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on

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Abstracts

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Foreword

This booklet contains the abstracts of the talks for the first part of the two-part workskop on Duality Theory in Algebra, Logic and Computer Science to be held in Oxford in the summer of 2012. The second part, with a greater focus on logic and algebra, is secheduled for 15–17 August.

We are grateful to all our participants for the interest they have shown in the June workshop and especially to our speakers for their contributions to what promises to be a very stimulating programme. Thanks are also due to the Oxford Mathematical Institute for hosting the event.

We acknowledge with gratitude the financial support which has made it possible for the workshops to take place: funding from EPSRC, through a Platform Grant awarded to the Mathematical Institute, and the British Logic Colloquium. We are also grateful to Professor Georg Gottlob (Oxford University Department of Computer Science) for providing a contribution, from a DIADEM Project grant, targetted at bringing together the mathematical theory of bilattices and the applications of bilattice models in computer science and logic.

Leonardo M. Cabrer
M. Andrew Moshier
Hilary A. Priestley
Organisers

Oxford, 13 June 2012

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Shorter Contributions

A Tutorial on Bilattices

Ofer Arieli
Academic College of Tel-Aviv

Bilattices are algebraic structures containing two related partial orders. They were introduced in the mid 80's by Matthew Ginsberg as a general framework for many applications, and since then have been further investigated and applied for various goals by Fitting, Avron and others. In this talk, we describe the basic theory concerning bilattices and give a survey on their primary applications in logic and computer science. This includes a presentation of proof systems that correspond to bilattices in an essential way, some notes on the use of bilattices for reasoning with inconsistency, and an overview of bilattice-based fixpoint semantics for logic programs.

An Algebraic Approach to Gelfand Duality

Guram Bezhanishvili
New Mexico State University
Joint work with P.J. Morandi and B. Olberding

By the celebrated Gelfand duality, the category **KHaus** of compact Hausdorff spaces is dually equivalent to the category $\mathbf{C^*Alg}$ of commutative (real) C*-algebras. I'll discuss the history and modern account of Gelfand duality. The approach I will present is more algebraic/categorical than analytic. I will show that $\mathbf{C^*Alg}$ forms a full subcategory of the category **bal** of bounded Archimedean ℓ-algebras, and give several algebraic and categorical characterizations of $\mathbf{C^*Alg}$ within **bal**. Among other things, we will see that there is a contravariant adjunction between the categories **bal** and **KHaus** that restricts naturally to Gelfand duality between $\mathbf{C^*Alg}$ and **KHaus**.

Modal Compact Hausdorff Spaces

Nick Bezhanishvili
Imperial College

and

John Harding
New Mexico State University

[Two linked talks with a common title and common abstract]

Joint work with Guram Bezhanishvili

We introduce modal compact Hausdorff spaces as generalizations of modal spaces, and show these are coalgebras for the Vietoris functor on compact Hausdorff spaces. We also introduce modal compact regular frames and modal de Vries algebras as algebraic counterparts of modal compact Hausdorff spaces, and give dualities for the categories involved. These extend the familiar Isbell and de Vries dualities for compact Hausdorff spaces, as well as the duality between modal spaces and modal algebras. As the first step in the logical treatment of modal compact Hausdorff spaces, we give a version of Sahlqvist correspondence for the positive modal language.

