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On the Logic of Perfect MV-algebras: Projectivity, Unification, Structural Completeness

Revaz Grigolia
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The variety of $MV(C)$-algebras generated by perfect MV-algebras and the logic of this variety is investigated. Perfect $MV(C)$-algebras are those MV-algebras generated by their infinitesimal elements or, equivalently, generated by their radical, where radical is the intersection of all maximal ideals. Dual objects of these algebras are constructed. It is shown that finitely generated $MV(C)$-algebra is projective if it is finitely presented. Unification type of the variety of $MV(C)$-algebras is 1. Moreover, the variety of $MV(C)$-algebras is structurally complete.

Some Remarks on the Metric Vietoris Monad

Dirk Hofmann
University of Aveiro

The main source of inspiration for this talk is the work of R. Rosebrugh and R.J. Wood on constructive complete distributive lattices where the authors elegantly employ the concepts of adjunction and module [2, 3]. We recall that an order relation on a set X defines a monotone map of type

$$X^{\mathrm{op}} \times X \to \mathsf{2},$$

and from that one obtains the Yoneda embeddings

$$X \to \mathsf{2}^{X^{\mathrm{op}}} =: PX \qquad \text{and} \qquad X \to (\mathsf{2}^{X})^{\mathrm{op}} =: VX.$$

Furthermore, PX and VX are part of monads $\mathbb{P}$ and $\mathbb{V}$ on Ord (the category of ordered sets and monotone maps), hence one obtains full embeddings

$$\mathsf{Ord}_{\mathbb{P}} \to \mathsf{Ord}^{\mathbb{P}} \qquad \text{and} \qquad \mathsf{Ord}_{\mathbb{V}} \to \mathsf{Ord}^{\mathbb{V}}$$

from the Kleisli categories into the Eilenberg–Moore categories $\mathsf{Ord}^{\mathbb{P}} \simeq \mathsf{Sup}$ (the category of complete lattices and sup-preserving maps) and $\mathsf{Ord}^{\mathbb{V}} \simeq \mathsf{Inf}$ (the category of complete lattices and inf-preserving maps) respectively. From that one obtains an equivalence

$$\mathrm{kar}(\mathsf{Ord}_{\mathbb{P}}) \simeq \mathsf{CCD}_{\mathrm{sup}} \qquad \text{and} \qquad \mathrm{kar}(\mathsf{Ord}_{\mathbb{V}}) \simeq \mathsf{CCD}_{\mathrm{inf}}$$

between the idempotent split completion of the Kleisli categories on one side, and the categories of completely distributive complete lattice and sup- respectively inf-preserving maps on the other. These equivalences restrict to

$$\mathsf{Ord}_{\mathbb{P}} \simeq \mathsf{TAL}_{\mathrm{sup}} \qquad \text{and} \qquad \mathsf{Ord}_{\mathbb{V}} \simeq \mathsf{TAL}_{\mathrm{inf}},$$

where “TAL” stands for totally algebraic lattices. Finally, both sides lead to the equivalence

$$\mathsf{Ord}^{\mathrm{op}} \simeq \mathsf{TAL}$$

between the dual category of Ord and the category TAL of totally algebraic lattices and sup- and inf-preserving maps.

Employing a formal analogy between order sets and topological (and other kinds of) spaces, in this talk we will follow the path described above, but now with topological and approach spaces in lieu of ordered sets (the latter representing “metric” topological spaces, see [1]). To illustrate this analogy, note that the ultrafilter convergence of a topological space defines a continuous map

$$(UX)^{\mathrm{op}} \times X \to \mathsf{2}$$

(where UX is the free ordered compact Hausdorff space over X, $(UX)^{\mathrm{op}}$ its Hochster dual, and $\mathsf{2}$ the Sierpiński space), which induces continuous maps

$$X \to \mathsf{2}^{(UX)^{\mathrm{op}}} =: PX \qquad \text{and} \qquad X \to (\mathsf{2}^{X})^{\mathrm{op}} =: VX.$$

As it turns out, PX is isomorphic to the filter-of-opens space of X, and VX is the upper Vietoris space. As above, both constructions are parts of monads, but, in contrast to the ordered case, the subsequent development is not symmetric. In this talk we will concentrate on the Vietoris monad, and show how these analogies lead to variations of Isbell, Stone, Priestley and Esakia duality. Finally, by writing $[0,\infty]$ instead of $\mathsf{2}$, we are automatically provided with metric variants of these constructions.