Dualities for Locally Hypercompact, Stably Hypercompact and Hyperspectral Spaces

Marcel Erné
Leibniz Universität Hannover

Several central notions of lattice and domain theory are or may be defined in terms of the Scott topology. For example, an element of a poset P is compact iff it generates a Scott-open principal filter; or, a poset is continuous iff its Scott topology is supercontinuous (= completely distributive). Replacing in such definitions the Scott topology by the weaker upper topology υP (generated by the complements of principal ideals), one arrives at analogous "hyper-notions". Thus, an element is hypercompact iff it generates an υ-open principal filter; specifically, a subset of a topological space is hypercompact iff its saturation is finitely generated; and a poset is hypercontinuous iff its upper topology is supercontinuous. Similarly, replacing "compact" with "hypercompact" in the respective definitions, one passes from locally compact spaces to locally hypercompact spaces, from stably compact spaces to stably hypercompact spaces, from spectral spaces to hyperspectral spaces, and so on. At first glance, the "hyper-versions" look quite similar to their classical counterparts. But a closer inspection reveals dramatic differences. For example, the category of locally compact sober spaces is dual to the category of continuous frames, and analogously, the category of locally hypercompact sober spaces is dual to the category of hypercontinuous frames. But the crucial difference between these two dualities is that there is an abundance of non-homeomorphic locally compact sober (or even T2) spaces with the same specialization order, whereas a locally hypercompact sober space is entirely determined by its specialization order: there is a concrete isomorphism between the category of locally hypercompact sober spaces and quasicontinuous domains—and a locally hyper-compact T1-space is already discrete. So one might guess that the topological theory of locally hypercompact spaces etc. is rather specific. But for Scott spaces (and even for the wider class of monotone determined spaces) a surprising coincidence arises: here every compact open set is already hypercompact. Moreover, the category of quasialgebraic domains is not only dual to the category of hyperalgebraic frames, but also concretely isomorphic to the category of compactly based sober Scott spaces—which answers in the negative Priestley's question of whether there might exist spectral spaces that carry the Scott topology but are not quasicontinuous. Indeed, the category of quasicoherent domains is concretely isomorphic to that of hyperspectral spaces and to the category of Priestley spaces carrying the Lawson topology; and these categories are dual to the category of hypercoherent frames and to the category of distributive lattices whose principal ideals are finite intersections of prime ideals.

Topological duality for lattices via canonical extensions

Sam van Gool
Radboud University Nijmegen
Joint work with Mai Gehrke

In this talk on our forthcoming paper [4], we describe a new topological duality for bounded lattices. The two main features of our duality are that it generalizes Stone duality for bounded distributive lattices, and that the morphisms on either side are not the standard ones. A positive consequence of the choice of morphisms is that those on the topological side are functional. We obtain the following results:

(a) canonical extensions of bounded lattices are the algebraic versions of the existing dualities for bounded lattices by Urquhart [7] and Hartung [5];
(b) there is a universal construction which associates to an arbitrary lattice two distributive lattice envelopes with an adjoint pair between them;
(c) we identify precisely which maps between bounded lattices admit functional duals on our newly defined dual spaces.

For the result mentioned under (a), we rely on previous work of Gehrke, Jónsson and Harding [2,3]. For the universal construction of (b), we modify a construction of the injective hull of a semilattice by Bruns and Lakser [1], adjusting their concept of 'admissibility' to the finitary case. For (c), we use Priestley duality for distributive lattices [6] and our own universal characterization in (b) of the distributive envelopes a bounded lattice.

[1] G. Bruns and H. Lakser, *Injective hulls of semilattices*, Canadian Mathematical Bulletin **13** (1970), no. 1, 115-118.
[2] Mai Gehrke and John Harding, *Bounded lattice expansions*, Journal of Algebra **238** (2001), no. 1, 345-371.
[3] Mai Gehrke and Bjarni Jónsson, *Bounded distributive lattices with operators*, Mathematica Japonica **40** (1994), no. 2, 207-215.
[4] Mai Gehrke and Sam van Gool, *Topological duality for lattices via canonical extensions*, forthcoming (2012).
[5] Gerd Hartung, An extended duality for lattices, *General algebra and applications* (K. Denecke and H.J. Vogel, eds.), Heldermann-Verlag, Berlin, 1993, pp. 126-142.
[6] Hilary Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, Bulletin of the London Mathematical Society **2** (1970), 186-190.
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Extending Functions to the Natural Extension

Georges Hansoul
Université de Liège

Introduction. The concept of canonical extension has been defined successively for various classes of algebras. Some kind of general paradigm arise rather easily: the types of the considered algebras have a well definite reduct—the "dominant", or base operations–and the other operations are somewhat secondary and expected to have some compatibility relations with respect to the base operations (in case of lattice-based algebras, the extra operations were first considered to be operators, then isotone functions, and now just arbitrary maps). One can find in litterature mainly two ways to obtain the canonical exten- sion. It sometimes happens that the canonical extension comes all in one, by a process in which all operations (base and extra) receive a single treatment—think of profinite or natural completions. But in other cases, we need a two step process. First construct the canonical extension of the designated reduct, and then extend each extra operation by some density procedure. In this case it may happen that more than one choice is available.

The lattice-based case has been up to now considered at length. But recently Davey, Gouveia, Haviar and Priestley have considered canonical extensions (under the name of natural extensions) of algebras lying in an internally residually finite prevariety $\mathcal{A}$, that is, a class of the form $\mathbb{ISP}(\mathcal{M})$ for some set $\mathcal{M}$ of finite algebras. For facility, we suppose here $\mathcal{A} = \mathbb{ISP}(\underline{P})$ where $\underline{P}$ is a finite algebra. In this case, the canonical or natural extension is obtained (in a single step) as a double dual construction. If $A \in \mathcal{A}$, its "proto-dual" is $A^* = \mathcal{A}(A, \underline{P})$ and A naturally embeds into P^A by the evaluation map $e(a \mapsto e_a(\varphi) = \varphi(a)$ for $\varphi \in A^*)$. Then the natural extension A^η of A is the closure of $e(A)$ within P^{A^*} (with the product topology of P endowed with the discrete topology). This situation will be the designated reduct of the situation of our talk. In other words, *we consider a map* $u\colon A \to B$ *between algebras of* $\mathcal{A}$ *and see how to lift* u *to the level of the natural extensions of* A *and* B.

If we want to parallel the lattice-based case, we first have to refine the topology on P^{A^*}.

1 The δ-topology

Suppose P and X are topological members of a subcategory $\mathcal{X}$ of Set. We introduce the δ-topology on P^X as follows. By a *partial continuous morphism* $f\colon X \to P$ we mean a continuous morphism $f\colon \operatorname{dom} f \to P$ where $\operatorname{dom} f$ is a closed subobject of X. The δ-topology on $\mathcal{X}(X, P)$ has as open basis the sets of the form

$$O_f = \{\, x \in \mathcal{X}(X, P) \mid f \subseteq x \,\}.$$

For instance, let P be the 2-element discrete space. if X has the cofinite topology, then you get the product topology. If X is the dual space of a Boolean algebra B, then the δ-topology on 2^X coincides with the usual δ-topology (also called σ-topology by Gehrke and Jónsson) on the canonical extension 2^X of B since the latter has for basis the intervals $[F, O]$ where $F \subseteq O$,

F closed and O open in X: we have $[F, O] = O_f$ for the partial continuous f which sends x to 0 if $x \in F$ and to 1 if $x \notin O$.

Though obvious, the following observations are essential:

1) *if x is a continuous morphism $X \to P$, then x is isolated in $\mathcal{X}(X, P)$;*
2) *if P is injective in $\mathcal{X}$, then the continuous morphisms are dense in $\mathcal{X}(X, P)$.*

2 Extensions of maps into finite algebras

In view of the results of the previous section, to make things easy and to avoid technicalities, we shall have the following standing assumptions for the rest of the paper: We work in a prevariety $\mathcal{A} = \mathbb{ISP}(\underline{P})$ generated by a finite algebra $\underline{P}$. The set P is endowed with a topological structure $\underset{\sim}{P}$ in which the topology is discrete and *which leads to a natural duality on $\mathcal{A}$*. On the dual class $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{P})$, we consider two categories. Firstly the dual category $\mathcal{X}$ of $\mathcal{A}$, whose morphisms respect $\underset{\sim}{P}$ and in particular are continuous. Secondly, if we denote by $\underset{\sim}{P}^b$ the non topological part of $\underset{\sim}{P}$, we also have the category $\mathcal{X}^b$, whose morphisms only respect $\underset{\sim}{P}^b$. In addition to $\underset{\sim}{P}$ leading a natural duality on A, we also assume that $\underset{\sim}{P}$ is injective in $\mathcal{X}$.