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Topological Duality for N4-lattices and N4-lattices with Modal Operators

Ramon Jansana
University of Barcelona

I will present joint work with Umberto Rivieccio. We develop a new topological duality for N4-lattices different form the one obtained by Odintsov. Our duality builds only on Esakia duality for Heyting algebras whereas Odintsov's builds also on Cornish and Fowler duality for De Morgan algebras. We extend our duality to obtain a duality for N4-lattices with modal operators such as those introduced by Rivieccio and by Odintsov.

A General Duality Theory for Clones

Sebastian Kerkhoff
Technische Universität Dresden

When it comes to dualising clones, the usual approach is to consider a clone as the set of term functions of a suitable algebra and then try to dualise this algebra, which may, or may not, be possible. Another approach was introduced by D. Mašulović in 2006, where clones are dualised by treating them as sets of homomorphisms in a quasivariety that is then understood and dualised as a category. Although new results were obtained by using this duality, the approach is somewhat limited. It only works for a tiny fraction of clones (the so-called centralizer clones on finite sets), and it does not give us any information about what happens to the Galois connection Pol-Inv, that is set up between operations and relations and is possibly the single most effective item in a clone theorist's toolbox.

The aim of the talk is to extend this approach such that both of these drawbacks are overcome. In other words, we want to build a duality theory that is general enough to work for any clone and also dualises all parts of the aforementioned Galois connection Pol-Inv. To do so, we will look at clones as sets of morphisms in categories, obtaining a notion essentially identical to that of a model of a Lawvere theory. We will then outline how to generalize relations to categories and construct a Galois theory that is similar to Pol-Inv but applicable in any (possibly abstract) category. With these notions, we will present a general duality theory for clones, allowing us to dualise any given clone together with its relational counterpart and the relationship between them.

Finally, we will put the approach to work and illustrate its application by producing some specific results for concrete examples as well as some general results that come from studying the duals of clones in a rather abstract fashion.

Adjunctions Induced by the Congruence Lattices of the Free Algebras in a Variety

Vincenzo Marra
University of Milan

Let A be a (finitary) variety of algebras, and let $\mathcal{F}_\kappa^{\mathsf{A}}$ be the free A-algebra generated by a set of cardinality κ. Fix an A-algebra A. For any subset $R \subseteq \mathcal{F}_\kappa^{\mathsf{A}} \times \mathcal{F}_\kappa^{\mathsf{A}}$, define its *solution set* (in A) to be

$$\mathbb{V}(R) := \{a \in A^\kappa \mid s(a) = t(a) \text{ holds in } A, \text{ for all } (s,t) \in R\}.$$

For any subset $S \subseteq A^\kappa$, define its *associated congruence* (on $\mathcal{F}_\kappa^{\mathsf{A}}$) to be

$$\mathbb{I}(S) := \left\{(s,t) \in \mathcal{F}_\kappa^{\mathsf{A}} \times \mathcal{F}_\kappa^{\mathsf{A}} \mid s(a) = t(a) \text{ holds in } A, \text{ for all } a \in S\right\}.$$

Then the pair $(\mathbb{V}, \mathbb{I})$ yields a Galois connection between the powersets of $\mathcal{F}_\kappa^{\mathsf{A}} \times \mathcal{F}_\kappa^{\mathsf{A}}$ and A^κ, for each cardinal κ. That is, we have

$$R \subseteq \mathbb{I}(S) \text{ if, and only if, } S \subseteq \mathbb{V}(R).$$

This, in turn, leads to a dual adjunction between A and the following category $\mathsf{Sub}_{\mathrm{def}}^{\mathsf{A}}$ of *subsets of A^κ and definable maps.* An object of $\mathsf{Sub}_{\mathrm{def}}^{\mathsf{A}}$ is an inclusion map $S \subseteq A^\kappa$, for some set S and some cardinal κ. A morphism between $S \subseteq A^\kappa$ and $T \subseteq A^\mu$ is a function $f\colon S \to T$ that agrees over S with some term-definable map $A^\kappa \to A^\mu$.