With these assumptions, we know that each A is isomorphic with its double dual $\mathcal{X}(A^*, \underset{\sim}{P})$ (with $A^* = \mathcal{A}(A, \underline{P})$), that the natural extension of A is $A^\eta = \mathcal{X}^b(A^*, \underset{\sim}{P}^b)$, and that the points of A are isolated and dense in A^η with respect to the product topology on A^η.

We now have a map $u\colon A \to B$ $(A, B \in \mathcal{A})$ where B is finite, and we want to extend it to A^η. Let $x \in A^\eta$. Then the family of all $u(V \cap A)$, where V runs through the δ-neighborhoods of x, is a lower directed family of non-empty finite sets: it has a least member that we shall denote by $\widetilde{u}(x)$.

Definition. The *natural* or canonical *extension* of $u\colon A \to B$ is the map $\widetilde{u}\colon A^\eta \to \mathcal{P}(B)$: $x \mapsto \widetilde{u}(x)$. Of course if $|\widetilde{u}(x)| = 1$ for all x, that is, if u is *smooth*, then $\widetilde{u}$ may be thought of as a map $A \to B$. But unfortunately, *if u is not smooth, it is not possible to choose continuously an element $u'(x)$ in each $\widetilde{u}(x)$.*

3 Extensions of maps

We now have a map $u\colon A \to B$ where A and B are arbitrary in $\mathcal{A}$. To reduce this situation to that of the previous section, we fix a finite subset F of the dual $B^* = \mathcal{A}(B, \underline{P})$ of B. Then $u_F\colon A \to B \to P^F$ defined as the composition with u of the projection pr_F along $F\colon B \cong \mathcal{X}(B^*, \underset{\sim}{P}) \subsetneq P^{B^*} \to P^F$ is a map from A into the finite algebra P^F, and therefore has an extension $\widetilde{u_F}$. Let $u(x, F) = \{\, y \in P^{B^*} \mid pr_F(y) \in \widetilde{u_F}(x) \,\}$. Then the following can be shown.

Theorem. *For each $x \in A^\eta$,*

$$\widetilde{u}(x) := \bigcap \{\, u(x, F) \mid F \text{ finite} \subseteq B^* \,\}$$

is a non-empty closed subset of B^η.

This leads to the following:

Definition. The *natural extension* of $u\colon A \to B$ is the map $\widetilde{u}\colon A^\eta \to \Gamma(B^\eta)\colon x \mapsto \widetilde{u}(x)$, where $\Gamma(B^\eta)$ is the space of closed subsets of B^η.

Theorem. *If A^η is endowed with the δ-topology and $\Gamma(B^\eta)$ with the co-Scott topology, then $\widetilde{u}$ is a continuous extension of u.*

The Categorical Duality between Complete (Semi)Lattices with Operators and Contexts with Relations

Peter Jipsen
School of Computational Sciences, Chapman University
Joint work with and M. Andrew Moshier

Since the seminal papers of Jónsson and Tarksi in the early 1950s, the categorical duality between complete and atomic Boolean algebras with operators and complete homomorphisms, and Kripke frames with bounded morphisms has been a corner stone of algebraic logic. In a paper with N. Galatos [1], we introduced residuated frames as objects dual to complete residuated lattices, but did not dualize complete residuated lattice homomorphisms. In a residuated frame the underlying lattice-order of a complete residuated lattice is represented by a context (in the terminology of formal concept analysis) or a polarity (in Birkhoff's terminology). Related work appears in [2,3] under the additional assumption that the contexts are reduced.

Recently M. A. Moshier [4] defined morphisms for contexts to obtain a category Cxt that is dual to the category INF of complete meet semilattices with completely meet-preserving homomorphisms. In this talk we define the category of contexts with relations by showing how the context morphisms, with suitable restrictions to bounded morphisms, correspond to homomorphisms between complete lattices with operators. Hence we obtain a categorical duality between complete (semi)lattices with operators and contexts with relations. This setting includes for example an equivalence between complete residuated lattices and residuated frames, which can be viewed as a basic duality of substructural logic.

Considering the subcategory of algebraic contexts with relations gives a duality to all semilattices with operators, while adding topology to the contexts leads to a bigger category that is dual to all lattices with operators, extending dualities of Urquhart, Hartung and Hartonas. One of the advantages of working on the context side is that objects can be logarithmically smaller and are simpler to construct. The notion of context morphism can also be loosened to a pair of Chu-like relations which compose with ordinary relation composition. This allows further freedom and efficiencies in the construction of operators and morphisms.

As application one has, for example, that the duality maps products of complete lattices to certain disjoint unions of contexts, and other constructions like ordinal sums and poset products can also be obtained by combinatorial means on the context side.

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[2] J. M. Dunn, M. Gehrke and A. Palmigiano, Canonical extensions and relational completeness of some substructural logics, J. Symbolic Logic, 70(3) 2005, 713–740.
[3] M. Gehrke, Generalized Kripke frames, Studia Logica, 84, 2006, 241–275.
[4] M. A. Moshier, A relational category of formal contexts, preprint.

The Duality Between Direct and Predicate Transformer Semantics

Klaus Keimel
Technische Universität Darmstadt

Together with G. D. Plotkin, R. Tix, Th. Streicher and others I have been involved in establishing a domain theoretical semantics for languages combining probability and nondeterminism. There is still one piece lacking in this enterprise. Everything takes place in the category of directed complete posets and Scott-continuous maps. In all of these cases one has an equivalence between a state transformer semantics and a predicate transformer semantics. In beginning to fill in the lacking piece I began to think about the general framework in which one can hope for an equivalence between the two types of semantics. As far as I can see this framework is rather narrow. The mothers for such equivalences are the continuation monads $R^{R^?}$ over a domain R of observations: There is a natural bijection between continuous maps $t\colon X \to R^{R^Y}$ (state transformers) and continuous maps $s\colon R^Y \to R^X$ (predicate transformers). Specifying an algebraic structure on R leads to two monads subordinate to the continuation monad, the monad $\mathcal{M}$, where $\mathcal{M}X$ is the dcpo of all algebra homomorphisms $h\colon R^X \to R$, and the free algebra monad $\mathcal{F}$, where $\mathcal{F}X$ is the the directed complete subalgebra of R^{R^X} generated by the projections $\widehat{x} = (f \mapsto f(x))\colon R^X \to R$. The equivalence between state and predicate transformer semantics works well for the monad $\mathcal{M}$, but the monad appropriate for semantics is the monad $\mathcal{F}$. We will discuss the question under which conditions the two monads agree. The notion of entropic algebras from universal algebra is crucial. The monads used for semantics of nondeterministic choice and for probabilistic choice fit into the above framework. In these examples one chooses R to be $(\mathbf{2}, \max, 0)$, $(\mathbf{2}, \min, 1)$, $(\overline{\mathbb{R}}_+, +, 0)$. When combining both kinds of choice one has to relax the above framework and use relaxed notions of homomorphism and entropicity.

Positive Coalgebraic Logic

Alexander Kurz
University of Leicester
Joint work with A. Balan and J. Velebil.

Positive modal logic as introduced by Dunn is the positive fragment of the basic modal logic K. We show that this is a special instance of a more general situation in which one replaces Kripke frames by coalgebras for an arbitrary (weak-pullback preserving) functor on the category of sets. This more general result is obtained via an analysis of the relationship between the adjunction relating sets and Boolean algebras on the one hand and the adjunction relating posets and distributive lattices on the other hand.