The construction above is a rather direct abstraction of the dual adjunction in algebraic geometry between affine varieties in k^n, for an algebraically closed field k, and ideals in the polynomial ring $k[X_1, \ldots, X_n]$. It arose in a recent joint paper with L. Spada, for the special case of MV-algebras. In that setting, it is crucial that two further facts hold. Take A to be the standard MV-algebra $[0,1]$. Then the closure operator on $[0,1]^\kappa$ given by the composition $\mathbb{V} \circ \mathbb{I}$ agrees with (and hence *defines*) topological closure in the standard Tychonoff (product) topology on $[0,1]^\kappa$, where $[0,1]$ carries its Euclidean topology. Moreover, the closure operator on $\mathcal{F}_\kappa^{\mathsf{A}} \times \mathcal{F}_\kappa^{\mathsf{A}}$ given by $\mathbb{I} \circ \mathbb{V}$ performs the construction of the *radical congruence* generated by R, i.e. the intersection of all maximal congruences extending R. (Compare Hilbert's Nullstellensatz for k-algebras.) This promptly leads to the main result that semisimple MV-algebras are dually equivalent to compact Hausdorff spaces embedded in Tychonoff cubes, endowed with definable (necessarily continuous) maps as morphisms.

I discuss the two basic questions, which conditions on the variety A and the A-algebra A guarantee that the closure operator $\mathbb{V} \circ \mathbb{I}$ is topological, and that the closure operator $\mathbb{I} \circ \mathbb{V}$ constructs the radical congruence in the sense above.

Fibred Universal Algebra: Many-Valued Duality Models of Predicate Logics

Yoshihiro Maruyama
University of Oxford

Universal Algebra provides foundations of algebraic *propositional* logic. We propose Fibred Universal Algebra or Indexed Universal Algebra as foundations of algebraic *predicate* logic, including both first-order logic and higher-order logic. Duality Theory then provides various models of predicate logic. Among them, Ω-valued models based on dualising objects Ω are striking and useful, relating to many-valued models of set theory (if one category involved is the category of sets); e.g., Boolean-valued, Heyting-valued, and quantum-valued models. Although the framework we develop is of quite universal character, we shall especially focus on substructural logics.

A fibred algebra is defined as a functorial bundle of algebras as follows. Let $\mathcal{V}$ be a variety of ordered algebras in the sense of Universal Algebra; $\mathcal{V}$ may also be taken to be a quasi-variety as in the theory of natural dualities. Let $\mathbf{V}$ denote the category of algebras in $\mathcal{V}$ and their homomorphisms. Then, a fibred algebra (or fibred $\mathcal{V}$-algebra to be precise) is defined as a variety-valued presheaf $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{V}$ for a category $\mathbf{C}$, which may be required to satisfy some conditions like being a CCC (or monoidal closed if $\mathbf{C}$ has a monoidal structure) or being a concrete category in Adámek-Herrlich-Strecker's sense. The former is the case when higher-order logic is concerned. The latter is the case in categorical duality theories as developed by Johnstone and Porst-Tholen. $P(C)$ is called a fibre of P. Such fibred algebras are main objects of study in Fibred Universal Algebra.

Building upon Lawvere's idea of quantifiers as adjoints, we require a fibred algebra $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{V}$ to have adjoints of $P(\pi)$ for all projections π in $\mathbf{C}$ (with so-called Beck-Chevalley conditions); more conditions may be required to represent other logical structures (e.g., equality and comprehension). Such special types of functors provide sound and complete semantics for substructural predicate logics in the sense of axiomatic extensions of Full Lambek Calculus; in these cases, $\mathcal{V}$ is taken to be varieties of residuated algebras. Thus, $\mathcal{V}$ can be the variety of Heyting algebras, and the corresponding soudness and completeness result in this case is well known in categorical logic, and can be extended to substructural predicate logics in general as shown in this work. In such a way, Fibred Universal Algebra provides foundations of algebraic predicate logic.

Duality models are those functors $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{V}$ with the logical conditions that give dual adjunctions between $\mathbf{C}$ and $\mathbf{V}$; e.g., the dual adjunction between spaces and frames provides a model of (topological) geometric logic; dual adjunctions between convex structures and continuous lattices, which we show are essentially the Hofmann-Mislove-Stralka duality, provide models of convex geometric logic; dual adjunctions between sets and Heyting algebras, induced by frames Ω as dualising objects, give rise to sheaf models of intuitionistic logic. The last dual adjunctions can be extended to substructural logics in general, providing models of substructural predicate logics; actually, substructural set theories. We may thus consider that duality for propositional logic is a Lawverian model of predicate logic, and even set theory.