Dualities of Stably Compact Spaces

Jimmie Lawson
Louisiana State University

Stably compact spaces have attracted increased attention in recent years for a variety of reasons. First of all they appear to be the closest analog in the T_0-setting to that fundamental topological class consisting of the compact Hausdorff spaces. Although they do not appear in the literature with the high frequency of compact Hausdorff spaces, yet they have a distinctive and substantial theory that is in many ways analogous to that of compact Hausdorff spaces and in some other ways is more interesting and intricate, for example, the theory of their associated partial orders. A second feature that has proved important from the perspective of theoretical computer science is the ability to transfer a variety of constructions that have arisen in domain theory to the setting of stably compact spaces and the resulting stability or robustness of the property of being stably compact under such constructions. Thus the "stably" of "stably compact" has broader connotations than its original definition. We note, however, that in moving from traditional domain theory to a theory centered on stably compact spaces, one often needs to replace order-theoretic structures with analogous topological ones, i.e., one is pushed from an order-theoretic to a topological perspective. However, strong connections remain between the theory of stably compact spaces and aspects of domain theory, which we will seek to emphasize.

A third feature of stably compact spaces is a basic duality which they themselves exhibit, the so-called de Groot duality, which manifests itself in various guises in the constructions that one carries out on them. There are various formulations of this basic duality and a variety of such extended dualities arising from it. For example, in the setting of stably compact spaces, certain dualities of an intuitive or informal nature such as angelic vs. demonic nondeterminism become explicit. More recent and exotic extended dualities involving capacities have been established by Jean Goubauldt-Larrecq. We suggest the measure-theoretic idea of input/output pavings as an alternative formulation of de Groot duality and a convenient framework for these more exotic dualities.

Stone Duality in Unexpected Places

Michael Mislove
Tulane University

Domain theory provides a remarkably rich set of mathematical constructs for computation. But one construct - the probabilistic power domain, Prob, has proved to be especially problematical: Prob doesn't combine well with other monads on domains, and despite repeated attempts, Prob is yet to be shown to leave any CCC of (continuous) domains invariant. A recent and related line of research has produced an alternative to Prob. It uses random variables over domains to model probabilistic choice. In this talk I will describe how random variables over domains are constructed, and I'll also show how Stone duality arises naturally to explain one of the key steps. I'll also describe how the same approach provides a uniform model for the classical channels of Shannon information theory.

Categories of Formal Contexts

M. Andrew Moshier
Chapman University

Formal contexts (or *polarities* in Birkhof's terminology) provide a convenient combinatorial way to present closure operators on sets. They have been studied extensively for their applications to concept analysis, particularly for finite contexts, and are used regularly as a technical device in general lattice theory (for example, to describe the Dedekind-MacNeille completion of a lattice). In the most common uses, *morphisms* of contexts do not play a role. Although various scholars (most thoroughly, Marcel Erné) have considered certain notions of context morphisms, these efforts have generally concentrated either on special kinds of contexts that closely match certain "nice" lattices or on special kinds of lattice morphisms.

Here we propose a category of formal contexts in which morphisms are relations that satisfy a certain natural combinatorial property. The idea is to take our cue from the fact that a formal context is simply a binary relation between two sets. So the identity morphism of such an object should be that binary relation itself. From this, we get the combinatorial properties of morphisms more or less automatically.

The first main result of the talk is that the category of contexts is dually equivalent to the category INF, of complete meet lattices with meet-preserving maps. To get a duality with complete lattices, the second result characterizes those context morphisms that correspond to complete lattice homomorphisms.

We also consider various constructions that are well-known in INF to illustrate that formal contexts yield remarkably simple, combinatorial descriptions of many common constructions.

Implicative Twist-Structures

Umberto Rivieccio
Birmingham University

The twist-structure construction is used to represent algebras related to non-classical logics (e.g., Nelson algebras, bilattices) as a special kind of power of some better-known algebraic structure (distributive lattices, Heyting algebras). I will introduce a special type of twist-structure that is built as a power of a generalized Boolean algebra and whose algebraic language is restricted to an implication and a negation. The class of algebras thus obtained is a variety that is semisimple, arithmetical, finitely generated and has equationally definable principal congruences. I will present a characterization of the congruences of each algebra in the variety in terms of the congruences of the corresponding generalized Boolean algebra and, using this result, I will describe the lattice of all subvarieties of implicative twist-structures.

Dualities Induced by Canonical Extensions

Rukiye Cavus
Université de Liège

Whenever a concept of canonical extension makes sense, it can usually be obtained as a double dual process. Using a duality in one direction and coming back with a discretised version of it. In fact the existence of the canonical extension is often shown to be equivalent to the existence of duality. So the maxim "under each canonical extension is hidden a duality" may come as triviality. However it may be considered as interesting in certain circumstances. Here we want to work out details of this process in the case of Boolean algebras with operations (mind you : not operators)(and it can be easily extended to all lattice expansions). As there exists an upper and lower extension for these non-normal operations, we are conforonted to two dualities, and upper and lower one. The comparison of these two non natural dualities gives us a better understanding of canonical extensions.

Beyond $\mathcal{FOUR}$: Representations of Non-interlaced Bilattices Using Natural Duality

Andrew Craig
University of Oxford
Joint work with L.M. Cabrer and H.A. Priestley

A bilattice **B** is said to be *interlaced* if the truth lattice operations are monotonic with respect to the knowledge order, and the knowledge lattice operations are monotonic with respect to the truth order. As the existing product representations only apply to interlaced bilattices, we look at natural dualities for quasivarieties generated by non-interlaced bilattices. Examples include the seven-element bilattice which was used by Ginsberg [1] for applications in default logic.

Given an algebra **M** with M its underlying set, an n-ary relation on M is said to be *algebraic* over **M** if it forms a subalgebra of $\mathbf{M}^n$. The theory of natural dualities uses topological spaces with additional algebraic relations (and operations) as the dual structures. As the bilattices we consider are not interlaced, the knowledge order is no longer an algebraic binary relation on the bilattice and thus it cannot be used as a relation in the dual structures given by the theory of natural dualities.

However, the knowledge order is still intrinsic to these dual structures, and we show how it is encoded in a relational structure which yields a duality. The nature of the obtained dual structures suggest an alternative algebraic semantics which may include a change in the signature.

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Banaschewski's Duality and Cube Bundles over Compact Hausdorff Spaces

Sebastian Kerkhoff and Friedrich Martin Schneider
Technische Universität Dresden

In 1981, B. Banaschewski established a duality between the category of compact Hausdorff spaces and that of separated, functionally complete I-lattices. In our talk, we want to dwell upon some details of this particular duality and discuss possible steps towards a Serre-Swan type result concerning finitely generated, projective modules of the I-lattice associated to a compact Hausdorff space via Banaschewski's duality

Distributive Bilattices and Their Cousins: Representations via Natural Dualities

Hilary Priestley
University of Oxford

Topological dual representations for the variety of distributive bilattices have been found by Mobasher *et al.* and by Jung and Rivieccio. This short talk presents a fresh perspective on the representation theory, within the distributive setting and embracing bilattices of various algebraic types. The approach, based on natural dualities set up by hom-functors, has a number of merits:

- it provides a uniform strategy applicable to algebras of different types and it elucidates the earlier representation theory;
- it supplies the dual representations very directly, using a bare minimum of algebraic machinery (with the fundamental product representation emerging as a consequence rather than being the starting point);
- the constructions are automatically functorial and new algebraic information is available (on free algebras, coproducts and quotients);
- the role played by the **knowledge order** is highlighted.

The most significant feature here is perhaps the last one. Certain logics associated with bilattices have the potential to be studied via a four-valued relational semantics, with the mathematical theory delineating the ontological and epistemic roles of the four values and revealing the manner in which 'truth' and 'knowledge' may be seen as dual to one another. The methodology used here also points the way to representations for quasivarieties generated by finite bilattices which are not distributive—the topic of Andrew Craig's talk.

[*Acknowledgements are due to the many people with whom I have discussed bilattice theory and whose expertise in the field has been invaluable.*]