Admissibility in Finite Algebras

George Metcalfe
University of Bern

The admissibility of a quasiequation in a finite algebra corresponds to the validity of that quasiequation in a finite free algebra, and is hence decidable. However, a naive approach to checking admissibility leads to computationally unfeasible procedures even for small algebras, and tells us little about the properties of admissible quasiequations for the algebra in question.

The aim of this talk is to explain, first, a uniform method for obtaining algorithms for checking admissibility in finite algebras, and, second, a strategy using natural dualities for axiomatizing admissibility in these cases.

Sahlqvist Correspondence for Intuitionistic Modal μ-Calculus

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Sahlqvist correspondence theory is among the most celebrated and useful results of the classical theory of modal logic, and one of the hallmarks of its success. Traditionally developedin a model-theoretic setting, it provides an algorithmic, syntactic identification of a class of modal formulas whose associated normal modal logics are strongly complete with respect to first-order definable classes of frames.

Sahlvist's result can equivalently be reformulated algebraically, via the well known duality between frames and complete atomic Boolean algebras with operators (BAO's). Using this duality-based approach, the Sahlqvist mechanism can be the motivated in terms of the order-theoretic properties on the algebraic interpretation of the logical connectives. This perspective gave rise to a research program [1, 2, 3, 4, 5] aimed at extending Sahlqvist-type results to wide families of logics the propositional base of which is non-classical. The highlight of this research program is the algorithm ALBA [2] for the elimination of monadic second order variables, which effectively extends and unifies the existing most general resultson correspondence.

The present talk reports on an ongoing work with Willem Conradie, Yves Fomatati, and Sumit Sourabh, which extends the duality-based approach to the intuitionistic modal μ-calculus. In particular, an enhanced version of ALBA has been defined so that μ-formulas can be treated in which all the variables to be eliminated might occur in the scope of fixpoint binders. This enhancement is proved to be sound thanks to the order-theoretic properties of the interpretation of fixpoint binders in the algebraic semantics for intuitionistic modal μ-calculus. The syntactically defined class of recursive formulas/inequalities is presented and an informal justification is given that the enhanced ALBA is successful on this class.

References

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Mundici's Γ-functor Theorem for Star-shaped Sets via Minkowski's Duality with Gauge Functions

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Extending previous results by C. C. Chang, in 1986 D. Mundici proved that to any MV-algebra A there corresponds a uniquely determined lattice-ordered Abelian group with unit (G, u) such that A is (isomorphic to) the unit interval $[0, u]$ of (G, u). The functor that takes (G, u) back to $[0, u]$ is traditionally denoted Γ. The construction of (G, u) from A uses "good sequences" of elements in A that suffice to canonically represent each element of G. We consider here the special case that G is the vector lattice (that is lattice-ordered vector space) of all continuous real-valued positively homogeneous functions on $\mathbb{R}^n$, with unit a distinguished such function that is nowhere zero except at the origin. We first generalise the classical theory of Minkowski's gauge functions of convex bodies so as to obtain an order-reversing bijection between the set of appropriately defined "star-shaped bodies" in $\mathbb{R}^n$, and the elements in the positive cone of G. We are then able to prove geometrically a representation theorem for star-shaped bodies that, through the order-reversing bijection above, is seen to be the exact geometric counterpart of the Γ-functor theorem. These results are part of the speaker's research towards her Ph.D. diploma. The investigation is concerned with the construction of algebraic counterparts, in the braad sense of duality theory, of convex geometry in Euclidean spaces.

The Many-valued Coalgebraic Cover Modality

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The aim of Coalgebraic Logic is to find formalisms that allow reasoning about T-coalgebras uniformly in the functor T. Moss' seminal idea was to consider the set functor T as providing a modality ∇_T, the semantics of which is given in terms of the relation lifting of T. The latter exists whenever T preserves weak pullbacks.

In joint work with Marta Bílková, Alexander Kurz and Jiří Velebil, we introduce basic notions and results about relation liftings on categories enriched in a commutative quantale. We derive two necessary and sufficient conditions for a 2-functor T to admit a functorial relation lifting: one is the existence of a distributive law of T over the 'powerset monad' on categories, one is the preservation by T of 'exactness' of certain squares. Both characterizations are generalizations of classical results known for set functors: the first characterization generalizes the existence of a distributive law over the genuine powerset monad, the second generalizes preservation of weak pullbacks.

We then introduce a generalized power set functor on categories enriched in commutative quantales. Under certain assumption on the quantale, this functor admits a relation lifting and allows us to study the semantics of the coalgebraic cover modality in the enriched setting.